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AN ORBIT CLOSURE FOR A REPRESENTATION OF THE KRONECKER QUIVER WITH BAD SINGULARITIES

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Abstract. We give an example of a representation of the Kronecker quiver for which the closure of the corresponding orbit contains a singularity smoothly equivalent to the isolated singularity of two planes crossing at a point. Therefore this orbit closure is neither Cohen–Macaulay nor unibranch.

1. Introduction and the main result. Throughout the paper, k denotes a fixed algebraically closed field. Let $Q = (Q_0, Q_1, s, e)$ be a finite quiver, that is, Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, and $s, e : Q_1 \to Q_0$ are functions such that any arrow $\alpha \in Q_1$ has the starting vertex $s(\alpha)$ and the ending vertex $e(\alpha)$. Let $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. We define the vector space

$$\mathrm{rep}_Q(\mathbf{d}) = \prod_{\alpha\in Q_1} \mathbb{M}_{d_{e(\alpha)}\times d_{s(\alpha)}}(k),$$

where $\mathbb{M}_{d'\times d''}(k)$ denotes the set of $d' \times d''$-matrices with coefficients in k for any positive integers d' and d''. The product $\mathrm{Gl}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{Gl}_{d_i}(k)$ of general linear groups acts on $\mathrm{rep}_Q(\mathbf{d})$ via

$$g \star V = (g_{e(\alpha)} V_\alpha g^{-1}_{s(\alpha)})_{\alpha\in Q_1}$$

for any $g = (g_i)_{i\in Q_0} \in \mathrm{Gl}(\mathbf{d})$ and $V = (V_\alpha)_{\alpha\in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$. The orbits of this action correspond to the isomorphism classes of the representations of Q with dimension vector $\mathbf{d}$.

Let M be a representation of Q with dimension vector $\mathbf{d}$. We will denote by $\mathcal{O}_M$ the corresponding $\mathrm{Gl}(\mathbf{d})$-orbit in $\mathrm{rep}_Q(\mathbf{d})$. An interesting problem is to study the geometry of the orbit closure $\overline{\mathcal{O}}_M$. For example we may ask when it is regular, normal or Cohen–Macaulay. The orbit closure $\overline{\mathcal{O}}_M$ is Cohen–Macaulay and normal if Q is a Dynkin quiver of type $\mathbb{A}_n$ or $\mathbb{D}_m$ ([3], [4]). For the remaining Dynkin quivers of type $\mathbb{E}_l$, $l = 6, 7, 8$, we know at least that $\overline{\mathcal{O}}_M$ is *unibranch* ([14]), that is, its normalization map is bijective.

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The Dynkin quivers are the only quivers Q for which the variety $\operatorname{rep}_Q(\mathbf{d})$ has only finitely many $\mathrm{Gl}(\mathbf{d})$-orbit for any dimension vector $\mathbf{d}$. The simplest quiver admitting infinite families of orbits is a point with a loop. Then the points of $\operatorname{rep}_Q(\mathbf{d})$ are square matrices and the orbit $\mathcal{O}_M$ is a conjugacy class. Hence $\overline{\mathcal{O}}_M$ is normal and Cohen–Macaulay ([6], [10], [11]).

Another distinguished example is given by the Kronecker quiver $Q: 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$. It has been proved recently that then $\overline{\mathcal{O}}_M$ is regular in codimension one, and moreover it is Cohen–Macaulay and normal at any point N such that there is no point W satisfying $\overline{\mathcal{O}}_N \subsetneq \overline{\mathcal{O}}_W \subsetneq \overline{\mathcal{O}}_M$ ([2]). In fact, [2] gives a classification of the types of singularities $\operatorname{Sing}(\overline{\mathcal{O}}_M, N)$ for such points N. Recall that following Hesselink (see [9, (1.7)]), the types of singularities $\operatorname{Sing}(\mathcal{X}, x_0)$ and $\operatorname{Sing}(\mathcal{Y}, y_0)$ of two pointed varieties $(\mathcal{X}, x_0)$ and $(\mathcal{Y}, y_0)$ coincide if there are smooth morphisms $f : \mathcal{Z} \to \mathcal{X}$, $g : \mathcal{Z} \to \mathcal{Y}$ and a point $z_0 \in \mathcal{Z}$ with $f(z_0) = x_0$ and $g(z_0) = y_0$. If $\operatorname{Sing}(\mathcal{X}, x_0) = \operatorname{Sing}(\mathcal{Y}, y_0)$ then the variety $\mathcal{X}$ is regular (respectively, normal, Cohen–Macaulay, unibranch) at x_0 if and only if the same is true for the variety $\mathcal{Y}$ at y_0 (see [8, Section 17] for more information about smooth morphisms).

Let $\mathcal{V}$ be the set of points $(x, y, z, t) \in k^4$ such that $xz = xt = yz = yt = 0$. Thus $\mathcal{V}$ is a union of two planes intersecting at the point 0. Consequently, the variety $\mathcal{V}$ is neither unibranch nor normal at 0. It is also not difficult to show that $\mathcal{V}$ is not Cohen–Macaulay (see for instance [7, p. 459]). The main result of the paper shows that $\operatorname{Sing}(\mathcal{V}, 0)$ appears as the type of singularity of an orbit closure in $\operatorname{rep}_Q(\mathbf{d})$, where Q is the Kronecker quiver.

THEOREM 1. *Let Q be the Kronecker quiver $1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$ and $\mathbf{d} = (3, 3)$. Let $M = (M_\alpha, M_\beta)$ and $N = (N_\alpha, N_\beta)$ be two points of $\operatorname{rep}_Q(\mathbf{d})$ given by*

$$M_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ M_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ N_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ N_\beta = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix}$$

for some scalars $\lambda_1 \neq \lambda_2$. Then $N \in \overline{\mathcal{O}}_M$ and $\operatorname{Sing}(\overline{\mathcal{O}}_M, N) = \operatorname{Sing}(\mathcal{V}, 0)$.

Note that this theorem gives the first example (to the author's knowledge) of an orbit closure in a variety of representations of a quiver which is not Cohen–Macaulay.

2. Transversal slices. Let $Q = (Q_0, Q_1, s, e)$ be a finite quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector and $N = (N_\alpha)_{\alpha \in Q_1}$ be a point of $\operatorname{rep}_Q(\mathbf{d})$. We identify the tangent space $\mathcal{T}_{\operatorname{rep}_Q(\mathbf{d}), N}$ with the vector space $\operatorname{rep}_Q(\mathbf{d})$, the tangent space $\mathcal{T}_{\mathcal{O}_N, N}$ with a subspace of $\operatorname{rep}_Q(\mathbf{d})$ and the tangent space $\mathcal{T}_{\mathrm{Gl}(\mathbf{d}), 1}$ with the product $\prod_{i \in Q_0} \mathbb{M}_{d_i \times d_i}(k)$. Let $\mu : \mathrm{Gl}(\mathbf{d}) \to \mathcal{O}_N$ denote the orbit map sending g to $g \star N$. Then the induced linear map of tangent spaces $\mu' : \mathcal{T}_{\mathrm{Gl}(\mathbf{d}),1} \to \mathcal{T}_{\mathcal{O}_N,N}$ is given by the formula

$$\mu'(h) = (h_{e(\alpha)}N_\alpha - N_\alpha h_{s(\alpha)})_{\alpha\in Q_1}$$

for any $h = (h_i)_{i\in Q_0} \in \mathcal{T}_{\mathrm{Gl}(\mathbf{d}),1}$. The kernel of μ' is just the endomorphism space $\mathrm{End}_Q(N)$ of the representation N, and the stabilizer $\mu^{-1}(N)$ of the point N is just the automorphism group $\mathrm{Aut}_Q(N)$ of the representation N. Since $\mathrm{Aut}_Q(N)$ is a non-empty open subset of the vector space $\mathrm{End}_Q(N)$, we have

$$\begin{aligned}\dim \mathrm{Im}\,\mu' &= \dim \prod_{i\in Q_0} \mathbb{M}_{d_i\times d_i}(k) - \dim \mathrm{End}_Q(N)\\ &= \dim \mathrm{Gl}(\mathbf{d}) - \dim \mathrm{Aut}_Q(N) = \dim \mathcal{O}_N = \dim \mathcal{T}_{\mathcal{O}_N,N}.\end{aligned}$$

Consequently, μ' is a surjective map, which means that the orbit map μ is separable. This enables us to apply the transversal slice method explained in [13, Section 5.1] (see also [5, Section 6.2]). Namely, let $\mathcal{S}$ be a $\mathrm{Gl}(\mathbf{d})$-invariant subvariety of $\mathrm{rep}_Q(\mathbf{d})$ containing N. We choose a linear complement $\mathcal{C}$ of $\mathcal{T}_{\mathcal{O}_N,N}$ in $\mathcal{T}_{\mathrm{rep}_Q(\mathbf{d}),N} = \mathrm{rep}_Q(\mathbf{d})$. Then

$$\mathrm{Sing}(\mathcal{S}, N) = \mathrm{Sing}(\mathcal{S} \cap (N + \mathcal{C}), N).$$

For instance, we may apply this for any orbit closure $\mathcal{S} = \overline{\mathcal{O}}_M$ containing the point N.

3. The proof of Theorem 1. Let Q be the Kronecker quiver $1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$ and $\mathbf{d} = (3,3)$. We consider the representation M in $\mathrm{rep}_Q(\mathbf{d})$ given in Theorem 1. The following lemma characterizes the orbit $\mathcal{O}_M$.

Lemma 2. *Let $V = (V_\alpha, V_\beta)$ be a point of* $\mathrm{rep}_Q(\mathbf{d})$. *Then V belongs to $\mathcal{O}_M$ if and only if*

$$\mathrm{rk}\,[\,V_\alpha \quad V_\beta\,] = \mathrm{rk}\begin{bmatrix} V_\alpha \\ V_\beta \end{bmatrix} = 3, \qquad \mathrm{rk}\begin{bmatrix} V_\alpha & V_\beta & 0 \\ 0 & V_\alpha & V_\beta \end{bmatrix} = \mathrm{rk}\begin{bmatrix} V_\alpha & 0 \\ V_\beta & V_\alpha \\ 0 & V_\beta \end{bmatrix} = 5.$$

Proof. We will use some basic facts concerning finite-dimensional representations of the Kronecker quiver which can be found in [1] or [12]. Observe that the above conditions are invariant under the action of the group $\mathrm{Gl}(\mathbf{d})$ and hold for $V = M$. Thus one implication is proved.

We consider the following representations of Q:

$$P_1 = k \underset{0}{\overset{0}{\leftleftarrows}} 0, \qquad P_2 = k^2 \underset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\leftleftarrows}} k, \qquad I_1 = k \underset{[1\,0]}{\overset{[0\,1]}{\leftleftarrows}} k^2, \qquad I_2 = 0 \underset{0}{\overset{0}{\leftleftarrows}} k.$$

In particular, $M \simeq P_2 \oplus I_1$. Assume that V satisfies the above rank conditions. It is easy to check that the equality $\operatorname{rk}[\, V_\alpha \; V_\beta \,] = 3$ means that $\dim \operatorname{Hom}_Q(V, P_1) = 0$, and $\operatorname{rk} \left[\begin{smallmatrix} V_\alpha & V_\beta & 0 \\ 0 & V_\alpha & V_\beta \end{smallmatrix}\right] = 5$ means that $\dim \operatorname{Hom}_Q(V, P_2) = 1$. Observe that the radical $\operatorname{rad} P_2$ of the representation P_2 is isomorphic to $P_1 \oplus P_1$. Since P_2 is a projective representation, it follows that

$$1 = \dim \operatorname{Hom}_Q(V, P_2) - \dim \operatorname{Hom}_Q(V, \operatorname{rad} P_2)$$

is the multiplicity of P_2 as a direct summand of V. By duality, the representation I_1 occurs as a direct summand of V. Hence $V \simeq P_2 \oplus I_1 \oplus V'$ for some representation V'. Comparing the dimension vectors of the above representations we get $V' \simeq 0$, and consequently, $V \simeq M$. ∎

We fix two different scalars $\lambda_1, \lambda_2 \in k$. Let N be the representation in $\operatorname{rep}_Q(\mathbf{d})$ given in Theorem 1. It is easy to calculate that the tangent space $\mathcal{T}_{\mathcal{O}_N, N}$ consists of the points

$$\left(\begin{bmatrix} c_{1,1} & c_{1,2} & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{bmatrix}, \begin{bmatrix} \lambda_1 c_{1,1} & \lambda_2 c_{1,2} & 0 \\ \lambda_1 c_{2,1} & d_{2,2} & \lambda_1 c_{2,3} \\ d_{3,1} & \lambda_2 c_{3,2} & \lambda_2 c_{3,3} \end{bmatrix} \right),$$

where $c_{i,j}, d_{i,j} \in k$. We choose the following linear complement of $\mathcal{T}_{\mathcal{O}_N, N}$ in $\mathcal{T}_{\operatorname{rep}_Q(\mathbf{d}), N}$:

$$\mathcal{C} = \left\{ \left(\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & 0 & * \\ 0 & * & * \end{bmatrix} \right) \right\},$$

where each $*$ stands for an arbitrary scalar. Thus each element $V = (V_\alpha, V_\beta)$ of $N + \mathcal{C}$ has the form

$$(1) \qquad V_\alpha = \begin{bmatrix} 0 & 0 & a_{1,3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad V_\beta = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & 0 & b_{2,3} \\ 0 & b_{3,2} & b_{3,3} \end{bmatrix}$$

for some scalars $a_{1,3}, b_{i,j}$. Let $\mathcal{U}$ denote the open subset of $\overline{\mathcal{O}}_M \cap (N + \mathcal{C})$ given by the inequality $b_{2,1} \neq b_{3,2}$.

Lemma 3. *$\mathcal{U}$ consists of the points V of the form* (1) *such that*

$$(2) \qquad a_{1,3} = b_{1,3} = b_{1,1}b_{1,2} = b_{1,1}b_{2,3} = b_{3,3}b_{1,2} = b_{3,3}b_{2,3} = 0, \qquad b_{2,1} \neq b_{3,2}.$$

Proof. We denote by $\mathcal{W}$ the set of points V of the form (1) satisfying (2). Let $V \in \mathcal{U}$. Observe that $\operatorname{rk}(M_\alpha + \lambda M_\beta) \le 2$ for any scalar λ. Since this is a closed condition invariant under the action of $\operatorname{Gl}(\mathbf{d})$, we have $\operatorname{rk}(V_\alpha + \lambda V_\beta) \le 2$ for any $\lambda \in k$. Thus the coefficients of the polynomial $\det(V_\alpha + \lambda V_\beta)$ of the variable λ vanish. After standard calculations we get

$$a_{1,3} = b_{1,3} = b_{1,1}b_{2,3} = b_{3,3}b_{1,2} = 0.$$

From Lemma 2 we conclude that

$$\operatorname{rk}\begin{bmatrix} V_\alpha & V_\beta & 0 \\ 0 & V_\alpha & V_\beta \end{bmatrix}, \operatorname{rk}\begin{bmatrix} V_\alpha & 0 \\ V_\beta & V_\alpha \\ 0 & V_\beta \end{bmatrix} \le 5.$$

Next standard calculations give the remaining two equalities

$$b_{1,1}b_{1,2} = b_{3,3}b_{2,3} = 0.$$

Thus $\mathcal{U} \subseteq \mathcal{W}$.

In order to prove the reverse inclusion it suffices to show that $\mathcal{W} \cap \mathcal{O}_M$ is a dense subset of $\mathcal{W}$. The variety $\mathcal{W}$ is the union of two four-dimensional irreducible components:

$$\mathcal{W}' = \left\{ \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} b_{1,1} & 0 & 0 \\ b_{2,1} & 0 & 0 \\ 0 & b_{3,2} & b_{3,3} \end{bmatrix} \right) : b_{2,1} \neq b_{3,2} \right\},$$

$$\mathcal{W}'' = \left\{ \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_{1,2} & 0 \\ b_{2,1} & 0 & b_{2,3} \\ 0 & b_{3,2} & 0 \end{bmatrix} \right) : b_{2,1} \neq b_{3,2} \right\}.$$

Applying Lemma 2 we can calculate that an element V in $\mathcal{W}$ belongs to $\mathcal{O}_M$ if and only if

$$\operatorname{rk}\,[\, b_{1,1} \quad b_{1,2} \,] = \operatorname{rk}\begin{bmatrix} b_{2,3} \\ b_{3,3} \end{bmatrix} = 1. \tag{3}$$

It is easy to see that there is a point in $\mathcal{W}'$ as well as a point in $\mathcal{W}''$ satisfying the open condition (3). Hence $\mathcal{W}' \cap \mathcal{O}_M$ is a dense subset of $\mathcal{W}'$ and $\mathcal{W}'' \cap \mathcal{O}_M$ is a dense subset of $\mathcal{W}''$. ∎

By the above lemma, $N \in \mathcal{U} \subset \overline{\mathcal{O}}_M$. Applying the transversal slice method we get

$$\operatorname{Sing}(\overline{\mathcal{O}}_M, N) = \operatorname{Sing}(\mathcal{U}, N).$$

It follows from Lemma 3 that $\mathcal{U}$ is isomorphic to the product of the smooth variety

$$\{(b_{2,1}, b_{3,2}) \in k^2 : b_{2,1} \neq b_{3,2}\}$$

and the variety $\mathcal{V}$ introduced in Section 1. Hence

$$\operatorname{Sing}(\mathcal{U}, N) = \operatorname{Sing}(\mathcal{V}, 0),$$

which finishes the proof of Theorem 1. ∎

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