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VECTOR-VALUED ERGODIC THEOREMS FOR MULTIPARAMETER ADDITIVE PROCESSES II

BY

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Abstract. Previously we obtained stochastic and pointwise ergodic theorems for a continuous d-parameter additive process F in $L_1((\Omega, \Sigma, \mu); X)$, where X is a reflexive Banach space, under the condition that F is bounded. In this paper we improve the previous results by considering the weaker condition that the function $W(\cdot) = \operatorname{ess\,sup}\{\|F(I)(\cdot)\| : I \subset [0,1)^d\}$ is integrable on Ω.

1. Introduction and result. Let X be a reflexive Banach space, and (Ω, Σ, μ) be a σ-finite measure space. For $1 \le p \le \infty$, let $L_p(\Omega; X) = L_p((\Omega, \Sigma, \mu); X)$ denote the usual Banach space of all X-valued strongly measurable functions f on Ω with the norm

$$\|f\|_p := \Big(\int \|f(\omega)\|^p \, d\mu(\omega)\Big)^{1/p} < \infty \qquad \text{if } 1 \le p < \infty,$$
$$\|f\|_\infty := \operatorname{ess\,sup}\{\|f(\omega)\| : \omega \in \Omega\} < \infty \quad \text{if } p = \infty.$$

If $d \ge 1$ is an integer, we let $\mathbb{R}_d^+ = \{u = (u_1, \ldots, u_d) : u_i \ge 0,\ 1 \le i \le d\}$ and $\mathbf{P}_d = \{u = (u_1, \ldots, u_d) : u_i > 0,\ 1 \le i \le d\}$. Further, $\mathcal{I}_d$ is the class of all bounded intervals I in $\mathbb{R}_d^+$ of the form

$$I = [a_1, b_1) \times \ldots \times [a_d, b_d),$$

where $0 \le a_i < b_i < \infty$, $1 \le i \le d$ (we note that $\mathcal{I}_d$ is somewhat different from that of [12], but this does not matter), and λ_d denotes the d-dimensional Lebesgue measure. In this paper, we consider a strongly measurable d-parameter semigroup $T = \{T(u) : u \in \mathbb{R}_d^+\}$ of linear contractions on $L_1(\Omega; X)$. Thus, T is strongly continuous on $\mathbf{P}_d$ (cf. Lemma VIII.7.9 in [5]). A linear operator S defined on $L_1(\Omega; X)$ is said to *have a majorant* P defined on $L_1(\Omega; \mathbb{R})$ if P is a positive linear operator on $L_1(\Omega; \mathbb{R})$ with the property that $\|Sf(\omega)\| \le P\|f(\cdot)\|(\omega)$ holds for almost all $\omega \in \Omega$, for every

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$f \in L_1(\Omega; X)$. As in [12], we will assume below that each $T(u)$, $u \in \mathbb{R}_d^+$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$.

By a (continuous d-parameter) *process* F in $L_1(\Omega; X)$ we mean a set function $F : \mathcal{I}_d \to L_1(\Omega; X)$. It is called *bounded* if

$$K(F) := \sup\{\|F(I)\|_1/\lambda_d(I) : I \in \mathcal{I}_d\} < \infty, \tag{1}$$

and an *additive process* (with respect to T) if it satisfies the following conditions:

(i) $T(u)F(I) = F(u + I)$ for all $u \in \mathbb{R}_d^+$ and $I \in \mathcal{I}_d$,

(ii) if $I_1, \ldots, I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$, then $F(I) = \sum_{i=1}^k F(I_i)$.

For example, if $F(I) = \int_I T(u)f\,du$ for all $I \in \mathcal{I}_d$, where f is a fixed function in $L_1(\Omega; X)$, then $F(I)$ defines a bounded additive process in $L_1(\Omega; X)$. There are many bounded additive processes in $L_1(\Omega; X)$ which cannot have this integral form (cf. [3]).

It is immediate that if F is a bounded additive process in $L_1(\Omega; X)$, then the mapping $\mathbf{P}_d \ni u = (u_1, \ldots, u_d) \mapsto F([0, u_1) \times \ldots \times [0, u_d)) \in L_1(\Omega; X)$ becomes continuous, the function

$$W(\cdot) := \operatorname{ess\,sup}\{\|F(I)(\cdot)\| : I \subset [0, 1)^d\} \tag{2}$$

belongs to $L_1^+(\Omega; \mathbb{R})$ and we have $\|W\|_1 \le K(F)$. In fact, we can take a sequence $\{I_n : n \ge 1\}$ of intervals with $I_n \subset [0, 1)^d$ for each $n \ge 1$ satisfying

$$W(\omega) = \sup_{n \ge 1} \|F(I_n)(\omega)\| \quad \text{ for almost all } \omega \in \Omega.$$

Then take a sequence $\{\mathbf{D}_n : n \ge 1\}$ of finite decompositions of the interval $[0, 1)^d$ such that each $\mathbf{D}_n$ consists of intervals $\{J_1^n, \ldots, J_{k(n)}^n\}$ in $\mathcal{I}_d$ and $I_n = \bigcup_{i=1}^{l(n)} J_i^n$ for some $l(n)$, with $1 \le l(n) \le k(n)$, and $\mathbf{D}_{n+1}$ is a refinement of $\mathbf{D}_n$ for every $n \ge 1$. It follows that

$$\|F(I_n)(\omega)\| \le \sum_{i=1}^{l(n)} \|F(J_i^n)(\omega)\| \le \sum_{i=1}^{k(n)} \|F(J_i^n)(\omega)\|$$

for almost all $\omega \in \Omega$. Putting $V_n(\omega) = \sum_{i=1}^{k(n)} \|F(J_i^n)(\omega)\|$ for $\omega \in \Omega$, we then get $0 \le V_n(\omega) \le V_{n+1}(\omega)$ on Ω, and

$$\|V_n\|_1 \le \sum_{i=1}^{k(n)} K(F)\lambda_d(J_i^n) = K(F).$$

Hence, the function $V(\omega) = \lim_{n\to\infty} V_n(\omega)$ satisfies $0 \le W(\omega) \le V(\omega)$ on Ω, and

$$\|W\|_1 \le \|V\|_1 = \lim_{n\to\infty} \|V_n\|_1 \le K(F) < \infty.$$

On the other hand, as shown by examples in §4, there are many unbounded additive processes F in $L_1(\Omega; X)$ for which the functions W defined by (2) are integrable on Ω. Since we considered in [12] *bounded* additive processes, the theorems there cannot be applied to such unbounded additive processes F.

Here we recall that the condition $W \in L_1^+(\Omega; \mathbb{R})$ was originally introduced by Kingman in [8] to obtain his pointwise ergodic theorem for a continuous 1-parameter additive or *subadditive* separable process, because his theorem fails to hold in the general case. (See also Akcoglu and Krengel [4].) In view of these facts, the author thinks that obtaining our ergodic theorems under the condition $W \in L_1^+(\Omega; \mathbb{R})$ is preferable. This is the starting point for the study in this paper.

In the following, $q\text{-}\lim_{\alpha\to\infty}$ and $q\text{-}\limsup_{\alpha\to\infty}$ will mean that these limits are taken as α tends to infinity along a countable dense subset Q of the positive real numbers. We may assume that Q includes the positive rational numbers. A net (f_α) of strongly measurable X-valued functions on Ω is said to *converge stochastically* to a strongly measurable X-valued function f_∞ on Ω if for every $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$ we have

$$\lim_{\alpha} \mu(A \cap \{\omega : \|f_\alpha(\omega) - f_\infty(\omega)\| > \varepsilon\}) = 0.$$

The purpose of this paper is to prove the following ergodic theorem, which improves Theorem 1 of [12].

THEOREM 3. *Let X be a reflexive Banach space and $T = \{T(u) : u \in \mathbb{R}_d^+\}$ be a semigroup of linear contractions on $L_1(\Omega; X)$, strongly continuous on $\mathbf{P}_d$, such that each $T(u)$, $u \in \mathbb{R}_d^+$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. Let F be a (continuous d-parameter) additive process in $L_1(\Omega; X)$ with respect to T.*

(I) *If F is measurable in the sense that the vector-valued function $u = (u_1, \ldots, u_d) \mapsto F([0, u_1) \times \ldots \times [0, u_d))$ from $\mathbf{P}_d$ to $L_1(\Omega; X)$ is strongly measurable, then the averages $\alpha^{-d}F([0, \alpha)^d)$ converge stochastically to a function F_∞ in $L_1(\Omega; X)$, invariant under T, as α tends to infinity.*

(II) *If the function W defined by* (2) *is integrable on Ω, and the operators $P_i = P(e^i)$, with e^i the ith unit vector in $\mathbb{R}_d^+$, satisfy the additional hypothesis*

$$\|P_i\|_p \le 1 \quad \text{for some } p > 1, \tag{3}$$

then there exists a function F_∞ in $L_1(\Omega; X)$, invariant under T, such that

$$F_\infty(\omega) = q\text{-}\lim_{\alpha\to\infty} \alpha^{-d} F([0, \alpha)^d)(\omega) \quad \text{for almost all } \omega \in \Omega. \tag{4}$$

In Theorem 1 of [12], we saw the stochastic convergence for a bounded additive process F in $L_1(\Omega; X)$. But, as shown by examples in §4, there are measurable additive processes F in $L_1(\Omega; X)$ which are not bounded.

Therefore, Theorem 3 generalizes Theorem 1 of [12]. Furthermore, the hypothesis (3) is strictly weaker than the hypothesis $\|P_i\|_\infty \le 1$ for $1 \le i \le d$; and the latter hypothesis was assumed in the second part of Theorem 1 of [12]. Thus, in this sense, Theorem 3 generalizes Theorem 1 of [12] as well.

2. Lemmas. To prove Theorem 3 we need the following two lemmas.

LEMMA 1. *Let h be a real-valued function on $\mathcal{I}_d$ such that*

(i) $h(u + I) \le h(I)$ *for all* $u \in \mathbb{R}_d^+$ *and* $I \in \mathcal{I}_d$,

(ii) $h(I) \le \sum_{i=1}^k h(I_i)$ *whenever* $I_1, \ldots, I_k \in \mathcal{I}_d$ *are pairwise disjoint and satisfy* $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$.

If the function $\widetilde{h}$ *on* $\mathbf{P}_d$ *defined by* $\widetilde{h}(u) = h([0, u_1) \times \ldots \times [0, u_d))$ *for* $u = (u_1, \ldots, u_d) \in \mathbf{P}_d$ *is Lebesgue measurable, then it is bounded above in any compact subset* $I^* = [\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d]$ *of* $\mathbf{P}_d$.

Proof. This is an adaptation of the argument of Theorem 7.4.1 of [6]. Let $a = (a_1, \ldots, a_d) \in \mathbf{P}_d$, and let ζ be a real number such that $\zeta \le \widetilde{h}(a)$. We denote by $(0, a)$ the open interval $(0, a_1) \times \ldots \times (0, a_d)$ in $\mathbf{P}_d$. Let $x = (x_1, \ldots, x_d) \in (0, a)$. Then, since $[0, a_i) = [0, x_i) \cup [x_i, a_i)$ for $1 \le i \le d$, we see that there are pairwise disjoint intervals $J_1, \ldots, J_{2^d}$ in $\mathcal{I}_d$ such that

$$[0, a) = [0, a_1) \times \ldots \times [0, a_d) = \bigcup_{j=1}^{2^d} J_j,$$

where J_j has the form

$$J_j = [\alpha(j), \beta(j)) = [\alpha(j)_1, \beta(j)_1) \times \ldots \times [\alpha(j)_d, \beta(j)_d)$$

for some $\alpha(j) = (\alpha(j)_1, \ldots, \alpha(j)_d) \in \mathbb{R}_d^+$ and $\beta(j) = (\beta(j)_1, \ldots, \beta(j)_d) \in \mathbf{P}_d$, and we have $[\alpha(j)_i, \beta(j)_i) = [0, x_i)$ or $[x_i, a_i)$ for $1 \le i \le d$. Condition (ii) implies that

$$\zeta \le \widetilde{h}(a) = h([0, a)) \le \sum_{j=1}^{2^d} h(J_j).$$

Since each J_j can be written as $J_j = u(j) + [0, v(j))$ with $u(j) \in \mathbb{R}_d^+$ and $v(j) \in (0, a)$, it follows from condition (i) that

$$\zeta \le \widetilde{h}(a) \le \sum_{j=1}^{2^d} \widetilde{h}(v(j)),$$

whence there exists j, $1 \le j \le 2^d$, such that $\zeta/2^d \le \widetilde{h}(v(j))$. If we write $E(\zeta) := \{y : y \in (0, a), \widetilde{h}(y) \ge \zeta/2^d\}$, then $v(j) \in E(\zeta)$ follows for this j. And, by the definition of $v(j)$, we see that for each i with $1 \le i \le d$, it follows that

$$v(j)_i = x_i \quad \text{or} \quad v(j)_i = a_i - x_i.$$

That is, $x_i = v(j)_i$ or $x_i = a_i - v(j)_i$; consequently,

$$(0, a) = \bigcup \{E(K) : K \subset \{1, \dots, d\}\},$$

where $E(K)$ denotes the subset of $(0, a)$ corresponding to K as follows: $E(K)$ is the set consisting of the elements $(x_1, \dots, x_d) \in (0, a)$ such that there exists $y = (y_1, \dots, y_d) \in E(\zeta)$ satisfying $x_i = y_i$ when $i \in K$, and $x_i = a_i - y_i$ when $i \notin K$. Since $\lambda_d(E(K)) = \lambda_d(E(\zeta))$ for every $K \subset \{1, \dots, d\}$, it follows that

$$\prod_{i=1}^{d} a_i = \lambda_d((0, a)) \le 2^d \lambda_d(E(\zeta)).$$

If the conclusion of the lemma were not true, i.e., if $\widetilde{h}$ were not bounded above in I^*, then there would exist $a(n) \in I^*, n \ge 1$, such that $\widetilde{h}(a(n)) \ge n$ for every $n \ge 1$. Then, since $\alpha_i \le a(n)_i \le \beta_i$ for $1 \le i \le d$, it follows from the above fact that the set

$$F(n) := \{x : x \in (0, \beta_1) \times \dots \times (0, \beta_d),\, \widetilde{h}(x) \ge n/2^d\}$$

must satisfy $\lambda_d(F(n)) \ge 2^{-d} \prod_{i=1}^{d} \alpha_i > 0$, whence $\widetilde{h}$ would be equal to ∞ on a set of positive Lebesgue measure. This is a contradiction, and hence the proof is complete.

LEMMA 2. *Let X be a reflexive Banach space and $T_1, \dots, T_d$ be commuting linear contractions on $L_1(\Omega; X)$. Suppose $P_1, \dots, P_d$ are (not necessarily commuting) positive linear contractions on $L_1(\Omega; \mathbb{R})$ such that $\|T_i f(\omega)\| \le P_i \|f(\cdot)\|(\omega)$ for almost all $\omega \in \Omega$, for every $f \in L_1(\Omega; X)$ and $1 \le i \le d$. If there exists $p > 1$ such that $\|P_i\|_p \le 1$ for every $1 \le i \le d$, then the ergodic averages*

$$A_n(T_1, \dots, T_d) f := A_n(T_1) \dots A_n(T_d) f,$$

where

$$A_n(T_i) := \frac{1}{n} \sum_{k=0}^{n-1} T_i^k,$$

converge a.e. on Ω for all $f \in L_1(\Omega; X)$ as n tends to infinity.

Proof. As in the proof of Theorem 1 of [12], let U denote the Brunel operator corresponding to $P_1, \dots, P_d$ (see Theorem 6.3.4 of [9]). Thus there exists a constant $C_d > 0$, depending only on d, and a nondecreasing sequence $d(n)$, $n = 1, 2, \dots$, of positive integers, with $\lim_{n\to\infty} d(n) = \infty$, such that if $f \in L_1(\Omega; X)$ then

$$\|A_n(T_1,\dots,T_d)f(\omega)\| \le C_d A_{d(n)}(U)\|f(\cdot)\|(\omega) \tag{5}$$

for almost all $\omega \in \Omega$. Since $\|P_i\|_p \le 1$ by hypothesis, U is a positive linear contraction on $L_1(\Omega;\mathbb{R})$ such that $\|U\|_p \le 1$. We may assume here that $1 < p < \infty$, by the Riesz convexity theorem. Then the ergodic theorem of Akcoglu and Chacon [2] implies that the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k g$$

exists a.e. on Ω for all $g \in L_1(\Omega;\mathbb{R})$, whence by (5) the maximal function

$$f^\sharp(\omega) := \sup_{n\ge 1} \|A_n(T_1,\dots,T_d)f(\omega)\| \tag{6}$$

satisfies $f^\sharp(\omega) < \infty$ for almost all $\omega \in \Omega$, for all $f \in L_1(\Omega;X)$. Thus, by Banach's convergence principle, it suffices to show that the limit $\lim_{n\to\infty} A_n(T_1,\dots,T_d)f$ exists a.e. on Ω for a function f in a dense subset M of $L_1(\Omega;X)$.

To prove this we notice that Akcoglu's dominated ergodic theorem [1] together with an induction argument (cf. e.g. [10]) implies that if $g \in L_p(\Omega;\mathbb{R})$, then the averages

$$A_{n_1}(P_1)\dots A_{n_d}(P_d)g$$

converge a.e. on Ω and in L_p-norm as $n_1,\dots,n_d$ tend to infinity independently; and the maximal function

$$g^*(\omega) := \sup_{n_1,\dots,n_d\ge 1} A_{n_1}\dots A_{n_d}|g|(\omega)$$

satisfies $\|g^*\|_p \le (p/(p-1))^d$. Since the reflexivity of X implies that $L_p(\Omega;X)$ is a reflexive Banach space, an easy modification of the argument of Theorem 3 of [10] shows that if $f \in L_p(\Omega;X)$, then the averages

$$A_{n_1}(T_1)\dots A_{n_d}(T_d)f$$

converge a.e. on Ω and in L_p-norm as $n_1,\dots,n_d$ tend to infinity independently. (Incidentally, the function $f^\sharp$ defined by (6) belongs to $L_p^+(\Omega;\mathbb{R})$ when $f \in L_p(\Omega;X)$, because $f^\sharp(\omega) \le \|f(\cdot)\|^*(\omega)$ a.e. on Ω and $\|f(\cdot)\| \in L_p^+(\Omega;\mathbb{R})$.)

Consequently, if $f \in L_1(\Omega;X)\cap L_p(\Omega;X)$, then the limit $A_n(T_1,\dots,T_d)f$ exists a.e. on Ω. This completes the proof, since $M := L_1(\Omega;X)\cap L_p(\Omega;X)$ is a dense subset of $L_1(\Omega;X)$.

3. Proof of Theorem 3

Proof of (I). Since each $P_i, 1 \le i \le d$, is a contraction majorant of the operator $T_i = T(e^i)$, it follows from the proof of Theorem 1 of [12] that the averages $n^{-d}F([0,n)^d)$, where $n \in \{1,2,\dots\}$, converge stochastically to a function F_∞ in $L_1(\Omega;X)$ as n tends to infinity. The invariance of F_∞ under the semigroup $T = \{T(u)\}$ follows, as in Theorem 1 of [12], when we see that $\alpha^{-d}F([0,\alpha)^d)$ converges stochastically to F_∞ as $\alpha \to \infty$. Thus we only prove its stochastic convergence below.

For $\alpha > 0$, let $n = n(\alpha)$ denote the greatest integer not exceeding α. If $\alpha > 2$, then, since $n-1 = n(\alpha) - 1 \geq 1$, it follows that

$$\alpha^{-d}F([0,n-1)^d) - F_\infty = [((n-1)/\alpha)^d - 1](n-1)^{-d}F([0,n-1)^d) + [(n-1)^{-d}F([0,n-1)^d) - F_\infty] =: \mathrm{I}(\alpha) + \mathrm{II}(\alpha),$$

and for every $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$ we have

$$\lim_{\alpha\to\infty} \mu(A \cap \{\omega : \|\mathrm{II}(\alpha)(\omega)\| > \varepsilon\}) = 0. \tag{7}$$

Thus we can choose a constant $\Gamma > 0$ and an integer $N \geq 2$ so that if $n = n(\alpha) \geq N$, then

$$\mu(A \cap \{\omega : \|(n-1)^{-d}F([0,n-1)^d)(\omega)\| > \Gamma\}) < \varepsilon.$$

By this and the fact that $\lim_{\alpha\to\infty}((n-1)/\alpha)^d = 1$, we find

$$\limsup_{\alpha\to\infty} \mu(A \cap \{\omega : \|\mathrm{I}(\alpha)(\omega)\| > \varepsilon\}) < \varepsilon.$$

This proves the stochastic convergence of $\alpha^{-d}F([0,n-1)^d)$ to F_∞ as $\alpha \to \infty$. Therefore, it suffices to show that the functions

$$\mathrm{III}(\alpha) := \alpha^{-d}F([0,\alpha)^d) - \alpha^{-d}F([0,n-1)^d), \quad \text{with } n = n(\alpha),$$

converge stochastically to 0 as $\alpha \to \infty$.

To see this, we use Lemma 1 as follows. First, since $T = \{T(u)\}$ is a contraction semigroup on $L_1(\Omega; X)$ by hypothesis, the real-valued function h on $\mathcal{I}_d$ defined by

$$h(I) = \|F(I)\|_1 \quad \text{for } I \in \mathcal{I}_d$$

satisfies conditions (i) and (ii) of Lemma 1. By the measurability of F, the function $\widetilde{h}$ of Lemma 1 becomes Lebesgue measurable. Thus we can apply Lemma 1 to infer that there exists a constant $C > 0$ such that $0 \leq \widetilde{h}(u) \leq C$ for all $u \in I^* := [2^{-1},2] \times \ldots \times [2^{-1},2] \subset \mathbf{P}_d$. It is elementary that if $\alpha > 2$, then since $n-1 = n(\alpha)-1 \geq 1$, the set $[0,\alpha)^d \setminus [0,n-1)^d$ has a decomposition $\{J_j : 1 \leq j \leq n^d - (n-1)^d\}$ into intervals in $\mathcal{I}_d$ such that each J_j has the form

$$J_j = u(j) + [0, v(j))$$

for some $u(j) \in \mathbb{R}_d^+$ and $v(j) \in I^*$. Therefore we deduce that

$$\begin{aligned}\|\mathrm{III}(\alpha)\|_1 &= \Big\|\alpha^{-d}\sum\{F(J_j) : 1 \leq j \leq n^d - (n-1)^d\}\Big\|_1 \\ &\leq \alpha^{-d}\sum\{\widetilde{h}(v(j)) : 1 \leq j \leq n^d - (n-1)^d\} \\ &\leq (1 - (1-n^{-1})^d) \cdot C \to 0\end{aligned}$$

as $\alpha \to \infty$, whence the desired conclusion follows.

Proof of (II). Here we assume that $W \in L_1^+(\Omega;\mathbb{R})$ and that the operators $P_i = P(e^i)$, $1 \le i \le d$, satisfy (3). We may assume as before that $1 < p < \infty$. Since

$$n^{-d}F([0,n)^d) = A_n(T_1,\dots,T_d)F([0,1)^d),$$

Lemma 2 implies that there exists a function F_∞ in $L_1(\Omega;X)$ such that

$$F_\infty(\omega) = \lim_{n\to\infty} n^{-d}F([0,n)^d)(\omega) \quad \text{for almost all } \omega \in \Omega. \tag{8}$$

Since F_∞ is invariant under $T_1,\dots,T_d$, we obtain the invariance of F_∞ under the semigroup $T = \{T(u)\}$ as soon as we show that $F_\infty(\omega) = q\text{-}\lim_{\alpha\to\infty} \alpha^{-d}F([0,\alpha)^d)(\omega)$ for almost all $\omega \in \Omega$. To prove this convergence result, we now introduce a new set function $F^1 : \mathcal{I}_d \to L_1^+(\Omega;\mathbb{R})$ as follows.

For $I \in \mathcal{I}_d$ we define

$$F^1(I)(\cdot) := \operatorname{ess\,sup}\{\|F(J)(\cdot)\| : J \subset I\}. \tag{9}$$

Since $W = F^1([0,1)^d) \in L_1^+(\Omega;\mathbb{R})$ by hypothesis, it follows that

(i) $F^1(I) \in L_1^+(\Omega;\mathbb{R})$,

(ii) $I \subset J$ implies $F^1(I)(\omega) \le F^1(J)(\omega)$ for almost all $\omega \in \Omega$,

(iii) $F^1(u+I)(\omega) \le P(u)F^1(I)(\omega)$ for almost all $\omega \in \Omega$, for every $u \in \mathbb{R}_d^+$ and $I \in \mathcal{I}_d$,

(iv) if $I_1,\dots,I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$, then $F^1(I)(\omega) \le \sum_{i=1}^k F^1(I_i)(\omega)$ for almost all $\omega \in \Omega$.

As in (I), we let $n = n(\alpha)$ for $\alpha > 0$. Then for almost all $\omega \in \Omega$ we have

$$\begin{aligned}
&\|\alpha^{-d}F([0,\alpha)^d)(\omega) - n^{-d}F([0,n)^d)(\omega)\| \\
&\quad \le \alpha^{-d}\|F([0,\alpha)^d)(\omega) - F([0,n)^d)(\omega)\| + (n^{-d} - \alpha^{-d})\|F([0,n)^d)(\omega)\| \\
&\quad \le n^{-d}\sum\{F^1(u+[0,1)^d)(\omega) : u \in \{0,1,\dots,n\}^d \setminus \{0,1,\dots,n-1\}^d\} \\
&\qquad + (1-(n/\alpha)^d)n^{-d}\|F([0,n)^d)(\omega)\| =: \mathrm{IV}(\alpha)(\omega) + \mathrm{V}(\alpha)(\omega),
\end{aligned}$$

and (8) implies that

$$q\text{-}\lim_{\alpha\to\infty} V(\alpha)(\omega) = 0 \quad \text{for almost all } \omega \in \Omega. \tag{10}$$

Therefore the proof will be completed if we show that $q\text{-}\lim_{\alpha\to\infty} \mathrm{IV}(\alpha)(\omega) = 0$ for almost all $\omega \in \Omega$.

To see this, let ε be a positive real number. Take a function $g \in L_1^+(\Omega;\mathbb{R}) \cap L_p^+(\Omega;\mathbb{R})$ so that

$$g \le W = F^1([0,1)^d) \quad \text{and} \quad \|W - g\|_1 < \varepsilon. \tag{11}$$

Using this g, we define a function $F_g(I)$ in $L_1(\Omega; X)$ for $I \in \mathcal{I}_d$, with $I \subset [0,1)^d$, by

$$F_g(I)(\omega) := \begin{cases} F(I)(\omega) & \text{if } \|F(I)(\omega)\| \le g(\omega), \\ g(\omega) \cdot \operatorname{sgn} F(I)(\omega) & \text{otherwise,} \end{cases}$$

where $\operatorname{sgn} x = x/\|x\|$ if $0 \ne x \in X$, and $\operatorname{sgn} 0 = 0$. Thus we have

$$\|F_g(I)(\omega)\| \le g(\omega) \quad \text{and} \quad \|F(I)(\omega) - F_g(I)(\omega)\| \le W(\omega) - g(\omega) \quad \text{on } \Omega,$$

where the last inequality comes from the fact that $\|F(I)(\omega)\| \le W(\omega)$ on Ω.

If

$$u = (n_1, \ldots, n_d) \in \{0, 1, \ldots\}^d \quad \text{and} \quad u \ne (0, \ldots, 0), \tag{12}$$

then let $k = \sum_{l=1}^d n_l$ (≥ 1) and denote by $\mathcal{S}(u)$ the set of all elements $(i(1), \ldots, i(k)) \in \{1, \ldots, d\}^k$ such that $n_l = \operatorname{card}\{m : i(m) = l,\ 1 \le m \le k\}$ for each $1 \le l \le d$ ($\operatorname{card} A$ is the number of elements of A). Since

$$F(u+I) = T_1^{n_1} \ldots T_d^{n_d} F(I) = T_1^{n_1} \ldots T_d^{n_d} F_g(I) + T_1^{n_1} \ldots T_d^{n_d} (F(I) - F_g(I)),$$

and $T_1, \ldots, T_d$ commute with each other, it follows that if $(i(1), \ldots, i(k)) \in \mathcal{S}(u)$, then

$$\|F(u+I)(\omega)\| \le P_1^{n_1} \ldots P_d^{n_d} g(\omega) + P_{i(1)} \ldots P_{i(k)} (W - g)(\omega)$$

for almost all $\omega \in \Omega$. Therefore if we put, for $u = (n_1, \ldots, n_d) \in \{0, 1, \ldots\}^d \setminus \{(0, \ldots, 0)\}$,

$$\begin{aligned}(W - g; u)(\omega) \\ := \min\{P_{i(1)} \ldots P_{i(k)} (W - g)(\omega) : (i(1), \ldots, i(k)) \in \mathcal{S}(u)\},\end{aligned} \tag{13}$$

then, by the definition of $F^1(u + [0,1)^d)$ (cf. (9)), we find

$$F^1(u + [0,1)^d)(\omega) \le P_1^{n_1} \ldots P_d^{n_d} g(\omega) + (W - g; u)(\omega) \tag{14}$$

for almost all $\omega \in \Omega$. Thus, by putting $(W - g; (0, \ldots, 0))(\omega) = (W - g)(\omega)$ if $u = (0, \ldots, 0) \in \mathbb{R}_d^+$, it follows that for almost all $\omega \in \Omega$,

$$\begin{aligned}\|\mathrm{IV}(\alpha)(\omega)\| &\le [(1 + 1/n)^d A_{n+1}(P_1, \ldots, P_d) g(\omega) - A_n(P_1, \ldots, P_d) g(\omega)] \\ &\quad + n^{-d} \sum \{(W - g; u)(\omega) : u \in \{0, 1, \ldots, n\}^d\} \\ &=: \widetilde{\mathrm{I}}(\alpha)(\omega) + \widetilde{\mathrm{II}}(\alpha)(\omega),\end{aligned}$$

and since $\lim_{n\to\infty} A_n(P_1, \ldots, P_d) g(\omega)$ exists for almost all $\omega \in \Omega$ (cf. the proof of Lemma 2), we have $q\text{-}\lim_{\alpha\to\infty} \widetilde{\mathrm{I}}(\alpha)(\omega) = 0$ for almost all $\omega \in \Omega$.

It remains to estimate the function

$$(W - g)^{\sim}(\omega) := q\text{-}\limsup_{\alpha\to\infty} \widetilde{\mathrm{II}}(\alpha)(\omega). \tag{15}$$

To do this, we use again the Brunel operator U corresponding to $P_1, \ldots, P_d$. By (13) and the property of the Brunel operator U (cf. e.g. the proof of

Theorem 6.3.4 of [9]), it follows that

$$(W-g)^{\sim}(\omega) = \limsup_{n\to\infty} n^{-d} \sum\{(W-g;u)(\omega) : u \in \{0,1,\dots,n\}^d\}$$
$$\le C_d \lim_{n\to\infty} A_n(U)(W-g)(\omega)$$

for almost all $\omega \in \Omega$, where we used the facts that $\|U\|_1 \le 1$ and that $\|U\|_p \le 1$ to deduce the almost everywhere convergence of the averages $A_n(U)(W-g)(\omega)$ as $n \to \infty$. Thus, Fatou's lemma implies that

$$\int_\Omega (W-g)^{\sim}(\omega)\, d\mu(\omega) \le C_d \liminf_{n\to\infty} \int_\Omega A_n(U)(W-g)(\omega)\, d\mu(\omega)$$
$$\le C_d\|W-g\|_1 < C_d\varepsilon.$$

It follows that if we set

$$\mathrm{IV}^\sharp(\omega) := q\text{-}\limsup_{\alpha\to\infty} \|\mathrm{IV}(\alpha)(\omega)\| \quad (\omega \in \Omega),$$

then

$$\mathrm{IV}^\sharp(\omega) \le q\text{-}\limsup_{\alpha\to\infty} \widetilde{\mathrm{II}}(\alpha)(\omega) = (W-g)^{\sim}(\omega)$$

for almost all $\omega \in \Omega$, and so

$$\int_\Omega \mathrm{IV}^\sharp(\omega)\, d\mu(\omega) \le \int_\Omega (W-g)^{\sim}(\omega)\, d\mu(\omega) \le C_d\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that $\mathrm{IV}^\sharp(\omega) = 0$ for almost all $\omega \in \Omega$, and hence the proof is complete.

We easily see from the above proof that Theorem 2 of [12] can be improved as follows when the set-valued function $F^1 : \mathcal{I}_d \to L_1^+$ defined by (9) is used in its proof. We omit the details.

THEOREM 4. *Let X, $T = \{T(u) : u \in \mathbb{R}_d^+\}$, and F be the same as in Theorem* 3. *Assume that the positive operators $P_i = P(e^i)$, $1 \le i \le d$, commute.*

(I) *If F is measurable in the sense of Theorem* 3, *then the averages*

$$F([0,\alpha_1) \times \ldots \times [0,\alpha_d)) / \prod_{i=1}^d \alpha_i$$

converge stochastically to a function F_∞ in $L_1(\Omega; X)$, invariant under $T = \{T(u) : u \in \mathbb{R}_d^+\}$, as α_i tends to infinity independently for each i with $1 \le i \le d$.

(II) *If the function W defined by* (2) *belongs to $L_1^+(\Omega;\mathbb{R})$, and if the averages*

$$A_n(P_1,\dots,P_d)f$$

converge a.e. for all $f \in L_1(\Omega;\mathbb{R})$ as n tends to infinity, then there exists a function F_∞ in $L_1(\Omega;X)$, invariant under T, such that (4) *holds.*

4. Examples. In this section we give three examples of additive processes F to show that (a) the measurability hypothesis on F cannot be omitted for the stochastic convergence of the averages $\alpha^{-d}F([0,\alpha)^d)$, (b) the hypothesis $W \in L_1^+(\Omega;\mathbb{R})$ is necessary for the a.e. convergence of the averages, and (c) there are many F, with $W \in L_1^+(\Omega;\mathbb{R})$, for which $K(F) = \infty$. For simplicity we restrict ourselves to the case $d = 2$ below.

EXAMPLE 1. Let $\Omega = \{\omega_0\}$ with $\mu(\{\omega_0\}) = 1$, and $T = \{T(u) : u \in \mathbb{R}_2^+\}$ be the semigroup consisting of the identity operator on $L_1(\Omega;\mathbb{R})$ alone. Take an additive real-valued function f on $\mathbb{R}$ (i.e., $f(s+t) = f(s)+f(t)$ for all $s,t \in \mathbb{R}$) such that

$$\sup\{|f(t)| : 0 < t < 1\} = \infty. \tag{16}$$

The existence of such an f is well known (see e.g. Lemma 1.14 of [13]). We recall that (16) is a necessary and sufficient condition for f to be nonmeasurable with respect to the Lebesgue measure on $\mathbb{R}$ (see e.g. Theorem 1 of [7]). Thus, our f is not measurable. Using this f, let

$$F(I) := (f(a_2) - f(a_1)) \cdot (f(b_2) - f(b_1))$$

for $I = [a_1,a_2) \times [b_1,b_2) \in \mathcal{I}_2$; then $F(I)$ defines an additive process in $L_1(\Omega;\mathbb{R}_2)$ which is not measurable in the sense of Theorem 3, by Fubini's theorem. From (16) we can choose real numbers t_1 and t_2, with $0 < t_1, t_2 < 1$, so that $f(t_1)/t_1 \neq f(t_2)/t_2$. Then, if we put $Q = \{r_1t_1+r_2t_2 : r_1, r_2$ are positive rationals$\}$, $\alpha^{-2}F([0,\alpha)^2) = f^2(\alpha)/\alpha^2$ fails to converge as α tends to infinity along the set Q.

EXAMPLE 2. Let $\Omega = [0,1)^2$, with the Lebesgue measure λ_2, and $T = \{T(u) : u \in \mathbb{R}_2^+\}$ be the semigroup of operators on $L_1([0,1)^2;\mathbb{R})$ defined by

$$T(u)f(x) := f(u \dot{+} x) \quad \text{for } x \in [0,1)^2,$$

where $u \dot{+} x$ denotes the element of $[0,1)^2$ equivalent to $u + x \bmod \mathbb{Z}_2$. Take an increasing nonnegative continuous function $g(t)$ on the interval $[0,1) \subset \mathbb{R}_1^+$ such that $g(0) = 0$, $\lim_{t\to 1-0} g(t) = \infty$, and also such that the function $f(s,t) := sg(t)$ for $(s,t) \in [0,1)^2$ is integrable on $[0,1)^2$ (e.g. $g(t) = (1-t)^{-1/2}-1$). Then define, for $I = [a_1,a_2)\times[b_1,b_2) \in \mathcal{I}_2$, a function $F(I)(x)$ on $[0,1)^2$ by

$$F(I)(x) := f((a_1,b_1)\dot{+}x) + f((a_2,b_2)\dot{+}x) - f((a_1,b_2)\dot{+}x) - f((a_2,b_1)\dot{+}x).$$

Thus, $F(I)$ defines a real-valued additive process in $L_1([0,1)^2)$ which is measurable in the sense of Theorem 3. By the definition of $F(I)$ we observe that

$$(17)\qquad \begin{aligned} &\text{either} \quad q\text{-}\liminf_{\alpha\to\infty} \alpha^{-2}F([0,\alpha)^2)(x) = -\infty,\\ &\text{or} \quad q\text{-}\limsup_{\alpha\to\infty} \alpha^{-2}F([0,\alpha)^2)(x) = \infty, \end{aligned}$$

for all $x = (x_1, x_2) \in [0,1)^2$ with $x_2 \neq 0$. Hence it follows from Theorem 3 (or directly) that $W \notin L_1^+([0,1)^2)$.

Example 3. Let $\Omega = \mathbb{R}_2$, with the Lebesgue measure λ_2, and $T = \{T(u) : u \in \mathbb{R}_2^+\}$ be the semigroup of translation operators $T(u)$ on $L_1(\mathbb{R}_2)$. Thus, $T(u)f(x) = f(u+x)$ for $x \in \mathbb{R}_2$. Take a real-valued continuous bounded function f on $\mathbb{R}_2$ such that $\{x : |f(x)| \neq 0\} \subset [0,1)^2$. Then define, for $I = [a_1, a_2) \times [b_1, b_2) \in \mathcal{I}_2$, a function $F(I)(x)$ on $\mathbb{R}_2$ by

$$F(I)(x) := f((a_1,b_1)+x) + f((a_2,b_2)+x) - f((a_1,b_2)+x) - f((a_2,b_1)+x).$$

It follows that $F(I)$ defines a real-valued additive process in $L_1(\mathbb{R}_2)$, measurable in the sense of Theorem 3, such that $W(x) \in L_1(\mathbb{R}_2)$. But, as is easily seen, it is possible to choose a function f so that

$$(18)\qquad \sup\Big\{\sum_{i=1}^{k} |F(I_i)(x)| : \{I_1,\ldots,I_k\} \text{ is a decomposition of } [0,1)^2\Big\} = \infty$$

for all $x \in [0,1)^2$. To find a concrete such function f, let e.g.

$$g(t) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n t) \qquad \text{for } t \in \mathbb{R},$$

where ϕ is a nonnegative periodic function on $\mathbb{R}$ with period 2 such that $\phi(t) = t$ if $0 \le t \le 1$ and $\phi(t) = 2 - t$ if $1 \le t \le 2$. Then g is a positive continuous function on $\mathbb{R}$ which is nowhere differentiable (see e.g. Theorem 7.18 of [11]). Thus, g is not of bounded variation on any bounded closed interval in $\mathbb{R}$. Using this g, let

$$h(t) = \begin{cases} tg(t) & \text{if } 0 \le t \le 1/2,\\ (1-t)g(t) & \text{if } 1/2 \le t \le 1,\\ 0 & \text{otherwise,} \end{cases}$$

$$e(t) = \begin{cases} t|\sin t^{-1}| & \text{if } 0 < t \le 1/2,\\ (1-t)|\sin t^{-1}| & \text{if } 1/2 \le t < 1,\\ 0 & \text{otherwise.} \end{cases}$$

Lastly, define $f(s,t) := h(s)e(t)$ for $(s,t) \in \mathbb{R}_2$. It is now routine to check that f is a real-valued continuous function on $\mathbb{R}_2$, with $\{x : f(x) \neq 0\} \subset [0,1)^2$, such that (18) holds for all $x \in [0,1)^2$. Thus, in this case, we must have $K(F) = \infty$.

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