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EXPONENTIALS OF NORMAL OPERATORS AND COMMUTATIVITY OF OPERATORS: A NEW APPROACH

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Abstract. We present a new approach to the question of when the commutativity of operator exponentials implies that of the operators. This is proved in the setting of bounded normal operators on a complex Hilbert space. The proofs are based on some results on similarities by Berberian and Embry as well as the celebrated Fuglede theorem.

1. Introduction. Let A and B be two bounded linear operators on a *Banach* space H. The way of defining e^A, the exponential of A, is known at an undergraduate level. The functions $\sinh A$, $\cosh A$ can be defined similarly. It is also an easy exercise to show that $AB = BA$ implies $e^A e^B = e^B e^A$. While the converse is not always true, it is, however, true under the hypothesis that A and B are self-adjoint on a $\mathbb{C}$-Hilbert space. This result is stated in the following theorem (a proof may be found in [13]):

THEOREM 1. *Let A and B be two self-adjoint operators defined on a Hilbert space. Then*

$$e^A e^B = e^B e^A \Leftrightarrow AB = BA.$$

There have been several attempts to prove the previous theorem for non-self-adjoint operators using the $2\pi i$-congruence-free hypothesis (see e.g. [7, 9, 10, 11, 13]). See also [3] for some low dimensional results without the $2\pi i$-congruence-free hypothesis.

In this paper, we present a different approach to this problem using results about similarities due to Berberian [2] and Embry [5]. The main question asked here is under which assumptions we have

$$e^A e^B = e^B e^A \Rightarrow AB = BA$$

for normal A and B.

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Another result obtained in this article is about sufficient conditions implying $AB = BC$ given that $e^A e^B = e^B e^C$, where A, B and C are self-adjoint operators.

All operators considered in this paper are assumed to be bounded and defined on a separable complex Hilbert space. The notions of normal, self-adjoint and unitary operators are defined in the usual fashion. So is the notion of the spectrum (with the usual notation σ). It is, however, convenient to recall the notion of a cramped operator. A *unitary* operator U is said to be *cramped* if its spectrum is completely contained in some open semi-circle (of the unit circle), that is,

$$\sigma(U) \subseteq \{e^{it} : \alpha < t < \alpha + \pi\}.$$

While we assume the reader is familiar with other notions and results on bounded operators (some standard references are [4, 8]), on several occasions we will recall some results that might not be known to some readers. One of them is the following result (first established in [1]).

THEOREM 2 (Berberian, [2]). *Let U be a cramped operator and let X be a bounded operator such that $UXU^* = X^*$. Then X is self-adjoint.*

2. Main results. The following lemma is fundamental to our results. Its proof follows from the holomorphic functional calculus.

LEMMA 1. *Let A and B be two commuting normal operators, on a Hilbert space, having spectra contained in simply connected regions not containing* 0. *Then*

$$A^i B^i = B^i A^i$$

where $i = \sqrt{-1}$ is the usual complex number.

LEMMA 2. *Let A be a self-adjoint operator such that $\sigma(A) \subset (0, \pi)$. Then*

$$(e^{iA})^i = e^{-A}.$$

Proof. The proof follows from the functional calculus. ∎

Before stating and proving the main theorem, we first give an intermediate result.

PROPOSITION 1. *Let N be a normal operator with cartesian decomposition $A + iB$. Let S be a self-adjoint operator. If $\sigma(B) \subset (0, \pi)$, then*

$$e^S e^N = e^N e^S \;\Rightarrow\; SN = NS.$$

Proof. Since A and B are two commuting self-adjoint operators, we have $e^Ae^{iB} = e^{iB}e^A$. Consequently,

$$\begin{aligned} e^Se^N = e^Ne^S &\Leftrightarrow e^Se^Ae^{iB} = e^Ae^{iB}e^S \\ &\Leftrightarrow e^Se^Ae^{iB} = e^{iB}e^Ae^S \\ &\Leftrightarrow e^Se^Ae^{iB} = e^{iB}(e^Se^A)^*. \end{aligned}$$

Since B is self-adjoint, e^{iB} is unitary. It is also cramped by the spectral hypothesis on B. Now, Theorem 2 implies that e^Se^A is self-adjoint, i.e.

$$e^Se^A = e^Ae^S.$$

Theorem 1 then gives us $AS = SA$.

It only remains to show that $BS = SB$. Since $e^Se^A = e^Ae^S$, we immediately obtain

$$e^Se^N = e^Ne^S \Rightarrow e^Se^Ae^{iB} = e^Ae^{iB}e^S \text{ or } e^Ae^Se^{iB} = e^Ae^{iB}e^S$$

and so

$$e^Se^{iB} = e^{iB}e^S$$

by the invertibility of e^A.

Using Lemmas 1 & 2 we immediately see that

$$e^Se^{-B} = e^{-B}e^S.$$

Theorem 1 yields $BS = SB$ and thus

$$SN = S(A + iB) = (A + iB)S = NS.$$

The proof of the proposition is complete. ■

Remark. We could have bypassed Berberian's result by alternatively using some of Embry's results (see [5]).

Similar results can be obtained too by a result due to I. H. Sheth [12]. They all more or less deduce the self-adjointness of an operator N from the operational equation $AN = N^*A$ (obviously under extra conditions on N and/or A).

In the proof of the previous proposition, N is a product of self-adjoint operators. Then a result by the author [6] can also be applied.

Now, we state and prove the main theorem in this paper:

Theorem 3. *Let N and M be two normal operators with cartesian decompositions $A + iB$ and $C + iD$ respectively. If $\sigma(B), \sigma(D) \subset (0, \pi)$, then*

$$e^Me^N = e^Ne^M \Rightarrow MN = NM.$$

Proof. We have

$$e^Me^N = e^Ne^M \Rightarrow e^{M^*}e^N = e^Ne^{M^*}$$

by the Fuglede theorem since e^M is normal. Hence by using again the normality of M,

$$e^{M^*}e^Me^N = e^{M^*}e^Ne^M \;\Rightarrow\; e^{M^*}e^Me^N = e^Ne^{M^*}e^M$$

or

$$e^{M^*+M}e^N = e^Ne^{M^*+M}.$$

Since M^*+M is self-adjoint, Proposition 1 applies and gives

$$(M^*+M)N = N(M^*+M) \quad \text{or just} \quad CN = NC.$$

This implies that $N^*C = CN^*$ and thus $(N+N^*)C = C(N+N^*)$. Therefore, we have

$$AC = CA \quad \text{and hence} \quad BC = CB.$$

Doing the same work for N in lieu of M, very similar arguments and Proposition 1 yield

$$AM = MA \quad \text{and hence} \quad AD = DA.$$

To prove the remaining bit, we go back to the equation $e^Ne^M = e^Me^N$. Then by the commutativity of B and C and by that of A and D, we obtain

$$e^Ae^{iB}e^Ce^{iD} = e^Ce^{iD}e^Ae^{iB} \;\Leftrightarrow\; e^Ae^Ce^{iB}e^{iD} = e^Ce^Ae^{iD}e^{iB}.$$

Since A and C commute and since e^Ae^C is invertible, we are left with

$$e^{iB}e^{iD} = e^{iD}e^{iB}.$$

Lemmas 1 & 2 yield

$$e^{-B}e^{-D} = e^{-D}e^{-B},$$

which leads to $BD = DB$. Hence $BM = MB$.

Finally, we have

$$NM = (A+iB)M = AM + iBM = MA + iMB = M(A+iB) = MN,$$

completing the proof. ■

We finish this paper by a result on self-adjoint operators which generalizes Theorem 1 to the case of three operators. We have

THEOREM 4. *Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space. Let A, B and C be self-adjoint operators on $\mathcal{H}$. If*

$$\begin{cases} \cosh A\, e^B = e^B \cosh A, \\ \sinh A\, e^C = e^B \sinh A, \\ e^C \cosh A = \cosh A\, e^C, \\ e^C \sinh A = \sinh A\, e^B, \end{cases}$$

then

$$AC = BA.$$

Proof. Define on $\mathcal{H} \oplus \mathcal{H}$ the operators

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

One has

$$\tilde{A}^2 = \begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}, \quad \tilde{A}^3 = \begin{pmatrix} 0 & A^3 \\ A^3 & 0 \end{pmatrix}, \quad \ldots .$$

Hence

$$e^{\tilde{A}} = \begin{pmatrix} I + \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots & A + \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots \\ A + \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots & I + \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots \end{pmatrix} . = \begin{pmatrix} \cosh A & \sinh A \\ \sinh A & \cosh A \end{pmatrix}$$

Similarly, we can find that

$$e^{\tilde{B}} = \begin{pmatrix} e^B & 0 \\ 0 & e^C \end{pmatrix}.$$

The hypotheses of the theorem guarantee that $e^{\tilde{A}}e^{\tilde{B}} = e^{\tilde{B}}e^{\tilde{A}}$ and since $\tilde{A}$ and $\tilde{B}$ are both self-adjoint, Theorem 1 then implies that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$.

Examining the entries of the matrices $\tilde{A}\tilde{B}$ and $\tilde{B}\tilde{A}$, we see that $AC = BA$, establishing the result. ■

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