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THE IDEAL (a) *IS NOT* G_δ *GENERATED*

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Abstract. We prove that the ideal (a) defined by the density topology is not G_δ generated. This answers a question of Z. Grande and E. Strońska.

1. Preliminaries. The aim of this paper is to give a complete solution of the following problem from the paper [GS]: *Is there a set A satisfying the condition (a) such that for each G_δ-set $B \supset A$ the set B does not satisfy the condition (a)?*

The ideal (a) of subsets of the real line was defined in [G], [GS] and examined in [G] and [N]. In [GS] the authors proved that it is $F_{\sigma\delta}$ generated (for the definition see below) and asked whether it is G_δ generated.

In [**?**] we reformulated the notion of the ideal (a) in the case of the Cantor set 2^ω as a basic space, instead of the real line (notice that for the Cantor space we can also define the density topology, although these two topologies are not compatible). The reason for doing that was the possibility of using the natural combinatorial structure of the Cantor set. We proved that the ideal (a) defined on 2^ω is not G_δ generated. The crucial part of that proof is a construction of some special perfect set, namely a set $P \subseteq 2^\omega$ such that $\Phi(P) \cap [t]$ is $F_{\sigma\delta}$-complete, where $\Phi(P)$ denotes the set of density points of P and $[t]$ is any basic neighborhood such that $P \cap [t] \neq \emptyset$.

The main part of this article is devoted to constructing a perfect set $P \subseteq \mathbb{R}$ with similar properties. The construction, however, is completely different.

Having such a special set we prove (in much the same way as in [**?**]) that the ideal (a) is not G_δ generated.

Notice that the set $P \setminus \Phi(P)$ is of measure zero. Other properties of this set are also worth investigating. Recall that in [B1] and [B2] its smallness with respect to Hausdorff measure was considered. Then in [Bu, Theorem 1] it was proved (assuming that P is a nowhere dense perfect set) that $P \setminus \Phi(P)$ is a meager subset of P. It is interesting to ask about the Borel class of this

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set. Lemma 3.7 of the present article shows that $P \setminus \Phi(P)$ can even be a $F_{\sigma\delta}$-complete set.

2. Definitions

2.1. The ideal (a). Let μ denote the Lebesgue measure on the real line. For every measurable A we denote the set of density points of A by

$$\Phi(A) = \left\{ x \in \mathbb{R} \colon \liminf_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h} = 1 \right\}.$$

The classical *density topology* on the real line is denoted by τ_d. Let us recall that $U \in \tau_d \Leftrightarrow U$ is Lebesgue measurable and $U \subseteq \Phi(U)$. The density topology has been considered in many papers; some of its properties were discussed in [T].

First, let us recall the following definition:

DEFINITION 2.1 ([GS]). A set $A \subseteq \mathbb{R}$ satisfies the condition (a) if for each nonempty set $U \subseteq \mathrm{cl}(A)$ belonging to τ_d the intersection $A \cap U$ is a nowhere dense subset of U (in the euclidean topology).

We will need at the outset the simple characterization of the ideal (a) from [N, Observation 2.2]:

CHARACTERIZATION. *$A \in (a)$ iff for every $U \in \tau_d \setminus \{\emptyset\}$ there exists an open set W in the standard topology such that $U \cap W \neq \emptyset$ and $U \cap W \cap A = \emptyset$.*

We will also need the following result:

THEOREM 2.2 ([GS]). *For every closed (in the standard topology) set $E \subseteq 2^\omega$ we have $E \setminus \Phi(E) \in (a)$.*

2.2. Some facts from descriptive set theory. Suppose that X is a Polish space and let Γ be any class in the Borel or projective hierarchy. We say that a set $A \subseteq X$ is *Γ-complete* if $A \in \Gamma(X)$ and for every Polish space Y and $B \in \Gamma(Y)$ there exists a continuous function $f \colon Y \to X$ such that $f^{-1}[A] = B$.

DEFINITION 2.3. $C_3 = \{x \in \omega^\omega \colon \lim_{n\to\infty} x(n) = \infty\}$.

The following theorem will be very useful:

THEOREM 2.4 ([Ke, 23A]). *The set C_3 is Π^0_3-complete ($F_{\sigma\delta}$-complete). In particular, it is neither F_σ or G_δ.*

Suppose that $\mathcal{I}$ is an ideal and $\mathcal{F}$ is any class of sets. We say that $\mathcal{I}$ is $\mathcal{F}$ *generated* if for every $X \in \mathcal{I}$ there exists $Y \in \mathcal{I} \cap \mathcal{F}$ such that $X \subseteq Y$.

3. The perfect set. This section is devoted to the construction of a perfect set $P \subseteq \mathbb{R}$ such that for any open interval W, if $W \cap \Phi(P) \neq \emptyset$ then $W \cap \Phi(P)$ is $F_{\sigma\delta}$-complete.

DEFINITION 3.1. For $k, n \in \omega$ and $D, s \in \mathbb{Z}$ define

$$E_s(n,k,D) = \left(\frac{k \cdot s + D}{2^n}, \frac{k \cdot s + D + 1}{2^n}\right),$$
$$G(n,k,D) = \mathbb{R} \setminus \bigcup_{s \in \mathbb{Z}} E_s(n,k,D).$$

That is, we split the real line into intervals of length $1/2^n$, remove every kth interval, and $G(n,k,D)$ is what is left.

Suppose that $I = [a,b]$ with $b > a$.

ESTIMATE 3.2. *Assume that* $|\{s \in \mathbb{Z}\colon E_s(n,k,D) \cap I \neq \emptyset\}| \geq 2$. *Then*

$$\frac{\mu(I \cap G(n,k,D))}{\mu(I)} \geq \frac{k-1}{k+1}.$$

Proof. By suitably extending or shrinking I we can find an interval $J = [c,d]$ such that

$$\frac{\mu(I \cap G(n,k,D))}{\mu(I)} \geq \frac{\mu(J \cap G(n,k,D))}{\mu(J)},$$

and

$$c = \frac{k \cdot s_c + D}{2^n}, \qquad d = \frac{k \cdot s_d + D + 1}{2^n},$$

for some integers $s_c < s_d$. Then

$$\frac{\mu(J \cap G(n,k,D))}{\mu(J)} = \frac{(s_d - s_c)(k-1)}{(s_d - s_c) \cdot k + 1},$$

and this expression is minimal for $s_d - s_c = 1$. ∎

From now on we assume that $l \in \omega$ is such that $2l + 4 < k$.

ESTIMATE 3.3. *Suppose that* $i \in \{2, \ldots, l\}$, *let* $x_0 \in \mathrm{cl}(E_s(n,k,D+i))$ *and consider* $h > 0$ *such that*

$$|\{s \in \mathbb{Z}\colon E_s(n,k,D) \cap [x_0 - h, x_0 + h] \neq \emptyset\}| \leq 1 \tag{1}$$

(*it is sufficient to assume that* $h \leq (l+1)/2^n$). *Then*

$$\frac{\mu([x_0 - h, x_0 + h] \cap G(n,k,D))}{\mu([x_0 - h, x_0 + h])} \geq 1 - \frac{1}{2i} = \frac{2i-1}{2i}.$$

Proof. This follows from the fact that the minimal value of the above quotient is reached when $x_0 = (s \cdot k + D + i)/2^n$ and $h = i/2^n$. ∎

ESTIMATE 3.4. *Assume as before that* $2l + 4 < k$ *and* $x_0 \in \mathrm{cl}(E_s(n,k,D+i))$. *Then there exists* $h > 0$ *such that* (1) *holds* (*we may* *even assume that* $h \le (l+1)/2^n$) *and such that*

$$\frac{\mu([x_0-h,x_0+h]\cap G(n,k,D))}{\mu([x_0-h,x_0+h])} \le 1-\frac{1}{2i+2} = \frac{2i+1}{2i+2}.$$

Proof. Put $h = x_0 - (s\cdot k + D)/2^n$. Then

$$\frac{\mu([x_0-h,x_0+h]\cap G(n,k,D))}{\mu([x_0-h,x_0+h])} = \frac{2h-1/2^n}{2h} = 1-\frac{1}{2^n\cdot 2h} \le 1-\frac{1}{2i+2},$$

since $h \le (i+1)/2^n$. ∎

FACT 3.5. *Assume that* $n, n', k, k' \in \mathbb{Z}$ *are such that* $n-n' \ge 2$, $4\,|\,k$, $4\,|\,k'$. *Then the intervals* $E_s(n,k,2)$ *and* $E_{s'}(n',k',2)$ *are such that either one is contained in the other or they have disjoint closures.*

Proof. Striving for a contradiction, suppose that the beginning of $E_{s'}(n',k',2)$ is the end of $E_s(n,k,2)$. Then $(k\cdot s+2+1)/2^n = (k'\cdot s'+2)/2^{n'}$, hence $k\cdot s+2+1 = 2^{n-n'}\cdot(k'\cdot s'+2)$, which is impossible.

On the other hand, suppose that the beginning of $E_s(n,k,2)$ is equal to the end of $E_{s'}(n',k',2)$. Then $(k'\cdot s'+2+1)/2^{n'} = (k\cdot s+2)/2^n$, hence $2^{n-n'}\cdot(k'\cdot s'+2+1) = k\cdot s+2$, which is impossible. ∎

From now on we shall use the sequences $\langle 20^l : l\in\omega\rangle$, $\langle 10^{l+2} : l\in\omega\rangle$. The following can be easily proved:

ESTIMATE 3.6.

(i) $10(l+1) < 10^{l+2}$, *in particular* $2l+4 < 10^{l+2}$;
(ii) $20^{l+1}-20^l \ge 2$, $4\,|\,10^{l+2}$;
(iii) $\sum_{l=0}^{\infty}\frac{2}{10^{l+2}+1} < \frac{3}{100}$;
(iv) $\frac{2^{20^{l+1}-20^l}}{10^{l+3}} > 100$.

To simplify notation, set $G_l = G(20^l,10^{l+2},2)$, $E_{s,l} = E_s(20^l,10^{l+2},2)$ and for $i=2,\dots,l$, $E^{(i)}_{s,l} = E_s(20^l,10^{l+2},2+i)$.

Define $P = \bigcap_{l\in\omega} G_l$. We conclude from Estimates 3.6(iii),(iv) and from Fact 3.5 that P is a perfect set.

From our assumptions we can deduce that if I is any interval, then

$$|\{s\in\mathbb{Z}\colon E_{s,l}\cap I\ne\emptyset\}| \ge 2 \;\Rightarrow\; |\{s\in\mathbb{Z}\colon E_{s,l+1}\cap I\ne\emptyset\}| \ge 2. \tag{2}$$

Suppose that $(a,b)\subseteq\mathbb{R}$ is an interval such that $(a,b)\cap P\ne\emptyset$. Then there exist $L\in\omega$ and $J\in\{2,\dots,L\}$ and $s\in\mathbb{Z}$ such that $L>2$ and $E^{(J)}_{s,L}\subseteq(a,b)$. Define $K_\emptyset = \mathrm{cl}(E^{(J)}_{s,L})$.

Then find $s_\emptyset\in\mathbb{Z}$ such that

$$\bigcup_{i=-2}^{2} E_{s_\emptyset+i,L+1} \subseteq K_\emptyset$$

(use Estimate 3.6(iv)).

For $r = \langle n \rangle$ define

$$K_{\langle n \rangle} = \mathrm{cl}(E^{(\min\{n+2,L+1\})}_{s_\emptyset,L+1})$$

and then choose $s_{\langle n \rangle} \in \mathbb{Z}$ such that

$$\bigcup_{i=-2}^{2} E_{s_{\langle n \rangle}+i,L+2} \subseteq K_{\langle n \rangle}.$$

For $r = \langle n_0, n_1 \rangle$ define

$$K_{\langle n_0,n_1 \rangle} = \mathrm{cl}(E^{(\min\{n_1+2,L+2\})}_{s_{\langle n_0 \rangle},L+2}).$$

Then find $s_{\langle n_0,n_1 \rangle} \in \mathbb{Z}$ such that

$$\bigcup_{i=-2}^{2} E_{s_{\langle n_0,n_1 \rangle}+i,L+3} \subseteq K_{\langle n_0,n_1 \rangle}.$$

In general, for $r = \langle n_0, \dots, n_{q-1} \rangle$ define

$$K_{\langle n_0,\dots,n_{q-1} \rangle} = \mathrm{cl}(E^{(\min\{n_{q-1}+2,L+q\})}_{s_{\langle n_0,\dots,n_{q-2} \rangle},L+q}). \tag{3}$$

Then find $s_{\langle n_0,\dots,n_{q-1} \rangle} \in \mathbb{Z}$ such that

$$\bigcup_{i=-2}^{2} E_{s_{\langle n_0,\dots,n_{q-1} \rangle}+i,L+q+1} \subseteq K_{\langle n_0,\dots,n_{q-1} \rangle}. \tag{4}$$

Define $\phi \colon \omega^\omega \to P$ by $\{\phi(x)\} = \bigcap_{n\in\omega} K_{x\restriction n}$. It is easy to see that ϕ is continuous.

Suppose that $h > 0$ and $x_0 = \phi(x)$ for some $x \in \omega^\omega$. Assume for simplicity that $[x_0 - h, x_0 + h] \subseteq K_{\langle x(0) \rangle}$. Denote

$$\varXi = \min\{l \in \omega \colon |\{s \in \mathbb{Z} \colon E_{s,l} \cap [x_0 - h, x_0 + h] \neq \emptyset\} \geq 2\}.$$

Since $[x_0 - h, x_0 + h] \subseteq K_{\langle x(0) \rangle} \subseteq \mathrm{cl}(E^{(\min\{x(0)+2,L+1\})}_{s_\emptyset,L+1})$ we conclude that $[x_0 - h, x_0 + h] \subseteq G_{L+1}$ and moreover, since $x_0 \in P$ and $P \subseteq G_l$ for all $l \in \omega$, we have

$$[x_0 - h, x_0 + h] \subseteq G_l \quad \text{for } l \leq L. \tag{5}$$

Thus $\varXi > L + 1$. For $l_0 = \varXi - 1$ we have

$$|\{s \in \mathbb{Z} \colon E_{s,l_0} \cap [x_0 - h, x_0 + h] \neq \emptyset\}| \leq 1;$$

moreover, $x_0 \in K_{x\restriction l_0 - L}$ implies that

$$[x_0 - h, x_0 + h] \subseteq K_{x\restriction p} \quad \text{for } p < l_0 - L. \tag{6}$$

Indeed, put $p = l_0 - L - 1$ and $q = p + 1$ in (3). Then

$$\begin{aligned}x_0 \in K_{x\restriction l_0 - L} = K_{\langle x(0),\dots,x(l_0-L-1)\rangle} &= K_{\langle x(0),\dots,x(q-1)\rangle}\\ &= \mathrm{cl}(E^{(\min\{x(q-1)+2,L+q\})}_{s_{\langle x(0),\dots,x(q-2)\rangle},L+q}) = \mathrm{cl}(E^{(\min\{x(l_0-L-1)+2,l_0\})}_{s_{x\restriction q-1},l_0})\\ &= \mathrm{cl}(E^{(\min\{x(p)+2,l_0\})}_{s_{x\restriction p},l_0}).\end{aligned}$$

From (4) we deduce that (by putting $q = p$) $\bigcup_{i=-2}^{2} E_{s_{\langle x(0),\dots,x(p-1)\rangle}+i,l_0} \subseteq K_{\langle x(0),\dots,x(p-1)\rangle} = K_{x\restriction p}$. This proves (6).

Next, putting $q = \varXi - L - 1$ in (3) we obtain $x_0 \in E^{(\min\{x(\varXi-L-2),\varXi-1\})}_{s_{x\restriction \varXi-L-2},\varXi-1}$ and hence

$$\begin{aligned}\frac{\mu(G_{\varXi-1}\cap[x_0-h,x_0+h])}{2h} &\ge \frac{2\cdot\min\{x(\varXi-L-2)+2;\varXi-1\}-1}{2\cdot\min\{x(\varXi-L-2)+2;\varXi-1\}}\\ &= 1-\frac{1}{2\cdot\min\{x(\varXi-L-2)+2;\varXi-1\}}\end{aligned}$$

by Estimate 3.3.

For $l < \varXi - 1$ we have

$$\frac{\mu(G_l\cap[x_0-h,x_0+h])}{2h} = 1$$

by (5) and (6).

For $l \ge \varXi$ we have

$$\frac{\mu(G_l\cap[x_0-h,x_0+h])}{2h} \ge \frac{10^{l+2}-1}{10^{l+2}+1} = 1-\frac{2}{10^{l+2}+1}$$

by Estimate 3.2. Hence

$$\begin{aligned}\frac{\mu(P\cap[x_0-h,x_0+h])}{2h} &= \frac{\mu(\bigcap_{l\in\omega} G_l\cap[x_0-h,x_0+h])}{2h}\\ &= 1-\frac{\mu(\bigcup_{l\in\omega} G_l^c\cap[x_0-h,x_0+h])}{2h}\\ &\ge 1-\sum_{l\in\omega}\frac{\mu(G_l^c\cap[x_0-h,x_0+h])}{2h}\\ &= 1-\sum_{l\in\omega}\left(1-\frac{\mu(G_l\cap[x_0-h,x_0+h])}{2h}\right)\\ &\ge 1-\sum_{l\ge\varXi}\frac{2}{10^{l+2}+1}+\frac{1}{2\cdot\min\{x(\varXi-L-2)+2;\varXi-1\}}.\end{aligned}$$

Then if $h \to 0$ and $\lim_{n\to\infty} x(n) = \infty$ then $\varXi \to \infty$, thus $x_0 \in \varPhi(P)$. On the other hand, let $x_0 \in P$ be such that $\phi(x) = x_0$ for some $x \in \omega^\omega$ and $x \notin C_3$. Then for some subsequence $\langle m_\nu \colon \nu \in \omega\rangle$ and $M \in \omega$ we have $x(m_\nu) \le M$

for all $\nu \in \omega$. Since $x_0 \in K_{x\restriction(m_\nu+1)}$, we have

$$x_0 \in \mathrm{cl}(E^{(\min\{x(m_\nu)+2, L+m_\nu+1\})}_{s_{x\restriction m_\nu}, L+m_\nu+1})$$

(put $\langle n_0, \ldots, n_{q-1}\rangle = x\restriction(m_\nu+1)$ in (3), so $n_{q-1} = x(m_\nu)$ and $q-1 = m_\nu$). By Estimate 3.4 there exists $h_\mu > 0$ such that

$$\frac{\mu(G_{L+m_\nu+1} \cap [x_0 - h_\nu, x_0 + h_\nu])}{2h} \leq \frac{2\cdot\min\{x(m_\nu)+2, L+m_\nu+1\}+1}{2\cdot\min\{x(m_\nu)+2, L+m_\nu+1\}+2} \leq \frac{2M+5}{2M+6}.$$

Because of Estimate 3.4 we may assume $h_\nu \leq (L+m_\nu+1)/2^{20^{L+m_\nu+1}}$. Obviously $\lim_{l\to\infty} l/2^{20^l} \to 0$, so $\lim_{\nu\to\infty} h_\nu = 0$. Hence $x_0 \notin \Phi(P)$.

Putting all these results together, we get $\phi^{-1}[(a,b) \cap \Phi(P)] = C_3$.

Finally, we obtain the following lemma:

LEMMA 3.7. *There exists a perfect set $P \subseteq \mathbb{R}$ such that for every interval (a,b) with $(a,b) \cap P \neq \emptyset$ the set $\Phi(P) \cap (a,b)$ is $F_{\sigma\delta}$-complete.*

As a corollary we have:

COROLLARY 3.8. *There exists a perfect set $P \subseteq \mathbb{R}$ such that for every interval (a,b) with $(a,b) \cap P \neq \emptyset$ the set $\Phi(P) \cap (a,b)$ is neither F_σ nor G_δ.*

Notice that this result is similar to the following result of S. Głąb [Gl]:

THEOREM 3.9 (S. Głąb, [Gl, Theorem 6]). *The set D_1^+ is Π_3^0-complete. (D_1^+ denotes the set of all compact subsets of $\mathbb{R}$ whose right density at 0 is equal to 1.)*

In the proof of this theorem the author also uses the set C_3. Of course, our result is different, since we do not study the hyperspace $\mathcal{K}(\mathbb{R})$ of all nonempty compact sets.

4. The solution. We end this paper by showing that the ideal (a) is not G_δ generated. In the proof we use the perfect set constructed in the previous section and we mimic the argument from [**?**].

THEOREM 4.1. *Suppose that G is a G_δ-set such that $P\setminus\Phi(P) \subseteq G$. Then $G \notin (a)$.*

Proof. Suppose, towards a contradiction, that $G \in (a)$. Then by the Characterization from Section 2.1 there exists an open set $W \subseteq \mathbb{R}$ such that $W \cap \Phi(P) \neq \emptyset$ and $W \cap \Phi(P) \cap G = \emptyset$. We may assume that $W = (a,b)$ for some $a < b$. Hence $(a,b) \cap \Phi(P) \neq \emptyset$ and $(a,b) \cap \Phi(P) \cap G = \emptyset$. We have $(a,b) \cap P \setminus G = (a,b) \cap \Phi(P)$ (and this set is nonempty). Indeed, if $x \in (a,b) \cap P \setminus G$ and $x \notin \Phi(P)$ then $x \in P \setminus \Phi(P) \subseteq G$, which is impossible. On the other hand, if $x \in (a,b) \cap \Phi(P)$ and $x \in G$ then $x \in (a,b) \cap \Phi(P) \cap G = \emptyset$, which is impossible. But then $(a, b) \cap P \neq \emptyset$, thus from Corollary 3.8 we know that $(a, b) \cap \Phi(P)$ is not an F_σ-set, which contradicts the equality $(a, b) \cap P \setminus G = (a, b) \cap \Phi(P)$. ∎

From Theorem 2.2 we obtain the following corollary which is a solution of the Problem from [GS, p. 311]:

COROLLARY 4.2. *The ideal* (a) *is not* G_δ *generated.*

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