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# ON THE DIFFERENTIABILITY OF CERTAIN SALTUS FUNCTIONS

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**Abstract.** We investigate several natural questions on the differentiability of certain strictly increasing singular functions. Furthermore, motivated by the observation that for each famous singular function $f$ investigated in the past, $f'(\xi) = 0$ if $f'(\xi)$ exists and is finite, we show how, for example, an increasing real function $g$ can be constructed so that $g'(x) = 2^x$ for all rational numbers $x$ and $g'(x) = 0$ for almost all irrational numbers $x$.

**1. Introduction and statement of results.** Let $\Phi$ be the family of all bijective functions from $\mathbb{N}$ onto $\mathbb{Q}$. (We do not consider 0 to be a member of the set $\mathbb{N}$.) For $\varphi \in \Phi$ define the function $F_\varphi : \mathbb{R} \to \mathbb{R}$ by

$$F_\varphi(x) = \sum_{\varphi(n)<x} \frac{1}{2^n}$$

where the summation is extended over all $n \in \mathbb{N}$ with $\varphi(n) < x$. This is the well-known prototype of a strictly increasing real function which is discontinuous at each rational number and continuous at each irrational number. (This is a worst case scenario for the monotonic functions because their points of discontinuity are always countably many and $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.)

Naturally, the image $W_\varphi$ of $F_\varphi$ is a subset of the open interval $]0, 1[$ and 0 and 1 are limit points of $W_\varphi$. Moreover, $F_\varphi$ is a saltus function with a jump to the right of height $2^{\varphi^{-1}(r)}$ at each $r \in \mathbb{Q}$. Thus the open intervals $I_n = ]F_\varphi(\varphi(n)), F_\varphi(\varphi(n)) + 2^{-n}[$ $(n \in \mathbb{N})$ are mutually disjoint and disjoint from $W_\varphi$. As a consequence, the set $W_\varphi$ is null and nowhere dense. But trivially, $W_\varphi$ has the cardinality of the continuum.

As an increasing function, $F_\varphi$ is differentiable almost everywhere. Let $\mathcal{E}_\varphi$ be the set of all reals at which $F_\varphi$ is not differentiable. Thus $\mathcal{E}_\varphi$ is a null set containing $\mathbb{Q}$. As a consequence of Fort's theorem [2], $\mathcal{E}_\varphi$ is always *residual*, i.e. $\mathbb{R} \setminus \mathcal{E}_\varphi$ is of first category. (This can also be shown in a direct way: Since $F_\varphi$ is increasing and $|F_\varphi(x) - F_\varphi(y)| \geq 2^{-n}$ whenever $x < \varphi(n) < y$,

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the function $F_\varphi$ cannot be differentiable at any point in the residual set $\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty ]\varphi(n) - 3^{-n}, \varphi(n) + 3^{-n}[$.)

In particular, $F_\varphi$ is not differentiable at infinitely many points of continuity. Moreover, $[a, b] \cap \mathcal{E}_\varphi$ has the cardinality of the continuum for arbitrary $a < b$ and $\varphi \in \Phi$. This statement can be sharpened in the following way.

THEOREM 1. *For arbitrary $a < b$ one can find a nowhere dense null set $Z$ of irrational numbers in $[a, b]$ such that $Z \cap \mathcal{E}_\varphi$ has the cardinality of the continuum for each $\varphi \in \Phi$.*

Since obviously $F_\varphi$ is the limit of a series of monotonic step functions, $F_\varphi$ is a *singular* function, i.e. its first derivative exists and vanishes almost everywhere. But there is a stronger argument for $\{x \in \mathbb{R} \setminus \mathcal{E}_\varphi \mid F'_\varphi(x) \neq 0\}$ being a null set. In fact, this set is always empty! Moreover, the following is true.

THEOREM 2. *Independently of $\varphi \in \Phi$, there never exists a real $\xi$ such that $F_\varphi$ has a right or a left derivative at $\xi$ which is finite and non-vanishing.*

Let $\Phi_0$ be the family of all $\varphi \in \Phi$ such that $\varphi^{-1}(r) \geq q$ for every rational number $r$ with least positive denominator $q$. Note that $\varphi \in \Phi_0$ if $\varphi$ is either the standard numbering of the rational numbers using Farey sequences or the popular numbering of $\mathbb{Q}$ which uses a spiral path through all points in the lattice $\mathbb{Z}^2$ starting with $(0, 0)$. If $r_1, r_2, \ldots$ is the beautiful sequence $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots$ of all positive rational numbers given in [1], then $\varphi_0 \in \Phi_0$ where $\varphi_0$ is defined by $\varphi_0(1) = 0$ and $\varphi_0(2n) = r_n$ and $\varphi_0(2n + 1) = -r_n$ for every $n \in \mathbb{N}$.

THEOREM 3. *If $\varphi \in \Phi_0$ then the first derivative of $F_\varphi$ exists and vanishes at each algebraic irrational number.*

Despite Theorems 1 and 3 it is small wonder that the set $\mathcal{E}_\varphi$ depends strongly on the choice of $\varphi$.

THEOREM 4.

(i) *For every countable set $X \subset \mathbb{R}$ one can choose $\varphi \in \Phi$ such that $X \subset \mathcal{E}_\varphi$ and additionally $F'_\varphi(x) = \infty$ for all $x \in X$.*

(ii) *On the other hand, for every countable set $X$ of irrational numbers one can choose $\varphi \in \Phi_0$ such that $X \cap \mathcal{E}_\varphi = \emptyset$ and $F'_\varphi(x) = 0$ for all $x \in X$.*

For $\varphi \in \Phi$ let $\Omega_\varphi$ be the set of all points $\xi \in \mathbb{R}$ at which $F_\varphi$ has an infinite derivative. Since differentiability means the existence of a finite derivative, $\Omega_\varphi \subset \mathcal{E}_\varphi$. In particular, $\Omega_\varphi$ is always a null set. The following theorem shows that there are $\varphi$ such that the set $\Omega_\varphi$ is extremely small and rather large, respectively.

THEOREM 5.

(i) *If* $\varphi \in \Phi_0$ *then* $\Omega_\varphi = \emptyset$.
(ii) *If* $a < b$ *then* $[a, b] \cap \Omega_\varphi$ *has the cardinality of the continuum for some* $\varphi \in \Phi$.

By Theorem 4, for every countable $X \subset \mathbb{R}$ we can achieve $X \subset \Omega_\varphi$ for some $\varphi \in \Phi$. Since $\Omega_\varphi$ is null, in view of Theorem 5(ii) the question arises whether $X \subset \Omega_\varphi$ is possible for an arbitrary null set $X$ or at least for an arbitrary nowhere dense null set $X$. The following theorem gives a negative answer.

THEOREM 6. *Let* $\mathbb{D}$ *be the Cantor ternary set. Then for every* $\varphi \in \Phi$ *the set* $\mathbb{D} \setminus \Omega_\varphi$ *has the cardinality of the continuum.*

Let $\mathcal{F}$ be the family of all real monotonic functions $f$ defined on an arbitrary (nondegenerate) interval $I$ such that $f'(x) = 0$ for almost all $x \in I$. Let $\mathcal{F}^*$ be the family of all functions $f$ in $\mathcal{F}$ such that $f'(x) \neq 0$ for at least one point $x$ at which $f$ is differentiable. By Theorem 2 all functions $F_\varphi$ lie in $\mathcal{F} \setminus \mathcal{F}^*$. Further, the classical Cantor function (the *devil's staircase*) lies in $\mathcal{F} \setminus \mathcal{F}^*$. Also the famous Riesz–Nagy function (see [5, 18.8]) and Minkowski's *Fragefunktion* (see [6, p. 345]) and the interesting function $F_{3,2}$ recently investigated in [6], which are all strictly increasing and singular, have the property that at each point the derivative is 0 or $\infty$ or does not exist. Since no example of a function in $\mathcal{F}^*$ seems to be known, the question arises whether $\mathcal{F}^* = \emptyset$.

In order to solve this question we modify the definition of our functions $F_\varphi$ and consider saltus functions $G_\varphi : \mathbb{R} \to \mathbb{R}$ for $\varphi \in \Phi$ which are defined by

$$G_\varphi(x) = \sum_{\varphi(n) < x} \frac{1}{n^2}.$$

Of course, just as the functions $F_\varphi$, all functions $G_\varphi$ are strictly increasing and continuous precisely at the irrational numbers. (The image of $G_\varphi$ is a null and nowhere dense subset of $]0, \pi^2/6[$.) Certainly, all functions $G_\varphi$ lie in $\mathcal{F}$. Now, the following theorem implies that $\mathcal{F}^* \neq \emptyset$.

THEOREM 7. *For every sequence of distinct irrational numbers* $\xi_1, \xi_2, \ldots$ *and every sequence* $c_1, c_2, \ldots$ *of positive real numbers there is a* $\varphi \in \Phi$ *such that* $G_\varphi$ *is differentiable at* $\xi_k$ *and* $G'_\varphi(\xi_k) = c_k$ *for every* $k \in \mathbb{N}$.

By Theorem 7 we may choose $\varphi$ so that $G'_\varphi(r + \pi) = 2^r$ for every $r \in \mathbb{Q}$ and then $g(x) := G_\varphi(x + \pi)$ defines a function $g : \mathbb{R} \to \mathbb{R}$ as mentioned in the abstract.

Despite Theorem 7 it is not true that for every $\varphi \in \Phi$ there are points $\xi$ such that $0 < G'_\varphi(\xi) < \infty$. (For a counterexample choose any $\varphi \in \Phi$ which maps $\{2^n \mid n \in \mathbb{N}\}$ onto $\mathbb{Q} \setminus \mathbb{Z}$ and define $\psi \in \Phi$ anyhow so that

$\psi(m) = \varphi(2^{m/2})$ for every even $m \in \mathbb{N}$. Then for every $k \in \mathbb{Z}$ there is a constant $\tau_k$ such that $G_\varphi(x) = F_\psi(x) + \tau_k$ whenever $k < x \le k+1$.)

Let $\mathcal{E}'_\varphi$ be the set of all reals at which $G_\varphi$ is not differentiable. Theorem 1 remains true when $F_\varphi$ is replaced by $G_\varphi$ and $\mathcal{E}_\varphi$ is replaced by $\mathcal{E}'_\varphi$ because $\mathcal{E}_\varphi \subset \mathcal{E}'_\varphi$ for every $\varphi \in \Phi$. (Note that $G'_\varphi(\xi) = c$ with $0 \le c < \infty$ implies $F'_\varphi(\xi) = 0$ since $\lim_{k\to\infty} 2^k k^{-2} = \infty$ and $\sum_{n\in\mathcal{N}} n^{-2} \ge 2^m m^{-2} \cdot \sum_{n\in\mathcal{N}} 2^{-n}$ whenever $\emptyset \ne \mathcal{N} \subset \mathbb{N}$ and $3 \le m = \min \mathcal{N}$.) Further, in view of its proof it is not difficult to verify that the second statement of Theorem 4 remains true as well when $F_\varphi$ is replaced by $G_\varphi$. Trivially this is also the case concerning Theorem 5(ii) and the first statement of Theorem 4 since, naturally, $G'_\varphi(\xi) = \infty$ when $F'_\varphi(\xi) = \infty$. But Theorem 3 has no counterpart for the functions $G_\varphi$.

THEOREM 8. *For each irrational $\xi$ there exists a $\varphi \in \Phi_0$ such that $G_\varphi$ is not differentiable at $\xi$.*

**2. Proof of Theorem 1.** For irrational $\xi$ let $[b_n]_{n\ge0}$ be the continued fraction expressing $\xi$ and let $A_n/B_n = [b_0, \dots, b_n]$ denote the $n$th convergent to $\xi$ where $A_n, B_n$ are coprime integers and $B_n > 0$. Consequently, $0 < (-1)^n(\xi - A_n/B_n) < (B_nB_{n+1})^{-1}$ for every $n \in \mathbb{N}$.

LEMMA 1. *If $\log B_{n+1} > \varphi^{-1}(A_n/B_n)$ for infinitely many $n \in \mathbb{N}$, then $F_\varphi$ is not differentiable at $\xi$.*

*Proof.* Put $h_n = \frac{9}{8}(A_n/B_n - \xi)$ for $n \in \mathbb{N}$. Naturally, the sequence $h_n$ tends to 0 as $n \to \infty$. Further, for arbitrary $h \ne 0$ and $m \in \mathbb{N}$ we have $|F_\varphi(\xi+h) - F_\varphi(\xi)| \ge 2^{-m}$ when the rational number $\varphi(m)$ lies between $\xi$ and $\xi + h$. Since $\xi < A_n/B_n < \xi + h_n$ when $h_n > 0$ and $\xi + h_n < A_n/B_n < \xi$ when $h_n < 0$, and since $\frac{8}{9}|h_n| < (B_nB_{n+1})^{-1}$, we have, for every $n \in \mathbb{N}$,

$$h_n^{-1}(F_\varphi(\xi+h_n) - F_\varphi(\xi)) \ge \tfrac{8}{9}B_nB_{n+1}2^{-m_n}$$

where $\varphi(m_n) = A_n/B_n$. This concludes the proof of Lemma 1 because $B_{n+1}2^{-m_n} \ge 1$ for infinitely many $n \in \mathbb{N}$, and certainly $B_n \to \infty$ as $n \to \infty$. ∎

*Proof of Theorem 1.* For fixed $a < b$ we construct a null and nowhere dense subset $Z$ of $[a,b] \setminus \mathbb{Q}$ such that for each $\varphi \in \Phi$ there is a set $S \subset Z$ with the cardinality of the continuum such that Lemma 1 can be applied to all numbers in $S$. First we choose $\delta > 0$ and an irrational $\xi$ expressed by the continued fraction $[b_n]_{n\ge0}$ so that $a < \xi \pm \delta < b$. Now fix $N \in \mathbb{N}$ large enough that $B_N > \sqrt{2/\delta}$. Then every irrational number $\xi' = [b'_n]_{n\ge0}$ lies between $a$ and $b$ when $b_n = b'_n$ for every $n = 0, 1, \dots, N$ because then $|\xi - \xi'| \le |\xi - A_N/B_N| + |\xi' - A_N/B_N| < 2/B_N^2 < \delta$ since $(A_N, B_N) = (A'_N, B'_N)$.

Now let $Z$ be the set of all irrational numbers $[b_0, b_1, \ldots, b_N, z_{N+1}, z_{N+2}, \ldots]$ with $z_n > n^2$ for every $n > N$. Then $Z \subset [a, b]$ and $Z$ is nowhere dense because the closure of $Z$ is a subset of $Z \cup \mathbb{Q}$ and every interval of positive length certainly contains an irrational number with continued-fraction expansion $[a_0, a_1, a_2, \ldots]$ having $a_n = 1$ for some $n > N$. In view of [3, Theorem 197] it is clear that $Z$ is a null set.

Starting with our sequence $b_0, b_1, \ldots, b_N$ we define recursively two sequences $b^1_{N+1}, b^1_{N+2}, \ldots$ and $b^2_{N+1}, b^2_{N+2}, \ldots$ of integers such that $n^2 < b^1_n < b^2_n$ for every $n > N$. Then we consider all sequences $(c_n)_{n\ge 0}$ with $c_n = b_n$ for every $n \le N$ and $c_n \in \{b^1_n, b^2_n\}$ for every $n > N$. Clearly, the family of all these sequences has the cardinality of the continuum and yields an equipotent set $S$ of irrational numbers in $Z$ by associating to each sequence $(c_n)_{n\ge 0}$ the continued fraction $[c_n]_{n\ge 0}$. It remains to show that this can be done so that Lemma 1 can be applied to each number in $S$.

Put $b^1_N = b^2_N = b_N$ and suppose that $b^1_k$ and $b^2_k$ are already defined for $n \ge k \ge N$. Then choose integers $b^1_{n+1}, b^2_{n+1}$ so that $b^2_{n+1} > b^1_{n+1} > (n+1)^2$ and

$$\min\{\log B([b_0, \ldots, b_N^{i(N)}, \ldots, b_{n+1}^{i(n+1)}]) \mid i(k) \in \{1,2\}\ (N \le k \le n+1)\}$$
$$> \max\{\varphi^{-1}([b_0, \ldots, b_N^{i(N)}, \ldots, b_n^{i(n)}]) \mid i(k) \in \{1,2\}\ (N \le k \le n)\},$$

where $B(r) = q$ when $r = p/q$ with coprime $p, q \in \mathbb{Z}$ and $q > 0$. By construction, for each continued fraction $[b_n]_{n\ge 0}$ with $b_n \in \{b^1_n, b^2_n\}$ for every $n > N$ we have $\log B_{n+1} > \varphi^{-1}(A_n/B_n)$ and therefore we may apply Lemma 1. ∎

**3. Proof of Theorem 2.** It is enough to deal with the right derivative case. Suppose that the right derivative of $F_\varphi$ at $\xi$ equals a real number $c \neq 0$. Since $F_\varphi$ is increasing, $c$ is positive. Let $x \in \mathbb{R}$ be such that $c = 2^x$. For each $m \in \mathbb{N}$ define

$$\mathcal{N}(m) := \{n \in \mathbb{N} \mid \xi \le \varphi(n) < \xi + 2^{-m}\} \quad \text{and} \quad \mu(m) := \min \mathcal{N}(m).$$

Then for every $\varepsilon > 0$ there is a positive integer $N_\varepsilon$ such that $2^{x-\varepsilon} < \Delta_m < 2^{x+\varepsilon}$ for every integer $m \ge N_\varepsilon$ where

$$\Delta_m := \frac{F_\varphi(\xi + 2^{-m}) - F_\varphi(\xi)}{2^{-m}} = 2^m \cdot \sum_{n\in\mathcal{N}(m)} 2^{-n}.$$

If $\mathcal{N}$ is a nonempty subset of $\mathbb{N}$ with minimum $\mu$, then of course

$$2^{-\mu} \le \sum_{n\in\mathcal{N}} 2^{-n} \le 2^{1-\mu}.$$

Consequently, $m - x - \varepsilon < \mu(m) < 1 + m - x + \varepsilon$ for every integer $m \ge N_\varepsilon$. Now we distinguish between the two cases $x \in \mathbb{Z}$ and $x \notin \mathbb{Z}$. Suppose first that $x \notin \mathbb{Z}$ and fix $\varepsilon > 0$ so that $[x - \varepsilon, x + \varepsilon] \cap \mathbb{Z} = \emptyset$. Thus for each integer $m \geq N_\varepsilon$ the interval $[m-x-\varepsilon, 1+m-x+\varepsilon]$ contains precisely one integer, which must be $\mu(m)$. Hence $\mathcal{N}(N_\varepsilon) \supset [N_\varepsilon - x - \varepsilon, \infty[\, \cap \mathbb{Z}$ since $\mu(m) \in \mathcal{N}(m) \subset \mathcal{N}(N_\varepsilon)$ for every integer $m \geq N_\varepsilon$ and $\bigcup_{m \geq N_\varepsilon} [m-x-\varepsilon, 1+m-x+\varepsilon] \cap \mathbb{Z} = [N_\varepsilon - x - \varepsilon, \infty[\, \cap \mathbb{Z}$.

Therefore the set $\mathbb{N} \setminus \mathcal{N}(N_\varepsilon)$ must be finite, but this is impossible because there are infinitely many rationals outside the interval $[\xi, \xi + 2^{-N_\varepsilon}]$ which have to be numbered by $\varphi$.

Suppose secondly that $x \in \mathbb{Z}$. Then we fix $\varepsilon = 1/4$ in order to conclude from $m - x - \varepsilon < \mu(m) < 1 + m - x + \varepsilon$ that $\mu(m) \in \{m - x, 1 + m - x\}$ for every integer $m \geq N_\varepsilon$. Now we choose an integer $r \geq N_\varepsilon$ such that $1 + r - x \notin \mathcal{N}(N_\varepsilon)$. (This can be done because $\mathbb{N} \setminus \mathcal{N}(N_\varepsilon)$ is infinite.) Since $\mu(r), \mu(r+1) \in \mathcal{N}(N_\varepsilon)$, we have $\mu(r), \mu(r+1) \neq 1 + r - x$ and therefore we must have $r - x = \mu(r) \in \mathcal{N}(r)$ and $2 + r - x = \mu(r+1) \in \mathcal{N}(r+1) \subset \mathcal{N}(r)$. But then

$$\Delta_r = 2^r \cdot \sum_{n \in \mathcal{N}(r)} 2^{-n} > 2^r \cdot (2^{-(r-x)} + 2^{-(2+r-x)}) = \frac{5}{4} \cdot 2^x$$

contrary to $\Delta_m < 2^{\varepsilon + x} = \sqrt[4]{2} \cdot 2^x < \frac{5}{4} \cdot 2^x$ for every integer $m \geq N_\varepsilon$.

**4. Vanishing derivatives.** A proof of the following lemma is a nice exercise in analysis.

LEMMA 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic on $[x - \delta, x + \delta]$ with fixed $x \in \mathbb{R}$ and $\delta > 0$. If $(h_n)$ is a decreasing sequence of positive numbers tending to $0$ such that $(h_n / h_{n+1})$ is bounded and the sequence $(h_n^{-1} \cdot |f(x + h_n) - f(x - h_n)|)$ tends to $0$, then $f$ is differentiable at $x$ with a vanishing first derivative.*

For $\varphi \in \Phi$ and any interval $I$ of positive length define $m_\varphi(I)$ to be the least $m \in \mathbb{N}$ such that the rational number $\varphi(m)$ lies in $I$. Clearly we always have the estimate

$$\sum_{\varphi(n) \in I} \frac{1}{2^n} \leq 2^{1 - m_\varphi(I)}.$$

Consequently, for all $x \in \mathbb{R}$ and $h > 0$ we have

$$|F_\varphi(x + h) - F_\varphi(x - h)| \leq 2^{1 - M}$$

with $M = m_\varphi([x - h, x + h[)$.

Therefore, since for $x$ fixed $F_\varphi(x + h) - F_\varphi(x - h)$ increases when $h$ increases, Lemma 2 implies

LEMMA 3. *Let $\xi$ be an irrational number and $\varphi \in \Phi$ and fix $k \in \mathbb{N}$. Let $M_n = m_\varphi([\xi - n^{-k}, \xi + n^{-k}[)$ for every $n \in \mathbb{N}$. If the sequence $(n^{-k} 2^{M_n})$ tends to $\infty$ as $n \to \infty$, then $F_\varphi$ is differentiable at $\xi$ with a vanishing first derivative.*

*Proof of Theorem 3.* We make use of the following lemma which is clearly true if $\xi \in \mathbb{Q}$, and a straightforward consequence of Liouville's theorem (cf. [3, 11.7]) if $\xi \notin \mathbb{Q}$.

LEMMA 4. *If $\xi \in \mathbb{R}$ is algebraic then there exists a positive integer $k$ such that for each $n \in \mathbb{N}$ the estimate $0 \neq |\xi - r/s| \leq n^{-k}$ is only possible for $r, s \in \mathbb{Z}$ and $s > 0$ if $s \geq n$.*

Now suppose that $\varphi \in \Phi_0$. Let $\xi, k, M_n$ be as in Lemma 3 and (with $\xi \notin \mathbb{Q}$) Lemma 4. By Lemma 4 we must have $M_n \geq n$ for every $n \in \mathbb{N}$ since $\varphi(m) = r/s$ with $r, s \in \mathbb{Z}$ and $0 < s \leq m$ for every $m \in \mathbb{N}$. Thus $(n^{-k} 2^{M_n})$ tends to $\infty$ and therefore Theorem 3 follows from Lemma 3. ∎

REMARK. More generally, Theorem 3 is true for every irrational number which is not a Liouville number. Indeed, by definition (cf. [7]), $\xi \in \mathbb{R}$ is *Liouville* if and only if for every $k \in \mathbb{N}$ there are integers $p, q$ with $q \geq 2$ such that $0 \neq |\xi - p/q| < q^{-k}$. (An equivalent definition of the Liouville numbers which uses continued fractions and is useful for concrete constructions can be found in [8, §35]. Every Liouville number is transcendental and (cf. [7]) the set of all Liouville numbers is both null and residual.) Consequently, if $\xi \in \mathbb{R}$ is not a Liouville number then the conclusion of Lemma 4 is true for $\xi$ even when $\xi$ is transcendental. (Famous examples of transcendental numbers which are not Liouville are $\pi, e, \ln 2$, cf. [4].)

*Proof of Theorem 4(ii).* We will prove a little more than claimed. Let $X$ be any $F_\sigma$-set of irrational numbers, i.e. the union of a sequence $X_1, X_2, \ldots$ of closed sets of irrational numbers. (So $X$ may be uncountable and even $\mathbb{R} \setminus X$ may be a null set.) For $\emptyset \neq S \subset \mathbb{R}$ and $a \in \mathbb{R}$ let $d(a, S) = \inf\{|a - s| \mid s \in S\}$ be the Euclidian distance between the point $a$ and the set $S$. Naturally, if $S$ is closed then $d(a, S) = 0$ if and only if $a \in S$. In particular $d(r, X_n) > 0$ for all $r \in \mathbb{Q}$ and $n \in \mathbb{N}$. We get an appropriate $\varphi \in \Phi_0$ in the following way. We define an injective function $\psi$ from $\mathbb{Q} \setminus \mathbb{Z}$ to $\mathbb{N}$ such that $\mathbb{N} \setminus \psi(\mathbb{Q} \setminus \mathbb{Z})$ is infinite, whence $\psi$ can be extended to a bijection from $\mathbb{Q}$ onto $\mathbb{N}$. Then we define $\varphi$ to be the inverse of this bijection. Specifically, if $p/q \notin \mathbb{Z}$ with coprime $p, q \in \mathbb{Z}$ and $q \geq 2$ then we put

$$\psi(p/q) := \sqrt{2}^{1+|p|/p} \cdot 3^{|p|} \cdot 5^q \cdot 7^{\delta(p/q)}$$

where $\delta(p/q)$ is the least positive integer which is not smaller than

$$\max\{d(p/q, X_i)^{-1} \mid i = 1, \ldots, q\}.$$

Obviously, the function $\psi$ is well defined and injective on $\mathbb{Q} \setminus \mathbb{Z}$ and can be extended to a bijection $\psi : \mathbb{Q} \to \mathbb{N}$. Naturally, $\varphi := \psi^{-1}$ lies in the family $\Phi_0$. In order to verify that $F'_\varphi(\xi) = 0$ for each $\xi \in X$ we fix $m \in \mathbb{N}$ so that $\xi \in X_m$ and apply Lemma 3 with $k = 1$. Certainly, for sufficiently large $n \in \mathbb{N}$ no integer lies in the interval $[\xi - 1/n, \xi + 1/n]$ and a rational $p/q$ lies in this interval only if $q \geq m$. By definition, for such a rational we always have $\delta(p/q) \geq d(p/q, X_m)^{-1} \geq |p/q - \xi|^{-1} \geq n$. Therefore $M_n > 7^n$ for all sufficiently large $n$, and this completes the proof since $(n^{-1}2^{7^n})$ tends to infinity. ∎

REMARK. As we have just seen, Theorem 4(ii) remains true when $X$ is assumed to be a subset of an $F_\sigma$-set of irrational numbers. Although such a set $X$ must be meager, Theorem 1 does not allow us to replace *countable* with *meager* in (ii). Such a replacement is also impossible in (i) since any meager set $X \subset \mathbb{R}$ which is not null would naturally be a counterexample. (An even better counterexample is provided by Theorem 6 since $\mathbb{D}$ is a nowhere dense null set.)

With the help of vanishing left derivatives Theorem 8 is quickly proved.

*Proof of Theorem 8.* For $n \in \mathbb{N}$ let $A_n/B_n$ be the $n$th convergent to $\xi$, so that $|\xi - A_n/B_n| < (B_n B_{n+1})^{-1}$ for every $n \in \mathbb{N}$. Now fix $N \in \mathbb{N}$ large enough to enable a choice of $\varphi \in \Phi_0$ such that $\varphi(B_n) = A_n/B_n$ for every odd $n \geq N$. Since $A_n/B_n > \xi$ for every odd $n$, in view of the proof of Theorem 4(ii) we can certainly achieve that additionally the left derivative of $G_\varphi$ exists and vanishes at $\xi$. Then with $h_n = (B_n B_{n+1})^{-1}$ we have

$$\frac{G_\varphi(\xi + h_n) - G_\varphi(\xi)}{h_n} \geq \frac{1}{h_n} \cdot \frac{1}{B_n^2} = \frac{B_{n+1}}{B_n} \geq 1$$

for every odd $n \geq N$. Hence the right derivative of $G_\varphi$ at $\xi$ cannot vanish if it exists. ∎

**5. Infinite derivatives.** The following variation of Lemma 2 is evidently true.

LEMMA 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be increasing on $[\xi - \delta, \xi + \delta]$ with fixed $\xi \in \mathbb{R}$ and $\delta > 0$. If $(x_n)$ is a decreasing sequence of positive numbers tending to 0 such that*

$$\lim_{n\to\infty} \frac{f(\xi + x_{n+1}) - f(\xi)}{x_n} = \infty \quad \textit{resp.} \quad \lim_{n\to\infty} \frac{f(\xi) - f(\xi - x_{n+1})}{x_n} = \infty$$

*then the right resp. left derivative of $f$ at $\xi$ is $\infty$.*

*Proof of Theorem 4(i).* Let $\mathbb{P}$ denote the set of all primes and choose distinct reals $a_p (p \in \mathbb{P})$ so that $X \subset \{a_p \mid p \in \mathbb{P}\}$. If $X$ is infinite, we may assume that $X = \{a_p \mid p \in \mathbb{P}\}$. We want to define $\varphi \in \Phi$ so that for every $p \in \mathbb{P}$ and every $n \in \mathbb{N}$ the rational number $\varphi(p^n)$ lies in the interval $]a_p, a_p + 3^{-p^{n+2}}[$ when $n$ is even, and in $]a_p - 3^{-p^{n+2}}, a_p[$ when $n$ is odd. Then for each $a_p \in X$ with $h_n := (-1)^n 3^{-p^n}$ we have

$$|h_n|^{-1} \cdot |F_\varphi(a_p + h_{n+2}) - F_\varphi(a_p)| \geq 3^{p^n} \cdot 2^{-p^n} \to \infty \quad (n \to \infty)$$

and therefore $F'_\varphi(a_p) = \infty$ in view of Lemma 5.

Now to achieve this, for $p \in \mathbb{P}$ put $\mathcal{R}_p := \{m/p^n \mid m \in \mathbb{Z} \wedge n \in \mathbb{N}\} \setminus \mathbb{Z}$. Naturally, each set $\mathcal{R}_p$ is dense. Hence for every $p \in \mathbb{P}$ and every $n \in \mathbb{N}$ we may choose $\varphi(p^n)$ in $]a_p + 3^{-p^{n+4}}, a_p + 3^{-p^{n+2}}[ \cap \mathcal{R}_p$ when $n$ is even, and in $]a_p - 3^{-p^{n+2}}, a_p - 3^{-p^{n+4}}[ \cap \mathcal{R}_p$ when $n$ is odd. Doing so we get an injective function from $M := \{p^n \mid p \in \mathbb{P}, n \in \mathbb{N}\}$ into $\mathbb{Q} \setminus \mathbb{Z}$ since all the sets $\mathcal{R}_p$ are mutually disjoint and for each $p \in \mathbb{P}$ all the intervals $]a_p \pm 3^{-i}, a_p \pm 3^{-j}[$ are mutually exclusive. Since $\mathbb{N} \setminus M$ is infinite, this injection can easily be extended to a bijection $\varphi \in \Phi$ which fits automatically. This concludes the proof of the first statement of Theorem 4. ∎

*Proof of Theorem 5(i).* We will prove a little more than claimed. For $\varphi \in \Phi$ let $\Omega_\varphi^+$ resp. $\Omega_\varphi^-$ be the set of all points $\xi \in \mathbb{R}$ such that the right resp. left derivative of $F_\varphi$ is infinite at $\xi$. Then $\Omega_\varphi = \Omega_\varphi^+ \cap \Omega_\varphi^-$ and it is clear that always $\Omega_\varphi^+ \supset \mathbb{Q}$. Theorem 5(i) is an immediate consequence of

THEOREM 9. *If $\varphi \in \Phi_0$ then $\Omega_\varphi^- = \emptyset$ and $\Omega_\varphi^+ = \mathbb{Q}$.*

*Proof.* Let $\varphi \in \Phi_0$ and assume indirectly that $\Omega_\varphi^- \neq \emptyset$ and choose $\xi \in \Omega_\varphi^-$. The left derivative of $F_\varphi$ is infinite at $\xi$ and hence there is a lower bound $M \in \mathbb{N}$ such that

$$2^m \cdot \sum_{\xi - 2^{-m} \le \varphi(n) < \xi} \frac{1}{2^n} > 1$$

for every $m \geq M$. Suppose there were some $m \geq M$ such that $\varphi^{-1}(a) > m$ for every rational $a$ with $\xi - 2^{-m} \leq a < \xi$. Then

$$2^m \cdot \sum_{\xi - 2^{-m} \le \varphi(n) < \xi} \frac{1}{2^n} \le 2^m \cdot \sum_{n=m+1}^{\infty} \frac{1}{2^n} = 1,$$

contrary to the above. It follows that for every $m \geq M$ there exists $n \leq m$ such that $\xi - 2^{-m} \leq \varphi(n) < \xi$. Consequently, since $\varphi \in \Phi_0$, for every $m \geq M$ there are coprime integers $p, q$ such that $0 < q \leq m$ and $|\xi - p/q| \leq 2^{-m}$. In view of the lemma below this is impossible provided that $\xi \notin \mathbb{Q}$. And the following remark is a strong argument that $\xi \notin \Omega_\varphi^-$ whenever $\xi \in \mathbb{Q}$. In a similar way we get a contradiction from the assumption that $\Omega_\varphi^+$ contains an irrational number $\xi$. ∎

REMARK. By applying Lemma 4 for rational $\xi$ and in view of the proof of Theorem 3, if $\varphi \in \Phi_0$ then at each rational number the left derivative of $F_\varphi$ must exist and vanish.

LEMMA 6. *For each irrational number $\xi$ there exist infinitely many positive integers $m$ such that $|\xi - p/q| \geq 1/(2m^2)$ whenever $p, q \in \mathbb{Z}$ and $0 < q \leq m$.*

*Proof.* Let $\xi$ be an irrational number and for every $n \in \mathbb{N}$ let $A_n/B_n$ be the $n$th convergent to $\xi$ where $A_n, B_n$ are coprime and $B_n > 0$. For each $k \in \mathbb{N}$ put $m_k := (B_k + B_{k+1} + \tau_k)/2$ with $\tau_k \in \{0,1\}$ so that $m_k \in \mathbb{N}$. We have $m_1 < m_2 < \cdots$ since always $B_k < B_{k+1}$. Further, since always $B_{n+2} \geq B_{n+1} + B_n$, for every $k \geq 3$ we have $B_{k-1} \geq 2$. In order to prove Lemma 6 we verify for every $k \geq 3$ that $|\xi - p/q| \geq 1/(2m_k^2)$ whenever $p, q \in \mathbb{Z}$ and $0 < q \leq m_k$. Assume indirectly that there is $k \geq 3$ such that both $0 < q \leq m_k$ and $|\xi - p/q| < 1/(2m_k^2)$ for certain $p, q \in \mathbb{Z}$. Then $|\xi - p/q| < 1/(2q^2)$ and therefore, as a well-known consequence (cf. [3, Theorem 184]), the rational number $p/q$ must be a convergent to $\xi$, whence $p/q = A_n/B_n$ for some $n \in \mathbb{N}$. We have $B_n \leq q$ (with $B_n = q$ if $p, q$ are coprime) and hence $B_n \leq m_k$. Moreover, $n \leq k$ since $B_n \leq m_k < B_{k+1}$. (Note that from $B_{k+1} \geq B_k + B_{k-1} \geq B_k + 2$ we derive $2B_{k+1} \geq B_{k+1} + B_k + 2 > 2m_k$.) Consequently, $B_n + B_{n+1} \leq B_k + B_{k+1} \leq 2m_k$. Naturally (cf. [8, §13, (12)]),

$$\left|\xi - \frac{A_n}{B_n}\right| > \frac{1}{B_n(B_n + B_{n+1})}$$

and thus we arrive at the contradiction

$$\frac{1}{2m_k^2} > \left|\xi - \frac{p}{q}\right| = \left|\xi - \frac{A_n}{B_n}\right| > \frac{1}{m_k \cdot (2m_k)}. \blacksquare$$

*Proof of Theorem 5(ii).* Let $a < b$ and assume without loss of generality that $a, b \in \mathbb{Q}$. Put $\theta_m = (b-a) \cdot 257^{-2^m}$ and choose an injective function $\varphi$ from the positive even numbers into $\mathbb{Q} \cap [a,b]$ and, by analogy with the construction of a Cantor set, intervals $\mathcal{I}_{m,1}, \mathcal{I}_{m,2}, \ldots, \mathcal{I}_{m,2^m}$ $(m = 1, 2, \ldots)$ so that the following properties are satisfied for each $m \in \mathbb{N}$:

(1) $\mathcal{I}_{1,1} = [a, a+\theta_1]$, $\mathcal{I}_{1,2} = [b - \theta_1, b]$ and $\varphi(2) = a, \varphi(4) = b$.
(2) $\mathcal{I}_{m,1}, \mathcal{I}_{m,2}, \ldots, \mathcal{I}_{m,2^m}$ are mutually disjoint compact intervals of length $\theta_m$ each.
(3) The $2 \cdot 2^m$ endpoints of the intervals $\mathcal{I}_{m,i}$ are the rational numbers $\varphi(n)$ with $n$ running through the even numbers up to $4 \cdot 2^m$.
(4) For $m' = m+1$ the $2^{m'}$ intervals $\mathcal{I}_{m',j}$ are placed so that each $\mathcal{I}_{m',j}$ is a subinterval of some $\mathcal{I}_{m,i}$ with one common endpoint.

Naturally, the nonempty compact set

$$\mathcal{S} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{2^m} \mathcal{I}_{m,n}$$

is a perfect subset of $[a,b]$ and hence it has the cardinality of the continuum.

Extend $\varphi$ in any way to a numbering of all rational numbers. We conclude the proof by verifying $F'_\varphi(\xi) = \infty$ for all irrational $\xi \in \mathcal{S}$. Let $\xi \in \mathcal{S} \setminus \mathbb{Q}$. Then for every $m \in \mathbb{N}$ we can find an interval $\mathcal{I}_{m,j}$ which contains $\xi$. We have

$\mathcal{I}_{m,j} = [\varphi(n), \varphi(n')]$ for some even $n, n' \leq 4 \cdot 2^m$. Thus $\xi < \varphi(n') < \xi + \theta_m$ and $\xi - \theta_m < \varphi(n) < \xi$ for some $n, n' \leq 2^{m+2}$. Hence for every $m \in \mathbb{N}$, both

$$\frac{1}{\theta_{m-1}} \cdot \sum_{\xi \leq \varphi(n) < \xi + \theta_m} \frac{1}{2^n} \quad \text{and} \quad \frac{1}{\theta_{m-1}} \cdot \sum_{\xi - \theta_m \leq \varphi(n) < \xi} \frac{1}{2^n}$$

are not smaller than

$$\frac{1}{\theta_{m-1}} \cdot \frac{1}{2^{2^{m+2}}} = \frac{1}{b-a} \left( \frac{257}{256} \right)^{2^{m-1}} \to \infty \qquad (m \to \infty).$$

Consequently,

$$\lim_{m \to \infty} \frac{F_\varphi(\xi \pm \theta_m) - F_\varphi(\xi)}{\pm \theta_{m-1}} = \infty$$

and therefore $F'_\varphi(\xi) = \infty$ by applying Lemma 5. ∎

*Proof of Theorem 6.* Define mutually disjoint compact intervals

$$\mathcal{I}(m,1), \mathcal{I}(m,2), \ldots, \mathcal{I}(m, 2^m)$$

of length $3^{-m}$ for every $m \in \mathbb{N}$ in the usual way so that

$$\mathbb{D} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{2^m} \mathcal{I}(m,n)$$

and so that for arbitrary $m, k \in \mathbb{N}$ every interval $\mathcal{I}(m, \cdot)$ contains precisely $2^k$ intervals $\mathcal{I}(m+k, \cdot)$. Let $\varphi \in \Phi$ be arbitrary and put $Q(n) := \{\varphi(1), \ldots, \varphi(n)\}$. We claim that for every $m \geq 2$ we can find distinct intervals $\mathcal{I}_{m,k}$ ($k = 1, \ldots, 2^{m-1}$) disjoint from $Q(2m^2)$ in the collection $\{\mathcal{I}(m^2, n) \mid n \leq 2^{m^2}\}$ such that for every $m \geq 2$ each interval $\mathcal{I}_{m,k}$ contains precisely two disjoint intervals $\mathcal{I}_{m+1,i}, \mathcal{I}_{m+1,j}$.

In order to verify this, start with $m = 2$. At most eight of the sixteen intervals $\mathcal{I}(4, \cdot)$ meet the set $Q(8)$ and hence at least two intervals $\mathcal{I}(4, \cdot)$ are disjoint from $Q(8)$. For arbitrary $m \geq 2$ each interval $\mathcal{I}(m^2, \cdot)$ contains precisely $2^{(m+1)^2 - m^2}$ intervals $\mathcal{I}((m+1)^2, \cdot)$ of which at most $2(m+1)^2$ meet $Q(2(m+1)^2)$, whence at least two of them are disjoint from $Q(2(m+1)^2)$ because $2^{(m+1)^2 - m^2} - 2(m+1)^2 \geq 2$.

Naturally,

$$\mathcal{Y} := \bigcap_{m=2}^{\infty} \bigcup_{k=1}^{2^{m-1}} \mathcal{I}_{m,k}$$

is a subset of $\mathbb{D}$ and $\mathcal{Y}$ has the cardinality of the continuum. We finish the proof by showing that $F'_\varphi(\xi) = \infty$ is impossible for $\xi \in \mathcal{Y}$.

Let $\xi \in \mathcal{Y}$. Then $\{\xi\} = \bigcap_{m=2}^{\infty} [a_m, b_m]$ where for every $m \geq 2$ we have $[a_m, b_m] = \mathcal{I}(m^2, n)$ for some $n \leq 2^{m^2}$ with $\mathcal{I}(m^2, n) \cap Q(2m^2) = \emptyset$. Therefore every interval $[a_m, b_m]$ has length $3^{-m^2}$ and contains a rational $\varphi(n)$ only if $n > 2m^2$. Hence

$$3^{m^2} \cdot \sum_{a_m \le \varphi(n) < b_m} \frac{1}{2^n} \le 3^{m^2} \cdot \sum_{n=2m^2+1}^{\infty} \frac{1}{2^n} = \left(\frac{3}{4}\right)^{m^2} \to 0 \quad (m \to \infty).$$

Now for every $m \ge 2$ we may choose $c_m \in \{a_m, b_m\}$ so that $|c_m - \xi| \ge \frac{1}{2}(b_m - a_m) = \frac{1}{2}3^{-m^2}$. Then $\lim_{m\to\infty} c_m = \xi$ and for every $m \ge 2$ we have $c_m \ne \xi$ and

$$\begin{aligned}\frac{F_\varphi(\xi) - F_\varphi(c_m)}{\xi - c_m} &\le \frac{F_\varphi(b_m) - F_\varphi(a_m)}{\frac{1}{2}(b_m - a_m)} \\ &= 2 \cdot 3^{m^2} \cdot \sum_{a_m \le \varphi(n) < b_m} \frac{1}{2^n} \to 0 \quad (m \to \infty).\end{aligned}$$

Hence $F'_\varphi(\xi) = \infty$ is impossible. ■

**6. Positive derivatives.** Finally it remains to prove our probably most surprising theorem.

*Proof of Theorem 7.* Define three sequences $(a_k), (b_k), (d_k)$ of positive even numbers such that with $A_k = \{a_k + nd_k \mid n \in \mathbb{N}\}$ and $B_k = \{b_k + nd_k \mid n \in \mathbb{N}\}$ all the elements in the family $\{A_k \mid k \in \mathbb{N}\} \cup \{B_k \mid k \in \mathbb{N}\}$ are mutually disjoint sets of even numbers. (Choose for example $a_k = 2 \cdot 3^k$ and $b_k = 4 \cdot 3^k$ and $d_k = 2 \cdot 3^{k+1}$.) For each $k \in \mathbb{N}$ define a strictly decreasing sequence $(x_m^{(k)})$ which tends to 0 as $m \to \infty$ by $x_m^{(k)} := (c_k d_k^2 m)^{-1}$. Elementary asymptotic analysis yields

$$\begin{aligned}\text{(6.1)} \qquad \sum_{n=m}^{\infty} \frac{1}{(s + nd_k)^2} &= \frac{1}{d_k^2 m} + O\left(\frac{1}{m^2}\right) \\ &= c_k x_m^{(k)} + O((x_m^{(k)})^2) \quad (m \to \infty)\end{aligned}$$

for each $k, s \in \mathbb{N}$. Now put $\delta_1 = 1$ and $\delta_k := \min\{|\xi_i - \xi_k| \mid i < k\}$ for all integers $k \ge 2$. Then choose $m_k \in \mathbb{N}$ for every $k \in \mathbb{N}$ such that with $y_k := x_{m_k}^{(k)} = (c_k d_k^2 m_k)^{-1}$,

$$\text{(6.2)} \qquad y_k + \sqrt{c_k y_k} < \delta_k/2,$$

$$\text{(6.3)} \qquad 2c_{k+1} y_{k+1} \le c_k y_k,$$

$$\text{(6.4)} \qquad \max\left\{ \sum_{n=m_k}^{\infty} \frac{1}{(a_k + nd_k)^2}, \sum_{n=m_k}^{\infty} \frac{1}{(b_k + nd_k)^2} \right\} \le 2c_k y_k,$$

for each $k \in \mathbb{N}$. Define $X_k \subset A_k$ and $Y_k \subset B_k$ by $X_k = \{a_k + nd_k \mid m_k \le n \in \mathbb{N}\}$ and $Y_k = \{b_k + nd_k \mid m_k \le n \in \mathbb{N}\}$. Now we define our desired $\varphi \in \varPhi$ first on $D := \bigcup\{X_k \cup Y_k \mid k \in \mathbb{N}\}$ by choosing for each $k \in \mathbb{N}$ and every integer $n \geq m_k$,

$$\varphi(a_k + nd_k) \in \,]\xi_k + x_{n+1}^{(k)}, \xi_k + x_n^{(k)}[\, \cap \mathbb{Q} \setminus \mathbb{Z},$$
$$\varphi(b_k + nd_k) \in \,]\xi_k - x_n^{(k)}, \xi_k - x_{n+1}^{(k)}[\, \cap \mathbb{Q} \setminus \mathbb{Z}.$$

By using the disjoint sets $\mathcal{R}_p$ ($p \in \mathbb{P}$) of the proof of Theorem 4 these choices can be made so that $\varphi$ is injective on $D$. Then we extend $\varphi$ (well-defined and injective) by defining $\varphi^{-1}$ on $\mathbb{Q} \setminus (\mathbb{Z} \cup \varphi(D))$ via

$$\varphi^{-1}(p/q) := \sqrt{13}^{1+|p|/p} \cdot 3^{|p|} \cdot 5^q \cdot 7^{\delta(p/q)}$$

where $p, q$ are coprime integers and $q \geq 2$ and where $\delta(p/q)$ is the least positive integer not smaller than $\max\{|p/q - \xi_i|^{-1} \mid i = 1, \ldots, q\}$. This extension is clearly possible because $\varphi^{-1}(p/q)$ is always odd by definition and $D$ contains only even numbers. Finally we extend $\varphi$ in any way to a bijection from $\mathbb{N}$ onto $\mathbb{Q}$.

Now fix $\kappa \in \mathbb{N}$ and for abbreviation put $\xi := \xi_\kappa$ and $x_m := x_m^{(\kappa)}$ for every $m \in \mathbb{N}$. For $k, m \in \mathbb{N}$ let $\mathcal{I}_{m,k} := [\xi_k - y_k, \xi_k + y_k] \cap [\xi - x_m, \xi + x_m]$. We claim that

$$\forall m, k \in \mathbb{N}: \quad k > \kappa \wedge \mathcal{I}_{m,k} \neq \emptyset \;\Rightarrow\; c_k y_k \leq x_m^2. \tag{6.5}$$

Indeed, if $k > \kappa$ then $\xi \notin [\xi_k - y_k, \xi_k + y_k]$ since by (6.2), $y_k < \delta_k \leq |\xi_k - \xi|$. Therefore, if additionally $\mathcal{I}_{m,k} \neq \emptyset$ for any $m$ then we clearly must have $x_m + y_k \geq |\xi_k - \xi| \geq \delta_k$ and thus $c_k y_k > x_m^2$ would imply $\sqrt{c_k y_k} + y_k > x_m + y_k \geq \delta_k$ contrary to (6.2).

In order to conclude the proof by verifying $G'_\varphi(\xi) = c_\kappa$ we take into account the following three issues.

First, there clearly exists a bound $\delta > 0$ such that $\mathbb{Z} \cup \varphi(\bigcup_{k=1}^{\kappa-1}(X_k \cup Y_k))$ is disjoint from $[\xi - \delta, \xi + \delta]$. We claim that the set

$$K_m := \{k \in \mathbb{N} \mid k > \kappa \wedge \mathcal{I}_{m,k} \neq \emptyset\}$$

is empty for some $m \in \mathbb{N}$ if and only if $\xi$ is not a limit point of the set $\{\xi_1, \xi_2, \ldots\}$. Indeed, if $K_m$ is empty for some $m$ then for every $k > \kappa$ we have $\mathcal{I}_{m,k} = \emptyset$ and hence $\xi_k \notin [\xi - x_m, \xi + x_m]$, whence $\xi$ cannot be a limit point of $\{\xi_1, \xi_2, \ldots\}$. Conversely, if $\xi$ is not a limit point then we may choose $h > 0$ so that $\xi_k \notin [\xi - h, \xi + h]$ for every $k > \kappa$. Since by (6.2) we have $y_k < \frac{1}{2}|\xi_k - \xi|$ for every $k > \kappa$, we must have $\mathcal{I}_{m,k} = \emptyset$ for every $k > \kappa$, or equivalently $K_m = \emptyset$ if $m$ is chosen so that $x_m < h/2$.

So if $\xi$ is not a limit point of $\{\xi_1, \xi_2, \ldots\}$ then there exists $\tilde{m}$ such that $\mathcal{I}_{\tilde{m},k} = \emptyset$ for every $k > \kappa$ and hence $\tilde{\delta} = \min\{\delta, x_{\tilde{m}}\}$ is a bound such that even $\mathbb{Z} \cup \varphi(\bigcup_{k \neq \kappa}(X_k \cup Y_k))$ is disjoint from $[\xi - \tilde{\delta}, \xi + \tilde{\delta}]$.

Secondly, put $L_{m,k} := \{n \in (X_k \cup Y_k) \mid \xi - x_m \le \varphi(n) < \xi + x_m\}$ and assume that $\xi$ is a limit point of $\{\xi_1, \xi_2, \ldots\}$. Thus $K_m$ is never empty and we may define $\mu(m) := \min K_m$. Then in view of the definition of $\varphi$,

$$\sum_{k>\kappa} \sum_{n \in L_{m,k}} \frac{1}{n^2} \le \sum_{k \in K_m} \left( \sum_{n=m_k}^{\infty} \frac{1}{(a_k + nd_k)^2} + \sum_{n=m_k}^{\infty} \frac{1}{(b_k + nd_k)^2} \right)$$

for all $m \in \mathbb{N}$. Thus by applying (6.4),

$$\sum_{k>\kappa} \sum_{n \in L_{m,k}} \frac{1}{n^2} \le 4 \sum_{k \in K_m} c_k y_k.$$

Furthermore, since $c_{\mu(m)+n} y_{\mu(m)+n} \le 2^{-n} c_{\mu(m)} y_{\mu(m)}$ for $n = 0, 1, 2, \ldots$ due to (6.3),

$$\sum_{k \in K_m} c_k y_k \le \sum_{k=\mu(m)}^{\infty} c_k y_k \le \sum_{n=0}^{\infty} 2^{-n} c_{\mu(m)} y_{\mu(m)} = 2 c_{\mu(m)} y_{\mu(m)}.$$

By (6.5) we have $c_{\mu(m)} y_{\mu(m)} \le x_m^2$ and so altogether we arrive at

$$\text{(6.6)} \qquad \frac{1}{x_m} \cdot \sum_{k>\kappa} \sum_{n \in L_{m,k}} \frac{1}{n^2} \le 8 x_m \to 0 \quad (m \to \infty)$$

provided that $\xi$ is a limit point of $\{\xi_1, \xi_2, \ldots\}$.

Thirdly, define $N_m := \{n \in \mathbb{N} \setminus D \mid \xi - x_m \le \varphi(n) < \xi + x_m\}$ and let $M$ be the smallest positive integer such that the interval $[\xi - x_M, \xi + x_M]$ does not contain integers or reduced fractions $p/q$ with $|q| < \kappa$. Then for each $m \ge M$ we have $N_m \subset [5 \cdot 7^{1/x_m}, \infty[$ because if $n \in N_m$ and $\varphi(n) = p/q$ (where $q > 0$ and the fraction $p/q$ is reduced) then $p/q \in [\xi - x_m, \xi + x_m] \subset [\xi - x_M, \xi + x_M]$ and hence (by the definition of $M$) $\kappa \in \{1, \ldots, q\}$ so that $1/x_m \le |p/q - \xi|^{-1} \le \delta(p/q)$ and therefore $n \ge 3^{|p|} \cdot 5^q \cdot 7^{\delta(p/q)} \ge 5 \cdot 7^{1/x_m}$. Consequently, for $m \ge M$,

$$\text{(6.7)} \qquad \frac{1}{x_m} \cdot \sum_{n \in N_m} \frac{1}{n^2} \le \frac{1}{x_m} \cdot \sum_{n \ge 5 \cdot 7^{1/x_m}} \frac{1}{n^2} \le \frac{1}{x_m} \cdot \int_{4 \cdot 7^{1/x_m}}^{\infty} \frac{dx}{x^2} \to 0 \quad (m \to \infty).$$

To conclude the proof by verifying $G'_\varphi(\xi) = c_\kappa$ it is enough to check

$$\lim_{m \to \infty} \frac{G_\varphi(\xi + x_m) - G_\varphi(\xi)}{x_m} = \lim_{m \to \infty} \frac{G_\varphi(\xi) - G_\varphi(\xi - x_m)}{x_m} = c_\kappa$$

because $\lim_{m \to \infty} x_{m+1}/x_m = 1$ and $G_\varphi$ is increasing. Now, for every $m \in \mathbb{N}$

we can write

$$\frac{G_\varphi(\xi + x_m) - G_\varphi(\xi)}{x_m} = \frac{1}{x_m}\Bigg(\sum_{n\in S_m\cap X_\kappa} \frac{1}{n^2} + \sum_{n\in S_m\cap D\setminus X_\kappa} \frac{1}{n^2} + \sum_{n\in S_m\setminus D} \frac{1}{n^2}\Bigg)$$

where $S_m := \{n \in \mathbb{N} \mid \xi \le \varphi(n) < \xi + x_m\}$. (Recall that $X_\kappa \subset D$.) In view of $S_m \setminus D \subset N_m$ and (6.7) we have

$$\lim_{m\to\infty} \frac{1}{x_m} \sum_{n\in S_m\setminus D} \frac{1}{n^2} = 0.$$

In view of (6.1) and the definition of $\varphi$ we have

$$\lim_{m\to\infty} \frac{1}{x_m} \sum_{n\in S_m\cap X_\kappa} \frac{1}{n^2} = c_\kappa.$$

Since $S_m \cap D \setminus X_\kappa$ is a subset of $\bigcup_{k\neq\kappa}(X_k \cup Y_k)$, we have

$$\lim_{m\to\infty} \frac{1}{x_m} \sum_{n\in S_m\cap D\setminus X_\kappa} \frac{1}{n^2} = 0$$

in view of (6.6) and the consideration involving the bound $\delta$ and the potential bound $\tilde{\delta}$. (Clearly, if $\xi$ is not a limit point of $\{\xi_1, \xi_2, \ldots\}$ then $S_m \cap D \setminus X_\kappa = \emptyset$ for sufficiently large $m$.) Summing up,

$$\lim_{m\to\infty} \frac{G_\varphi(\xi + x_m) - G_\varphi(\xi)}{x_m} = c_\kappa.$$

Analogously,

$$\lim_{m\to\infty} \frac{G_\varphi(\xi) - G_\varphi(\xi - x_m)}{x_m} = c_\kappa,$$

and this finishes the proof of Theorem 7. ∎

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