

*STOCHASTIC DYNAMICAL SYSTEMS WITH WEAK
CONTRACTIVITY PROPERTIES
II. ITERATION OF LIPSCHITZ MAPPINGS*

BY

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Abstract. In this continuation of the preceding paper (Part I), we consider a sequence $(F_n)_{n \geq 0}$ of i.i.d. random Lipschitz mappings $X \rightarrow X$, where X is a proper metric space. We investigate existence and uniqueness of invariant measures, as well as recurrence and ergodicity of the induced stochastic dynamical system (SDS) $X_n^x = F_n \circ \cdots \circ F_1(x)$ starting at $x \in X$. The main results concern the case when the associated Lipschitz constants are log-centered. Principal tools are local contractivity, as considered in detail in Part I, the Chacon–Ornstein theorem and a hyperbolic extension of the space X as well as the process (X_n^x) .

The results are applied to a class of examples, namely, the reflected affine stochastic recursion given by $X_0^x = x \geq 0$ and $X_n^x = |A_n X_{n-1}^x - B_n|$, where (A_n, B_n) is a sequence of two-dimensional i.i.d. random variables with values in $\mathbb{R}_*^+ \times \mathbb{R}_*^+$.

6. Introduction. This is a direct continuation of our preceding paper [9]. For this reason, here the numbering starts with Section 6 instead of 1. In order to enable a reasonably self-contained access, we recollect the basic facts from [9].

We take a proper metric space (X, d) and the monoid \mathfrak{G} of all continuous mappings $X \rightarrow X$, equipped with the topology of uniform convergence on compact sets. On \mathfrak{G} , we consider a sequence $(F_n)_{n \geq 1}$ of i.i.d. \mathfrak{G} -valued random variables whose common distribution $\tilde{\mu}$ is a regular probability measure on \mathfrak{G} . That is, the F_n are random functions on X , defined on a suitable probability space $(\Omega, \mathfrak{A}, \Pr)$. They give rise to the stochastic dynamical system (SDS) $\Omega \ni \omega \mapsto X_n^x(\omega)$ defined by

$$(6.1) \quad X_0^x = x \in X, \quad \text{and} \quad X_n^x = F_n(X_{n-1}^x), \quad n \geq 1.$$

For general background references on this type of Markov processes, see the bibliography of Part I [9].

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In the present paper, we shall always assume that the F_n belong to the semigroup $\mathfrak{L} \subset \mathfrak{G}$ consisting of all $f : X \rightarrow X$ with finite Lipschitz constant

$$l(f) = \sup\{d(f(x), f(y))/d(x, y) : x, y \in X, x \neq y\}.$$

We choose a reference point $o \in X$. The real random variables

$$(6.2) \quad A_n = l(F_n) \quad \text{and} \quad B_n = d(F_n(o), o)$$

will play an important role. Our main assumption is that the SDS is non-expanding on the average:

$$(6.3) \quad \mathbb{E}(\log A_n) \leq 0.$$

The case when $\mathbb{E}(\log A_n) < 0$ strictly is well understood, because in this case, the SDS is strongly contractive in the sense of [9, Definition (2.1)], that is,

$$(6.4) \quad \Pr[d(X_n^x, X_n^y) \rightarrow 0 \text{ for all } y \in X] = 1.$$

In this paper, the main focus is on the critical case when A_n is log-centered:

$$(6.5) \quad \boxed{\mathbb{E}(\log A_n) = 0.}$$

The properties of the SDS that we are studying here are the following.

- Topological recurrence versus transience, where transience means that

$$\Pr[d(X_n^x, x) \rightarrow \infty] = 1 \quad \text{for all } x \in X,$$

while recurrence refers to a suitable non-empty set $L \subset X$ (the attractor) such that for all $x \in X$ ⁽¹⁾

$$(6.6) \quad \begin{aligned} &\text{for every open } U \subset X \text{ with } U \cap L \neq \emptyset, \\ &\Pr[X_n^x \in U \text{ for infinitely many } n] = 1. \end{aligned}$$

- Existence and uniqueness (up to multiplication with constants) of an invariant Radon measure ν on X , where invariance means that

$$\nu(U) = \int_X \Pr[X_1^x \in U] d\nu(x)$$

for any Borel set $U \subset X$.

- Ergodicity of the time shift T with respect to the extension of ν to the trajectory space of the SDS. The latter space, for the SDS starting at x , is

$$(X^{\mathbb{N}_0}, \mathfrak{B}(X^{\mathbb{N}_0}), \Pr_x),$$

where $\mathfrak{B}(X^{\mathbb{N}_0})$ is the product Borel σ -algebra on $X^{\mathbb{N}_0}$, and \Pr_x is the image of the measure \Pr under the mapping

$$\Omega \rightarrow X^{\mathbb{N}_0}, \quad \omega \mapsto (X_n^x(\omega))_{n \geq 0}.$$

⁽¹⁾ For topological recurrence on L , this should just hold for all $x \in L$, but in all our applications, we shall have this for all $x \in X$.

Given an invariant Radon measure ν , its above-mentioned extension to the trajectory space is the σ -finite measure

$$\Pr_\nu = \int_{\mathbf{L}} \Pr_x d\nu(x).$$

It is invariant with respect to the time shift $T : \mathbf{X}^{\mathbb{N}_0} \rightarrow \mathbf{X}^{\mathbb{N}_0}$.

A convenient property that provides tools for handling the log-centered case (6.5) is *local contractivity*, which means that for every $x \in \mathbf{X}$ and every compact $K \subset \mathbf{X}$,

$$(6.7) \quad \Pr[d(X_n^x, X_n^y) \cdot \mathbf{1}_K(X_n^x) \rightarrow 0 \text{ for all } y \in \mathbf{X}] = 1.$$

This property was introduced by Babillot, Bougerol and Elie [1] in the context of affine recursions and studied in detail by Benda [2], [3]. Local contractivity is one of the main themes of [9], in whose introduction the reader will find more background and references. Of course, when local contractivity cannot be verified, one needs to develop further methods. Our main results concern that situation.

This paper is structured as follows.

We impose suitable moment and non-degeneracy conditions on the i.i.d. 2-dimensional real random variables (A_n, B_n) which were defined in (6.2). Those standard assumptions are stated in (7.4) below. We first prove existence of a non-empty limit set \mathbf{L} on which the SDS is recurrent (§7, Theorem (7.6)). Following [9, Corollary (2.8)], \mathbf{L} is characterized as the smallest non-empty closed subset of \mathbf{X} with the property that

$$(6.8) \quad f(\mathbf{L}) \subset \mathbf{L} \quad \text{for every } f \in \text{supp}(\tilde{\mu}).$$

Then (§8) we introduce a *hyperbolic extension* of the space \mathbf{X} as well as of the SDS. The extended SDS turns out to be generated by Lipschitz mappings with Lipschitz constant 1 (Lemma (8.5)). The hyperbolic extension appears to be interesting in its own right, and we intend to come back to it in future work. It implies that the extended SDS is either transient or conservative, although in general typically not locally contractive.

First, in §9, we consider the case when the extended SDS is transient. In this case, we can show (9.4) that the original SDS is locally contractive, so that all results of [9, Section 2] apply. In particular, we get uniqueness of the invariant Radon measure ν (up to constant factors) and ergodicity of the shift on the associated trajectory space. It is worth mentioning that the “classical” instance of this situation is the affine stochastic recursion on \mathbb{R} :

$$(6.9) \quad Y_0^y = y \in \mathbb{R}, \quad \text{and} \quad Y_n^y = A_n Y_{n-1}^y + B_n, \quad n \geq 1.$$

Its hyperbolic extension is a random walk on the affine group, which is well known to be transient.

The hardest case turns out to be the one when the extended SDS is conservative (§10). In this case, we are able to obtain a result only under an additional assumption (10.8) on the original SDS that resembles the criterion used in [9, Section 4] for SDS of contractions. But then we even get ergodicity and uniqueness of the invariant Radon measure for the extended SDS (Theorem (10.15)).

In the final section (§11), we explain how to apply all those results to the reflected affine stochastic recursion on $\mathbf{X} = \mathbb{R}^+$,

$$(6.10) \quad X_0^x = x, \quad X_n^x = |A_n X_{n-1}^x - B_n|,$$

where $(A_n, B_n)_{n \geq 1}$ is a sequence of i.i.d. pairs of positive real random variables. This interesting SDS can be seen as the synthesis of the affine stochastic recursion (6.9) and the reflected random walk considered in [9]. We choose the minus sign in the recursion in order to underline the analogy with the reflected random walk. Here, we shall only consider the most typical situation where $B_n > 0$. When $A_n \equiv 1$, we are back in the case of the reflected random walk. It turns out that in the log-centered case, this SDS is not in general locally contractive—a fact that can serve as a motivation for the research undertaken in this paper.

7. The contractive case, and recurrence in the log-centered case.

With the use of the real random variables A_n and B_n of (6.2), we can compare our SDS (X_n^x) starting at $x \in \mathbf{X}$ with the affine SDS (Y_n^y) on \mathbb{R}^+ of (6.9). Namely,

$$(7.1) \quad d(X_n^x, o) \leq Y_n^{|x|}, \quad \text{where} \quad |x| = d(x, o).$$

Thus, we can use the results of [9, Section 3]. First of all, we have the following, which is of course well-known.

(7.2) COROLLARY. *Given the random i.i.d. Lipschitz mappings F_n , let A_n and B_n be as in (6.2). If $\mathbf{E}(\log^+ A_n) < \infty$ and $-\infty \leq \mathbf{E}(\log A_n) < 0$ then the SDS (X_n^x) generated by the F_n is strongly contractive on \mathbf{X} . If in addition $\mathbf{E}(\log^+ B_n) < \infty$ then the SDS has a unique invariant probability measure ν on \mathbf{X} , and it is (positive) recurrent on the limit set $\mathbf{L} = \text{supp}(\nu)$, which is characterized by (6.8) and satisfies (6.6). Also, the time shift on the trajectory space $\mathbf{X}^{\mathbb{N}_0}$ is ergodic with respect to the probability measure Pr_ν .*

Proof. Strong contractivity is obvious. When $\mathbf{E}(\log^+ B_n) < \infty$, the affine recursion $(Y_n^{|x|})$ is positive recurrent. We now use properness of \mathbf{X} : for some $r > 0$, we have $Y_n^{|x|} \in [0, r)$ infinitely often with probability 1, and the return time to that interval has finite expectation. By (7.1), (X_n^x) visits the relatively compact open ball $\mathbf{B}(r) = \mathbf{B}(o, r) = \{x \in \mathbf{X} : d(x, o) < r\}$ infinitely often with probability 1, and the return time to that ball also has

finite expectation. Thus, [9, Theorem (2.13)] applies in the stronger and simpler variant of strong contractivity. ■

The interesting and much harder case is the one of (6.5), where $\log A_n$ is integrable and centered. The assumptions of [9, Proposition (3.3)], applied to A_n and B_n of (6.2), will not in general imply that our SDS (X_n^x) is locally contractive. Also, it is not clear a priori that a non-empty set \mathbf{L} with (6.8) exists. We shall now show this with the help of the probabilistic arguments of Part I.

(7.3) PROPOSITION. *Suppose that $\mathbf{E}(\log^+ A_n) < \infty$ and $\mathbf{E}(\log^+ B_n) < \infty$. If $\Pr[A_n < 1] > 0$ then the non-empty set \mathbf{L} characterized by (6.8) is well-defined. Furthermore, the SDS is topologically irreducible on \mathbf{L} ; more generally, for every open $U \subset \mathbf{X}$ with $U \cap \mathbf{L} \neq \emptyset$ and for every $x \in \mathbf{X}$,*

$$\Pr[X_n^x \in U] > 0 \quad \text{for some } n = n_{x,U} \text{ } ^{(2)}.$$

Proof. Let $\alpha = \mathbf{E}(\log^+ A_n)$ and $\beta = \mathbf{E}(\log^- A_n)$, so that $\alpha < \infty$, $\beta > 0$ and $\mathbf{E}(\log A_n) = \alpha - \beta$. If $\beta = \infty$ then Corollary (7.2) applies and yields the stated results. So assume that $\beta < \infty$.

Let $t = \beta/(2\alpha + 2\beta)$. We modify the probability measure $\tilde{\mu}$ on \mathfrak{L} that governs our SDS, and define a new one, $\tilde{\mu}'$, by

$$d\tilde{\mu}'(f) = c \cdot (t \mathbf{1}_{\{\mathfrak{l}(f) \geq 1\}}(f) + (1-t) \mathbf{1}_{\{\mathfrak{l}(f) < 1\}}(f)) d\tilde{\mu}(f)$$

with the appropriate normalizing constant c . We consider a sequence $(F'_n)_{n \geq 1}$ of i.i.d. random Lipschitz mappings with common distribution $\tilde{\mu}'$, and write $((X'_n)^x)$ for the associated SDS. Also, we let $A'_n = \mathfrak{l}(F'_n)$ and $B'_n = d(F'_n(o), o)$. Then $\mathbf{E}(\log^+ B'_n) < \infty$ and $\mathbf{E}(\log A'_n) = c(t\alpha - (1-t)\beta) < 0$. Thus, the new SDS satisfies the hypotheses of Corollary (7.2). Let \mathbf{L} be its limit set. Since $\text{supp}(\tilde{\mu}') = \text{supp}(\tilde{\mu})$, the set \mathbf{L} is well-defined and characterized by (6.8) in terms of $\text{supp}(\tilde{\mu})$.

Let $U \subset \mathbf{X}$ be open and $U \cap \mathbf{L} \neq \emptyset$, and let $x \in \mathbf{X}$. Since $\tilde{\mu}' \leq c(1-t)\tilde{\mu}$, we have

$$\Pr[X_n^x \in U] \geq (c(1-t))^{-n} \Pr[(X'_n)^x \in U].$$

Since $((X'_n)^x)$ satisfies (6.6), there is n such that the right hand side in the last inequality is positive. ■

We now state, once for all, the standard assumptions that we will impose on our SDS in all main results concerning the log-centered case.

(7.4) STANDARD ASSUMPTIONS.

- (i) Non-degeneracy: $\Pr[A_n > 0] = 1$, $\Pr[A_n < 1] > 0$, and $\Pr[A_n y + B_n = y] < 1$ for all $y \in \mathbb{R}$.

⁽²⁾ For topological irreducibility on \mathbf{L} , one would only require this for all $x \in \mathbf{L}$.

- (ii) Moment conditions: $\mathbf{E}(|\log A_n|^2) < \infty$ and $\mathbf{E}((\log^+ B_n)^{2+\varepsilon}) < \infty$ for some $\varepsilon > 0$.
- (iii) Centered case: $\mathbf{E}(|\log A_n|) < \infty$ and $\mathbf{E}(\log A_n) = 0$.

(7.5) REMARKS. (a) Under the Standard Assumptions, we can apply [9, Proposition (3.3)] to deduce that $(Y_n^{|x|})$ is locally contractive and recurrent on its limit set $\mathbf{L}_{\mathbb{R}}$, which is contained in \mathbb{R}^+ by construction. Note that it depends on the reference point $o \in \mathbf{X}$ through the definition of B_n .

(b) A sufficient condition for the requirement of (7.4)(i) that $\Pr[A_n y + B_n = y] < 1$ to hold for all $y \in \mathbb{R}$ is that

$$\Pr[F_n(o) = o] < 1.$$

Indeed, if $y = o$, then $\Pr[A_n y + B_n = y] = \Pr[F_n(o) = o]$. If $y \neq o$ then observe that by our assumptions, $A_n - 1$ assumes both positive and negative values with positive probability, so that the requirement is again met. ■

In the following, we shall write

$$A_{m,m} = 1 \quad \text{and} \quad A_{m,n} = A_{m+1} \cdots A_{n-1} A_n \quad (n > m).$$

(7.6) THEOREM. *Under the Standard Assumptions (7.4), the SDS is topologically recurrent on the set \mathbf{L} of Proposition (7.3), and (6.6) holds for \mathbf{L} .*

Proof. The (non-strictly) descending ladder epochs are

$$\ell(0) = 0, \quad \ell(k+1) = \inf\{n > \ell(k) : A_{0,n} \leq A_{0,\ell(k)}\}.$$

Since $(A_{0,n})$ is a recurrent multiplicative random walk on \mathbb{R}_*^+ , these epochs are stopping times with i.i.d. increments. The induced SDS is $(\bar{X}_k^x)_{k \geq 0}$, where $\bar{X}_k^x = X_{\ell(k)}^x$. It is also generated by random i.i.d. Lipschitz mappings, namely

$$\bar{F}_k = F_{\ell(k)} \circ F_{\ell(k)-1} \circ \cdots \circ F_{\ell(k-1)+1}, \quad k \geq 1.$$

With the same stopping times, we also consider the induced affine recursion given by $\bar{Y}_k^{|x|} = Y_{\ell(k)}^{|x|}$. It is generated by the i.i.d. pairs $(\bar{A}_k, \bar{B}_k)_{k \geq 1}$, where

$$\bar{A}_k = A_{\ell(k-1), \ell(k)} \quad \text{and} \quad \bar{B}_k = \sum_{j=\ell(k-1)+1}^{\ell(k)} B_j A_{j, \ell(k)}.$$

It is known [6, Lemma 5.49] that under our assumptions, we have $\mathbf{E}(\log^+ \bar{A}_k) < \infty$, $\mathbf{E}(\log \bar{A}_k) < 0$ and $\mathbf{E}(\log^+ \bar{B}_k) < \infty$. Returning to (\bar{X}_k^x) , we have $l(\bar{F}_k) \leq \bar{A}_k$ and $d(\bar{F}_k(o), o) \leq \bar{B}_k$. Corollary (7.2) applies, and the induced SDS is strongly contractive. It has a unique invariant probability measure $\bar{\nu}$, and it is (positive) recurrent on $\bar{\mathbf{L}} = \text{supp}(\bar{\nu})$. Moreover, for every starting point $x \in \mathbf{X}$ and each open set $U \subset \mathbf{X}$ that intersects $\bar{\mathbf{L}}$, we conclude that almost surely, (\bar{X}_k^x) visits U infinitely often.

In view of the fact that the original SDS is topologically irreducible on \mathbf{L} , we have $\bar{\mathbf{L}} \subset \mathbf{L}$. We now define a sequence of subsets of \mathbf{L} by

$$\mathbf{L}_0 = \bar{\mathbf{L}} \quad \text{and} \quad \mathbf{L}_m = \bigcup \{f(\mathbf{L}_{m-1}) : f \in \text{supp}(\tilde{\mu})\}.$$

Then the closure of $\bigcup_m \mathbf{L}_m$ is a subset of \mathbf{L} that is mapped into itself by every $f \in \text{supp}(\tilde{\mu})$. The property (6.8) of \mathbf{L} , which holds by Proposition (7.3), yields

$$\mathbf{L} = \left(\bigcup_m \mathbf{L}_m \right)^-.$$

We now show by induction on m that for every starting point $x \in \mathbf{X}$ and every open set U that intersects \mathbf{L}_m ,

$$\Pr[X_n^x \in U \text{ for infinitely many } n] = 1,$$

and this will conclude the proof.

For $m = 0$, the statement is true. Suppose it is true for $m - 1$. Given an open set U that intersects \mathbf{L}_m , we can find an open, relatively compact set V that intersects \mathbf{L}_{m-1} such that $\tilde{\mu}(\{f \in \mathcal{L} : f(V) \subset U\}) = \alpha > 0$.

By the induction hypothesis, (X_n^x) visits U infinitely often with probability 1. We can now apply [9, Lemma (2.10)] with $\ell = 2$, $U_0 = U$ and $U_1 = V$ to conclude that also V is visited infinitely often with probability 1. ■

The transition operator of our SDS is given by

$$(7.7) \quad P\varphi(x) = \mathbf{E}(\varphi(X_1^x)) = \int_{\mathcal{L}} \varphi(f(x)) d\tilde{\mu}(f)$$

for any Borel function $\varphi : \mathbf{X} \rightarrow \mathbb{R}$ for which that integral exists. In particular, we may choose $\varphi \in \mathcal{C}_c(\mathbf{X})$, the space of compactly supported continuous functions $\mathbf{X} \rightarrow \mathbb{R}$.

(7.8) LEMMA.

- (a) If $\mathbf{E}(\log^+ A_n) < \infty$ and $\mathbf{E}(\log^+ B_n) < \infty$ then every invariant Radon measure ν of the SDS satisfies $\mathbf{L} \subset \text{supp}(\nu)$.
- (b) Under the Standard Assumptions (7.4), the SDS possesses an invariant Radon measure ν with $\text{supp}(\nu) = \mathbf{L}$. Furthermore, the transition operator P is a conservative contraction of $L^1(\mathbf{X}, \nu)$ for every invariant measure ν .

Proof. (a) Let ν be invariant. This means that for every Borel set $U \subset \mathbf{X}$,

$$\nu(U) = \int_{\mathbf{X}} \tilde{\mu}(\{f \in \mathcal{L} : f(x) \in U\}) d\nu(x).$$

Therefore $f(\text{supp}(\nu)) \subset \text{supp}(\nu)$ for all $f \in \text{supp}(\tilde{\mu})$. By Proposition (7.3), the set \mathbf{L} is the smallest non-empty set with that property, and statement (a) follows.

(b) Theorem (7.6) yields conservativity. Indeed, let $\mathbf{B}(r)$ be an open ball centered at o that intersects \mathbf{L} . For every starting point $x \in \mathbf{X}$, the SDS (X_n^x) visits $\mathbf{B}(r)$ infinitely often with probability 1. We can choose $\varphi \in \mathcal{C}_c^+(\mathbf{X})$ such that $\varphi \geq 1$ on $\mathbf{B}(r)$. Then

$$\sum_{k=1}^{\infty} P^k \varphi(x) = \infty \quad \text{for every } x \in \mathbf{X}.$$

The existence of an invariant Radon measure follows once more from Lin [8, Thm. 5.1], and conservativity of P on $L^1(\mathbf{X}, \nu)$ follows (see e.g. Revuz [10, Thm. 5.3]). If right from the start we consider the whole process only on \mathbf{L} with the induced metric, then we obtain an invariant measure ν with $\text{supp}(\nu) = \mathbf{L}$. ■

Note that unless we know that the SDS is locally contractive, we cannot argue right away that every invariant measure must be supported exactly by \mathbf{L} . The Standard Assumptions (7.4) will not in general imply local contractivity, as we shall see below. Thus, the question of uniqueness of the invariant measure is more subtle. For a sufficient condition that requires a more restrictive (Harris type) notion of irreducibility, see [8, Def. 5.4 & Thm. 5.5].

8. Hyperbolic extension. In order to get closer to answering the uniqueness question in a more “topological” spirit, we also want to control the Lipschitz constants A_n . We shall need to distinguish between two cases.

A. Non-lattice case. If the random variables $\log A_n$ are non-lattice, i.e., there is no $\kappa > 0$ such that $\log A_n \in \kappa\mathbb{Z}$ almost surely, then we consider the extended SDS

$$(8.1) \quad \widehat{X}_n^{x,a} = (X_n^x, A_n A_{n-1} \cdots A_1 a)$$

on the extended space $\widehat{\mathbf{X}} = \mathbf{X} \times \mathbb{R}_*^+$, with initial point $(x, a) \in \widehat{\mathbf{X}}$. We also extend ν to a Radon measure $\lambda = \lambda_\nu$ on $\widehat{\mathbf{X}}$ by

$$(8.2) \quad \int_{\widehat{\mathbf{X}}} \varphi(x, a) d\lambda(x, a) = \int_{\mathbf{X}} \int_{\mathbb{R}} \varphi(x, e^u) d\nu(x) du.$$

This is the product of ν with the multiplicative Haar measure on \mathbb{R}_*^+ .

B. Lattice case. Otherwise, there is a maximal $\kappa > 0$ such that $\log A_n \in \kappa\mathbb{Z}$ almost surely. Then we consider again the extended SDS (8.1), but now the extended space is $\widehat{\mathbf{X}} = \mathbf{X} \times \exp(\kappa\mathbb{Z})$, where of course $\exp(\kappa\mathbb{Z}) = \{e^{\kappa m} : m \in \mathbb{Z}\}$. The initial point (x, a) now has to be such that also $a \in \exp(\kappa\mathbb{Z})$.

In this case, we define λ by

$$(8.3) \quad \int_{\widehat{\mathbf{X}}} \varphi(x, a) d\lambda(x, a) = \int \sum_{\mathbf{X} \ m \in \mathbb{Z}} \varphi(x, e^{\kappa m}) d\nu(x).$$

In both cases, it is straightforward to verify that λ is an invariant Radon measure for the extended SDS on $\widehat{\mathbf{X}}$.

Consider the hyperbolic upper half-plane $\mathbb{H} \subset \mathbb{C}$ with the Poincaré metric

$$\theta(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|},$$

where $z, w \in \mathbb{H}$ and \bar{w} is the complex conjugate of w . We use it to define a hyperbolic metric on $\widehat{\mathbf{X}}$ by

$$(8.4) \quad \begin{aligned} \hat{d}((x, a), (y, b)) &= \theta(\mathbf{i}a, d(x, y) + \mathbf{i}b) \\ &= \log \frac{\sqrt{d(x, y)^2 + (a + b)^2} + \sqrt{d(x, y)^2 + (a - b)^2}}{\sqrt{d(x, y)^2 + (a + b)^2} - \sqrt{d(x, y)^2 + (a - b)^2}}. \end{aligned}$$

It is a good exercise, using the specific properties of θ , to verify that this is indeed a metric. The metric space $(\widehat{\mathbf{X}}, \hat{d})$ is again proper, and for any $a > 0$, the embedding $\mathbf{X} \rightarrow \widehat{\mathbf{X}}, x \mapsto (x, a)$, is a homeomorphism ⁽³⁾.

(8.5) LEMMA. *Let $f : \mathbf{X} \rightarrow \mathbf{X}$ be a Lipschitz mapping with Lipschitz constant $\mathbf{l}(f) > 0$. Then the mapping $\hat{f} : \widehat{\mathbf{X}} \rightarrow \widehat{\mathbf{X}}$ defined by*

$$\hat{f}(x, a) = (f(x), \mathbf{l}(f)a)$$

is a contraction of $(\widehat{\mathbf{X}}, \hat{d})$ with Lipschitz constant 1.

Proof. By the dilation invariance of the hyperbolic metric we have

$$\begin{aligned} \tilde{d}(\hat{f}(x, a), \hat{f}(y, b)) &= \theta(\mathbf{i}\mathbf{l}(f)a, d(f(x), f(y)) + \mathbf{i}\mathbf{l}(f)b) \\ &\leq \theta(\mathbf{i}\mathbf{l}(f)a, \mathbf{l}(f)d(x, y) + \mathbf{i}\mathbf{l}(f)b) \\ &= \theta(\mathbf{i}a, d(x, y) + \mathbf{i}b) = \hat{d}((x, a), (y, b)). \end{aligned}$$

Thus, $\mathbf{l}(\hat{f}) \leq 1$. Furthermore, if $\varepsilon > 0$ and $x, y \in \mathbf{X}$ are such that $d(f(x), f(y)) \geq (1 - \varepsilon)\mathbf{l}(f)d(x, y)$ then we obtain in the same way

$$\tilde{d}(\hat{f}(x, a), \hat{f}(y, b)) \geq \theta(\mathbf{i}a, (1 - \varepsilon)d(x, y) + \mathbf{i}b).$$

When $\varepsilon \rightarrow 0$, the right hand side tends to $\hat{d}((x, a), (y, b))$. Hence $\mathbf{l}(\hat{f}) = 1$. ■

Thus, with the sequence (F_n) , we associate the sequence (\widehat{F}_n) of i.i.d. Lipschitz contractions of $\widehat{\mathbf{X}}$ with Lipschitz constants 1. The associated SDS

⁽³⁾ During the final revision of this paper, we learned that we were not the first to invent the hyperbolic extension. See Kaimanovich [7, Prop. 3.20]. (His context is very different.)

on $\widehat{\mathbf{X}}$ is $(\widehat{X}_n^{x,a})$, as defined in (8.1). From [9, Lemma (2.2)], which is true for any SDS of contractions, we get the following, where $o \in \mathbf{X}$ and $\hat{o} = (o, 1)$.

(8.6) COROLLARY. $\Pr[\hat{d}(\widehat{X}_n^{x,a}, \hat{o}) \rightarrow \infty] \in \{0, 1\}$, and the value is the same for all $(x, a) \in \widehat{\mathbf{X}}$.

We shall now study separately the cases when the extended SDS is transient (the probability in Corollary (8.6) is 1), or conservative (that probability is 0) in order to deduce the results that we are aiming at, concerning uniqueness of the invariant Radon measure and ergodicity.

9. Transient extended SDS. We first consider the situation when $(\widehat{X}_n^{x,a})$ is transient. We shall use once more the comparison (7.1) of (X_n^x) with the affine stochastic recursion $(Y_n^{|x|})$. Recall that $|x| = d(o, x)$ and $B_n \geq 0$. The hyperbolic extension $(\widehat{Y}_n^{|x|,a})$ of $(Y_n^{|x|})$ is a random walk on the hyperbolic upper half-plane. It can also be viewed as a random walk on the affine group of all mappings $g_{a,b}(z) = az + b$. Under the non-degeneracy assumptions of (7.4)(i), this random walk is well-known to be transient.

(9.1) LEMMA. *Assume that the Standard Assumptions (7.4) hold. Then for all $r > 0$ and $s > 1$ which are sufficiently large there are $\alpha = \alpha_{r,s}$ and $\delta = \delta_{r,s} > 0$ such that, setting $K_{r,s} = [0, r] \times [1/s, s]$ and $Q_{r,\alpha} = [0, r] \times [\alpha, \infty)$, the affine recursion satisfies*

$$\Pr[\widehat{Y}_n^{y,a} \in K_{r,s} \text{ for some } n \geq 1] \geq \delta \quad \text{for all } (y, a) \in Q_{r,\alpha}.$$

Proof. In this proof only, we write ν for the invariant Radon measure associated with $(Y_n^{|x|})$. Its existence is guaranteed by [1] and [4]; see [9, Proposition (3.3)]. Let $\lambda = \lambda_\nu$ be its hyperbolic extension according to (8.2), resp. (8.3). We normalize ν , and consequently λ , so that ν is the measure which is denoted $m(f)$ in [1, p. 482].

The random walk $(\widehat{Y}_n^{y,a})$ on the affine group (parametrized by $\mathbb{R}_*^+ \times \mathbb{R}$) evolves on $\mathbb{R}_*^+ \times \mathbb{R}^+$, when $y \geq 0$. By [1], its potential kernel

$$\mathcal{U}\varphi(y, a) = \sum_{n=0}^{\infty} \mathbb{E}(\varphi(\widehat{Y}_n^{y,a})), \quad \varphi \in \mathcal{C}_c(\mathbb{R}_*^+ \times \mathbb{R}^+),$$

is finite and weakly compact as a family of Radon measures that are parametrized by (y, a) . Furthermore [1, Thm. 2.2],

$$\lim_{a \rightarrow \infty} \mathcal{U}\varphi(y, a) = \int \varphi d\lambda,$$

and convergence is uniform when y remains in a compact set. We fix $r > 1$ large enough so that $\nu([0, r']) > 0$, where $r' = r - 1$, and let $s > 1$ in the non-lattice case, resp. $s \geq 2e^\kappa$ in the lattice case. We set $s' = (s + 1)/2$ and $c_{r,s} = \lambda(K_{r',s'})/2$, which is strictly positive, and choose $\varphi \in \mathcal{C}_c^+(\mathbb{R}_*^+ \times \mathbb{R}^+)$

so that $\mathbf{1}_{K_{r',s'}} \leq \varphi \leq \mathbf{1}_{K_{r,s}}$. By the above, there is $\alpha = \alpha_{r,s} > 0$ such that $\mathcal{U}\varphi(y, a) \geq c_{r,s}$ for all $(y, a) \in Q_{r,\alpha}$. Given any starting point (y, a) , let

$$\tau = \inf\{n \geq 1 : \widehat{Y}_n^{y,a} \in K_{r,s}\}.$$

We know that

$$M_{r,s} = \sup \mathcal{U}\mathbf{1}_{K_{r,s}} < \infty.$$

Let $(y, a) \in Q_{r,\alpha}$. Just for the purpose of this proof, we consider the hitting distribution $\sigma_{(y,a)}$ on $K_{r,s}$ defined by $\sigma_{(y,a)}(B) = \Pr[\tau < \infty, \widehat{Y}_\tau^{y,a} \in B]$. Then by the Markov property,

$$\begin{aligned} \mathcal{U}\mathbf{1}_{K_{r,s}}(y, a) &= \mathbb{E}\left(\sum_{n=0}^{\infty} \mathbf{1}_{K_{r,s}}(\widehat{Y}_n^{y,a})\right) = \mathbb{E}\left(\mathbf{1}_{[\tau < \infty]} \sum_{n=\tau}^{\infty} \mathbf{1}_{K_{r,s}}(\widehat{Y}_n^{y,a})\right) \\ &= \int_{K_{r,s}} \mathbb{E}\left(\sum_{n=0}^{\infty} \mathbf{1}_{K_{r,s}}(\widehat{Y}_n^{z,b})\right) d\sigma_{(y,a)}(z, b) \\ &\leq M_{r,s}\sigma_{(y,a)}(K_{r,s}) = M_{r,s}\Pr_{(y,a)}[\tau < \infty], \end{aligned}$$

where the index (y, a) indicates the starting point. Therefore we can set $\delta = c_{r,s}/M_{r,s}$, and $\Pr_{(y,a)}[\tau < \infty] \geq \delta$ for all $(y, a) \in Q_{r,\alpha}$. ■

Let $\overline{B}(r)$ be the closed ball in X with center o and radius r . Set $B_{r,s} = \overline{B}(r) \times [1/s, s]$ and $C_{r,\alpha} = \overline{B}(r) \times [\alpha, \infty)$.

(9.2) LEMMA. *Assume that the Standard Assumptions (7.4) hold and that $(\widehat{X}_n^{x,a})$ is transient. Then for every sufficiently large $r > 0$, there is $\alpha > 0$ such that*

$$\Pr[\widehat{X}_n^{x,a} \in C_{r,\alpha} \text{ for infinitely many } n] = 0 \quad \text{for all } (x, a) \in \widehat{X}.$$

Proof. Let

$$\Lambda = \Lambda^{x,a} = \{\omega \in \Omega : \widehat{X}_n^{x,a}(\omega) \in C_{r,\alpha} \text{ for infinitely many } n\}.$$

Given r and s so large that Lemma (9.1) applies, let $\alpha, \delta > 0$ be as in that lemma. For each $(c, a) \in Q_{r,\alpha}$ there is an $N_{c,a} \in \mathbb{N}$ such that

$$(9.3) \quad \Pr[\widehat{Y}_n^{y,a} \in K_{r,s} \text{ for some } n \text{ with } 1 \leq n \leq N_{c,a}] \geq \delta/2.$$

If $(c, a) \notin Q_{r,\alpha}$ then we set $N_{c,a} = 0$. Since $B_{r,s}$ is compact, the transience assumption yields $\Pr(\bigcup_{j=2}^{\infty} \Omega_j) = 1$, where

$$\Omega_j = \Omega_j^{x,a} = \{\omega \in \Omega : \widehat{X}_n^{x,a}(\omega) \notin B_{r,s} \text{ for every } n \geq j\}.$$

Thus, we need to show that $\Pr(\Lambda \cap \Omega_j) = 0$ for every $j \geq 2$. We define a sequence of stopping times $\tau_k = \tau_k^{x,a}$ and (when $\tau_k < \infty$) associated pairs

$(x_k, a_k) = \widehat{X}_{\tau_k}^{x,a}$ by

$$\begin{aligned} \tau_1 &= \inf\{n > N_{|x|,a} : \widehat{X}_n^{x,a} \in \mathbb{C}_{r,\alpha}\}, \\ \tau_{k+1} &= \begin{cases} \inf\{n > \tau_k + N_{|x_k|,a_k} : \widehat{X}_n^{x,a} \in \mathbb{C}_{r,\alpha}\} & \text{if } \tau_k < \infty, \\ \infty & \text{if } \tau_k = \infty. \end{cases} \end{aligned}$$

Unless explained separately, we always use $\tau_k = \tau_k^{x,a}$. Note that $\omega \in \Lambda$ if and only if $\tau_k(\omega) < \infty$ for all k . Therefore

$$\Lambda \cap \Omega_j = \bigcap_{k \geq j} \Lambda_{j,k}, \quad \Lambda_{j,k} = [\tau_k < \infty, \widehat{X}_n^{x,a} \notin \mathbb{B}_{r,s} \text{ for all } n \text{ with } j \leq n \leq \tau_k].$$

We have $\Lambda_{j,k} \subset \Lambda_{j,k-1}$. Next, note that

$$\text{if } \widehat{X}_n^{x,a}(\omega) \notin \mathbb{B}_{r,s} \text{ then } \widehat{Y}_n^{|x|,a}(\omega) \notin K_{r,s}.$$

This follows from (7.1).

We have $\widehat{X}_{\tau_{k-1}}^{x,a} \in \mathbb{C}_{r,\alpha}$ for $k \geq 2$. Just for the purpose of the next few lines of the proof, we introduce the measure σ on $\mathbb{C}_{r,\alpha}$ given by $\sigma(\widehat{B}) = \Pr(\Lambda_{j,k-1} \cap [\widehat{X}_{\tau_{k-1}}^{x,a} \in \widehat{B}])$, where $\widehat{B} \subset \mathbb{C}_{r,\alpha}$ is a Borel set. Then, using the strong Markov property and (9.3),

$$\begin{aligned} \Pr(\Lambda_{j,k}) &= \Pr([\tau_k < \infty, \widehat{X}_n^{x,a} \notin \mathbb{B}_{r,s} \text{ for all } n \text{ with } \tau_{k-1} < n \leq \tau_k] \cap \Lambda_{j,k-1}) \\ &= \int_{\mathbb{C}_{r,\alpha}} \Pr[\tau_1^{y,b} < \infty, \widehat{X}_n^{y,b} \notin \mathbb{B}_{r,s} \text{ for all } n \text{ with } 0 < n \leq \tau_1^{y,b}] d\sigma(y, b) \\ &\leq \int_{\mathbb{C}_{r,\alpha}} \Pr[\tau_1^{y,b} < \infty, \widehat{Y}_n^{|y|,b} \notin K_{r,s} \text{ for all } n \text{ with } 0 < n \leq N_{|y|,b}] d\sigma(y, b) \\ &\leq \int_{\mathbb{C}_{r,\alpha}} (1 - \delta/2) d\sigma(y, b) = (1 - \delta/2)\Pr(\Lambda_{j,k-1}). \end{aligned}$$

We continue recursively downwards until we reach $k = 2$ (since $k = 1$ is excluded unless $(x, a) \in \mathbb{C}_{r,\alpha}$). Thus, $\Pr(\Lambda_{j,k}) \leq (1 - \delta/2)^{k-1}$, and as $k \rightarrow \infty$, we get $\Pr(\Lambda \cap \Omega_j) = 0$, as required. ■

(9.4) THEOREM. *Given the random i.i.d. Lipschitz mappings F_n , let A_n and B_n be as in (6.2). Suppose that the Standard Assumptions (7.4) hold, and that $\Pr[\hat{d}(\widehat{X}_n^{x,a}, \hat{\delta}) \rightarrow \infty] = 1$. Then the SDS induced by the F_n on \mathbb{X} is locally contractive.*

In particular, it has an invariant Radon measure ν that is unique up to multiplication with constants.

Also, the shift T on $(\mathbb{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathbb{X}^{\mathbb{N}_0}), \Pr_\nu)$ is ergodic, where \Pr_ν is the measure on $\mathbb{X}^{\mathbb{N}_0}$ associated with ν .

Proof. Fix any starting point (x, a) of the extended SDS. Let r be sufficiently large so that the last two lemmas apply, and such that

$$\Pr[X_n^x \in \bar{\mathbf{B}}(r) \text{ for infinitely many } n] = 1.$$

We claim that

$$(9.5) \quad \lim_{n \rightarrow \infty} A_{0,n} \mathbf{1}_{\bar{\mathbf{B}}(r)}(X_n^x) = 0 \quad \text{almost surely.}$$

We consider α associated with r as in Lemma (9.2). Then we choose an arbitrary $s \geq \alpha$. We know by transience of the extended SDS that

$$\Pr[\hat{X}_n^{x,a} \in \mathbf{B}_{r,s} \text{ for infinitely many } n] = 0.$$

We combine this with Lemma (9.2) to get

$$\Pr[\hat{X}_n^{x,a} \in \mathbf{B}_{r,s} \cup \mathbf{C}_{r,\alpha} \text{ for infinitely many } n] = 0.$$

Since $s \geq \alpha$, we have $\mathbf{B}_{r,s} \cup \mathbf{C}_{r,\alpha} = \bar{\mathbf{B}}(r) \times [1/s, \infty)$.

Thus, if $\mathbb{N}(x, r)$ denotes the a.s. infinite random set of all n for which $X_n^x \in \bar{\mathbf{B}}(r)$, then for all but finitely many $n \in \mathbb{N}(x, r)$, we have $A_{0,n} < 1/s$. This holds for every $s > \alpha$, and we have proved (9.5). We conclude that

$$d(X_n^x, X_n^y) \mathbf{1}_{\bar{\mathbf{B}}(r)}(X_n^x) \leq A_{0,n} d(x, y) \mathbf{1}_{\bar{\mathbf{B}}(r)}(X_n^x) \rightarrow 0 \quad \text{almost surely.}$$

Now that we have local contractivity, the remaining statements follow from [9, Theorem (2.13)]. ■

10. Conservative extended SDS. Now we assume that we are in the conservative case, i.e., the probability in Corollary (8.6) is 0. We start with an invariant measure ν for the SDS on \mathbf{X} . If the Standard Assumptions (7.4) hold, the existence of ν is guaranteed by Lemma (7.8). Then we extend ν to a measure $\lambda = \lambda_\nu$ on $\hat{\mathbf{X}}$ of (8.2), resp. (8.3).

We can realize the extended SDS, starting at $(x, a) \in \hat{\mathbf{X}}$, on the space

$$(\hat{\mathbf{X}}^{\mathbb{N}_0}, \mathfrak{B}(\hat{\mathbf{X}}^{\mathbb{N}_0}), \Pr_{x,a}),$$

where $\mathfrak{B}(\hat{\mathbf{X}}^{\mathbb{N}_0})$ is the product Borel σ -algebra, and $\Pr_{x,a}$ is the image of the measure \Pr under the mapping

$$\Omega \rightarrow \hat{\mathbf{X}}^{\mathbb{N}_0}, \quad \omega \mapsto (\hat{X}_n^{x,a}(\omega))_{n \geq 0}.$$

Then we consider the Radon measure on $\hat{\mathbf{X}}^{\mathbb{N}_0}$ defined by

$$\Pr_\lambda = \int_{\hat{\mathbf{X}}} \Pr_{x,a} d\lambda(x, a).$$

The integral with respect to \Pr_λ is denoted \mathbf{E}_λ . We write \hat{T} for the time shift on $\hat{\mathbf{X}}^{\mathbb{N}_0}$. Since λ is invariant for the extended SDS, \hat{T} is a contraction of $L^1(\hat{\mathbf{X}}^{\mathbb{N}_0}, \Pr_\lambda)$. Also, in this section, \mathfrak{I} stands for the σ -algebra of \hat{T} -invariant sets in $\mathfrak{B}(\hat{\mathbf{X}}^{\mathbb{N}_0})$. As before, any function $\varphi : \hat{\mathbf{X}}^\ell \rightarrow \mathbb{R}$ can be extended to $\hat{\mathbf{X}}^{\mathbb{N}_0}$

by setting $\varphi(\mathbf{x}, \mathbf{a}) = \varphi((x_0, a_0), \dots, (x_{\ell-1}, a_{\ell-1}))$ if $(\mathbf{x}, \mathbf{a}) = ((x_n, a_n))_{n \geq 0}$. In analogy with [9, (2.3)], we define the extended SDS starting in (x, a) at time m

$$\widehat{X}_{m,n}^{x,a} = (X_{m,n}^x, A_{m,n}a) \quad (n \geq m).$$

We now set, for $n \geq m$ and $\varphi : \widehat{X}^{\mathbb{N}_0} \rightarrow \mathbb{R}$,

$$S_{m,n}^{x,a} \varphi(\omega) = \sum_{k=m}^n \varphi((\widehat{X}_{m,k}^{x,a}(\omega))_{k \geq m})$$

and in particular $S_n^{x,a} \varphi(\omega) = S_{0,n}^{x,a} \varphi(\omega)$. Consider the sets

$$(10.1) \quad \Omega_r = \{\omega \in \Omega : \liminf \widehat{d}(\widehat{X}_n^{\hat{o}}(\omega), \hat{o}) \leq r\} \quad (r \in \mathbb{N}), \quad \Omega_\infty = \bigcup_r \Omega_r.$$

By our assumption of conservativity, $\Pr(\Omega_\infty) = 1$. For $r \in \mathbb{N}$, write $\widehat{B}(r)$ for the *closed* ball in $(\widehat{X}, \widehat{d})$ with center \hat{o} and radius r . Then for every $\omega \in \Omega_r$ and $s \in \mathbb{N}_0$, the set $\{n : \widehat{X}_n^{x,a}(\omega) \in \widehat{B}(r+s) \text{ for all } (x, a) \in \widehat{B}(s)\}$ is infinite. For each r , set $\psi_r(x, a) = \max\{1 - \widehat{d}((x, a), \widehat{B}(r)), 0\}$. Then $\psi_r \in C_c^+(\widehat{X})$ satisfies

$$(10.2) \quad \begin{aligned} \mathbf{1}_{\widehat{B}(r+1)} &\geq \psi_r \geq \mathbf{1}_{\widehat{B}(r)}, \\ |\psi_r(x, a) - \psi_r(y, b)| &\leq \widehat{d}((x, a), (y, b)) \quad \text{on } \widehat{X}, \\ S_n^{x,a} \psi_{r+s}(\omega) &\rightarrow \infty \quad \text{for all } \omega \in \Omega_r, (x, a) \in \widehat{B}(s). \end{aligned}$$

We now replace ψ_r by a continuous and strictly positive function Ψ on \widehat{X} in such a way that

$$(10.3) \quad \sum_n \Psi(\widehat{X}_n^{x,a}(\omega)) = \infty \quad \text{for all } \omega \in \Omega_\infty \text{ and } (x, a) \in \widehat{X}.$$

Indeed, we can find a decreasing sequence of numbers $c_r > 0$ such that $\sum_r c_r \max \psi_{r+2} < \infty$ and the functions

$$(10.4) \quad \Phi = \sum_r c_r \psi_{r+2} \quad \text{and} \quad \Psi = \sum_r c_r \psi_r$$

are in $L^1(\widehat{X}, \lambda)$ and thus their extensions to $\widehat{X}^{\mathbb{N}_0}$ are in $L^1(\widehat{X}^{\mathbb{N}_0}, \Pr_\lambda)$. Both Φ and Ψ will be used several times as reference functions in applications of the Chacon–Ornstein theorem. By construction, (10.3) holds. We have obtained the following, which justifies calling the non-transient case “conservative”.

(10.5) LEMMA. *When the extended SDS is conservative, the shift \widehat{T} is conservative.*

Next, for any $\varphi \in L^1(\widehat{X}^{\mathbb{N}_0}, \Pr_\lambda)$, consider the function

$$\mathbf{v}_\varphi = E_\lambda(\varphi | \mathfrak{I}) / E_\lambda(\Psi | \mathfrak{I})$$

on $\widehat{X}^{\mathbb{N}_0}$. A priori, the quotient of conditional expectations is defined only \Pr_λ -almost everywhere, and we consider a representative which is always finite. We turn this into the family of finite positive random variables

$$V_\varphi^{x,a}(\omega) = \mathbf{v}_\varphi((\widehat{X}_n^{x,a}(\omega))_{n \geq 0}), \quad (x, a) \in \widehat{X}.$$

(10.6) LEMMA. *In the conservative case, let $\tau : \Omega \rightarrow \mathbb{N}$ be any a.s. finite random time. Then, on the set where $\tau(\omega) < \infty$, for every $\varphi \in L^1(\widehat{X}^{\mathbb{N}_0}, \Pr_\lambda)$,*

$$\lim_{n \rightarrow \infty} \frac{S_n^{x,a} \varphi - S_\tau^{x,a} \varphi}{S_n^{x,a} \Psi - S_\tau^{x,a} \Psi} = V_\varphi^{x,a} \text{ Pr-almost surely, for } \lambda\text{-almost every } (x, a) \in \widehat{X}.$$

Proof. We know that $S_n^{x,a} \Psi(\omega) \rightarrow \infty$ for all $\omega \in \Omega_\infty$. By the ergodic theorem of Chacon and Ornstein [5] (see [10]), $S_n^{x,a} \varphi / S_n^{x,a} \Psi \rightarrow V_\varphi^{x,a}$ almost surely on Ω_∞ , for λ -almost every $(x, a) \in \widehat{X}$. Furthermore, both $S_\tau^{x,a} \varphi / S_n^{x,a} \Psi$ and $S_\tau^{x,a} \Psi / S_n^{x,a} \Psi$ tend to 0 on Ω_∞ as $n \rightarrow \infty$. When $n > \tau$,

$$\frac{S_n^{x,a} \varphi}{S_n^{x,a} \Psi} = \underbrace{\frac{S_\tau^{x,a} \varphi}{S_n^{x,a} \Psi}}_{\rightarrow 0 \text{ a.s.}} + \left(1 - \underbrace{\frac{S_\tau^{x,a} \Psi}{S_n^{x,a} \Psi}}_{\rightarrow 0 \text{ a.s.}} \right) \frac{S_n^{x,a} \varphi - S_\tau^{x,a} \varphi}{S_n^{x,a} \Psi - S_\tau^{x,a} \Psi}.$$

The statement follows. ■

When the extended SDS is conservative, we do not see how to involve local contractivity, but we can provide a reasonable additional assumption which will yield uniqueness of the invariant Radon measure. We set

$$(10.7) \quad D_n(x, y) = \frac{d(X_n^x, X_n^y)}{A_1 \cdots A_n}.$$

(Compare with the proof of [9, Theorem (4.2)], which corresponds to $A_n \equiv 1$.) The assumption is

$$(10.8) \quad \Pr[D_n(x, y) \rightarrow 0] = 1 \quad \text{for all } x, y \in X.$$

(10.9) REMARK. If we set $D_{m,n}(x, y) = d(X_{m,n}^x, X_{m,n}^y) / A_{m,n}$ then (10.8) implies that

$$\Pr \left[\lim_{n \rightarrow \infty} D_{m,n}(x, y) = 0 \text{ for all } x, y \in X, m \in \mathbb{N} \right] = 1.$$

Indeed, let X_0 be a countable, dense subset of X . Then (10.8) implies that

$$\Pr \left[\lim_{n \rightarrow \infty} D_{m,n}(x, y) = 0 \text{ for all } x, y \in X_0, m \in \mathbb{N} \right] = 1.$$

Let Ω_0 be the subset of Ω_∞ where this holds.

Note that $D_{m,n}(x, y) \leq d(x, y)$. Given arbitrary $x, y \in X$ and $x_0, y_0 \in X_0$, we get on Ω_0

$$D_{m,n}(x, y) \leq D_{m,n}(x_0, y_0) + d(x, x_0) + d(y, y_0),$$

and the statement follows. ■

In the next lemma, we provide a condition for (10.8) to hold. It will be useful in §11.

(10.10) LEMMA. *In the case when the extended SDS is conservative, suppose that for every $\varepsilon > 0$ and $r \in \mathbb{N}$ there is k such that $\Pr[D_k(x, y) < \varepsilon$ for all $x, y \in \mathbf{B}(r)] > 0$. Then (10.8) holds.*

Proof. We set $D_\infty(x, y) = \lim_n D_n(x, y)$ and $w(x, y) = \mathbb{E}(D_\infty(x, y))$. A straightforward adaptation of the argument used in the proof of [9, Theorem (4.2)] yields

$$(10.11) \quad \lim_{m \rightarrow \infty} \frac{w(X_m^x, X_m^y)}{A_1 \cdots A_m} = D_\infty(x, y) \quad \text{almost surely.}$$

Again, we claim that $\Pr[D_\infty(x, y) \geq \varepsilon] = 0$. By conservativity, it is sufficient to show that $\Pr(A_r) = 0$ for every $r \in \mathbb{N}$, where

$$A_r = \bigcap_{m \geq k} \bigcup_{n \geq m} [\widehat{X}_n^x, \widehat{X}_n^y \in \mathbf{B}(r) \times [1/r, r], D_n(x, y) \geq \varepsilon].$$

By assumption, there is k such that the event $\Gamma_{k,r} = [D_k(x, y) < \varepsilon/2$ for all $x, y \in \mathbf{B}(r)]$ satisfies $\Pr(\Gamma_{k,r}) > 0$.

We now continue as in the proof of [9, Theorem (4.2)], and find that for all $u, v \in \mathbf{B}(r)$ with $d(u, v) \geq \varepsilon$,

$$w(u, v) \leq d(u, v) - \delta, \quad \text{where } \delta = \Pr(\Gamma_{k,r}) \cdot (\varepsilon/2) > 0.$$

This implies that on A_r , almost surely we have infinitely many $n \geq k$ for which $w(X_n^x, X_n^y) \leq d(X_n^x, X_n^y) - \delta$ and $A_1 \cdots A_n \leq r$, that is,

$$\frac{w(X_n^x, X_n^y)}{A_1 \cdots A_n} \leq D_n(x, y) - \frac{\delta}{r} \quad \text{infinitely often.}$$

Letting $n \rightarrow \infty$, we get $D_\infty(x, y) < D_\infty(x, y)$ almost surely on A_r , so that indeed $\Pr(A_r) = 0$. ■

We now elaborate the main technical prerequisite to handle the case when the extended SDS is conservative. Some care is advisable in order to have a clear picture regarding the dependencies of sets on which various “almost everywhere” statements hold. Let $\varphi \in L^1(\widehat{\mathbf{X}}^{\mathbb{N}_0}, \Pr_\lambda)$. Let Ω_0 be as in Remark (10.9). For λ -almost every $(x, a) \in \widehat{\mathbf{X}}$, there is a set $\Omega_\varphi^{x,a} \subset \Omega_0$ with $\Pr(\Omega_\varphi^{x,a}) = 1$ such that

$$\frac{S_n^{x,a} \varphi(\omega)}{S_n^{x,a} \Psi(\omega)} \rightarrow V_\varphi^{x,a}(\omega)$$

for every $\omega \in \Omega_\varphi^{x,a}$. For the remaining $(x, a) \in \widehat{\mathbf{X}}$, we set $\Omega_\varphi^{x,a} = \emptyset$.

(10.12) PROPOSITION. *In the case when the extended SDS is conservative, assume (10.8). Let $\varphi \in C_c^+(\widehat{\mathbf{X}}^\ell)$ with $\ell \geq 1$. Then for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \varphi) > 0$ with the following property.*

For all $(x, a), (y, b) \in \widehat{X}$ and any a.s. finite random time $\tau : \Omega \rightarrow \mathbb{N}_0 \cup \{\infty\}$, on the set of all $\omega \in \Omega_{\Phi}^{x,a}$ with $\tau(\omega) < \infty$ and $|\log(A_{0,\tau}(\omega)a/b)| < \delta$ one has

$$\limsup_{n \rightarrow \infty} \left| \frac{S_n^{x,a} \varphi}{S_n^{x,a} \Psi} - \frac{S_{\tau,n}^{y,b} \varphi}{S_{\tau,n}^{y,b} \Psi} \right| \leq \varepsilon W^{x,a},$$

where $W^{x,a} = V_{\Phi}^{x,a} + 1$.

Proof. Recall that Φ, Ψ, φ and ψ_r are also considered as functions on $\widehat{X}^{\mathbb{N}_0}$ via their extensions defined above.

Since Ψ is continuous and > 0 , there is $C = C_{\varphi} > 0$ such that $\varphi \leq C\Psi$. Also, there is some $r_0 \in \mathbb{N}$ such that the projection of $\text{supp}(\varphi)$ onto the first coordinate in \widehat{X} (i.e., the one with index 0) is contained in $\widehat{B}(r_0)$. We let $\varepsilon' = \min\{\varepsilon/2, \varepsilon/(2C), c_{r_0+1}\varepsilon/2, 1\}$, where c_{r_0+1} comes from the definition (10.4) of Φ and Ψ . Since φ is uniformly continuous, there is $\delta > 0$ with $2\delta \leq \varepsilon'$ such that

$$\begin{aligned} |\varphi((x_0, a_0), \dots, (x_{\ell-1}, a_{\ell-1})) - \varphi((y_0, b_0), \dots, (y_{\ell-1}, b_{\ell-1}))| &\leq \varepsilon' \\ \text{whenever } \hat{d}((x_j, a_j), (y_j, b_j)) &< 2\delta, j = 0, \dots, \ell - 1. \end{aligned}$$

We write

$$\left| \frac{S_n^{x,a} \varphi}{S_n^{x,a} \Psi} - \frac{S_{\tau,n}^{y,b} \varphi}{S_{\tau,n}^{y,b} \Psi} \right| \leq \underbrace{\frac{|S_n^{x,a} \varphi - S_{\tau,n}^{y,b} \varphi|}{S_n^{x,a} \Psi}}_{\text{Term 1}} + \underbrace{\frac{S_{\tau,n}^{y,b} \varphi}{S_{\tau,n}^{y,b} \Psi}}_{\leq C_{\varphi}} \underbrace{\frac{|S_n^{x,a} \Psi - S_{\tau,n}^{y,b} \Psi|}{S_n^{x,a} \Psi}}_{\text{Term 2}}.$$

We consider the random element $z = X_{\tau}^x$, so that $X_n^x = X_{\tau,n}^z$. Using the dilation invariance of hyperbolic metric, we have

$$\begin{aligned} \hat{d}(\widehat{X}_n^{x,a}, \widehat{X}_{\tau,n}^{y,b}) &= \theta(\mathbf{i}A_{0,n}a, d(X_{\tau,n}^z, X_{\tau,n}^y) + \mathbf{i}A_{\tau,n}b) \\ &= \theta(\mathbf{i}A_{0,\tau}a, D_{\tau,n}(z, y) + \mathbf{i}b) \\ &\leq |\log(A_{0,\tau}a/b)| + \theta(\mathbf{i}b, D_{\tau,n}(z, y) + \mathbf{i}b). \end{aligned}$$

By (10.8), for $\omega \in \Omega_{\Phi}^{x,a}$ with $\tau(\omega) < \infty$ there is a finite $\sigma(\omega) \geq \tau(\omega)$ in \mathbb{N} such that $\theta(\mathbf{i}b, D_{\tau,n}(z, y) + \mathbf{i}b) < \delta$ for all $n \geq \sigma(\omega)$. We will assume that our $\omega \in \Omega_{\Phi}^{x,a}$ also satisfies $|\log(A_{0,\tau}(\omega)a/b)| < \delta$.

Now, we first bound the lim sup of Term 1 by $\varepsilon/2$. If $n \geq \sigma$ and $|A_{0,\tau}(\omega)a/b| < \delta$, then we obtain

$$|\varphi(\widehat{X}_n^{x,a}, \widehat{X}_{n+1}^{x,a}, \dots, \widehat{X}_{n+\ell-1}^{x,a}) - \varphi(\widehat{X}_{\tau,n}^{y,b}, \widehat{X}_{\tau,n+1}^{y,b}, \dots, \widehat{X}_{\tau,n+\ell-1}^{y,b})| < \varepsilon' \leq \varepsilon/2.$$

Suppose moreover that at least one of $\varphi(\widehat{X}_n^{x,a}, \widehat{X}_{n+1}^{x,a}, \dots, \widehat{X}_{n+\ell-1}^{x,a})$ or $\varphi(\widehat{X}_{\tau,n}^{y,b}, \widehat{X}_{\tau,n+1}^{y,b}, \dots, \widehat{X}_{\tau,n+\ell-1}^{y,b})$ is positive. Then $\widehat{X}_n^{x,a}$ or $\widehat{X}_{\tau,n}^{y,b}$ belongs to $\widehat{B}(r_0)$,

and by the above (since $\delta < 1$) both belong to $\widehat{\mathbf{B}}(r_0 + 1)$. Thus, for $n \geq \sigma$,

$$\begin{aligned} |\varphi(\widehat{X}_n^{x,a}, \widehat{X}_{n+1}^{x,a}, \dots, \widehat{X}_{n+\ell-1}^{x,a}) - \varphi(\widehat{X}_{\tau,n}^{y,b}, \widehat{X}_{\tau,n+1}^{y,b}, \dots, \widehat{X}_{\tau,n+\ell-1}^{y,b})| \\ \leq \varepsilon' \psi_{r_0+1}(\widehat{X}_n^{x,a}) \leq (\varepsilon/2) \Psi(\widehat{X}_n^{x,a}). \end{aligned}$$

We get

$$\frac{|(S_n^{x,a} \varphi - S_\sigma^{x,a} \varphi) - (S_{\tau,n}^{y,b} \varphi - S_{\tau,\sigma}^{y,b} \varphi)|}{S_n^{x,a} \Psi - S_\sigma^{x,a} \Psi} \leq \varepsilon/2.$$

Since $S_n^{x,a} \Psi \rightarrow \infty$ almost surely, when passing to the limsup, we can omit all terms in the last inequality that contain a σ ; see Lemma (10.6). This yields the bound on the limsup of Term 1.

Next, we bound the limsup of Term 2 by $\varepsilon/2$. We start in the same way as above, replacing φ with any of the functions ψ_r and replacing ℓ with 1. Using the specific properties (10.2) of ψ_r (in particular, Lipschitz continuity with constant 1), and replacing $\widehat{\mathbf{B}}(r_0)$ with $\widehat{\mathbf{B}}(r + 1) = \text{supp}(\psi_r)$, we arrive at the inequality

$$|\psi_r(\widehat{X}_n^{x,a}) - \psi_r(\widehat{X}_{\tau,n}^{y,b})| \leq \frac{\varepsilon}{2C} \psi_{r+2}(\widehat{X}_n^{x,a}).$$

It holds for all $n \geq \sigma$, with probability 1. We deduce

$$|\Psi(\widehat{X}_n^{x,a}) - \Psi(\widehat{X}_{\tau,n}^{y,b})| \leq \frac{\varepsilon}{2C} \Phi(\widehat{X}_n^{x,a})$$

and

$$\frac{|(S_n^{x,a} \Psi - S_\sigma^{x,a} \Psi) - (S_{\tau,n}^{y,b} \Psi - S_{\tau,\sigma}^{y,b} \Psi)|}{S_n^{x,a} \Psi - S_\sigma^{x,a} \Psi} \leq \frac{\varepsilon}{2C} \frac{S_n^{x,a} \Phi - S_\sigma^{x,a} \Phi}{S_n^{x,a} \Psi - S_\sigma^{x,a} \Psi}.$$

Passing to the lim sup as above, and using the Chacon–Ornstein theorem, we see that the limsup of Term 2 is bounded almost surely by $\frac{\varepsilon}{2C} V_\Phi^{x,a}$ on $\Omega_\Phi^{x,a}$. ■

Below, when we sloppily say “for almost every $a > 0$ ”, we shall mean “for Lebesgue-almost every $a > 0$ ” in the non-lattice case, resp. “for every $a = e^{-\kappa m}$ ($m \in \mathbb{Z}$)” in the lattice case.

(10.13) COROLLARY. *Let $\varphi \in \mathcal{C}_c^+(\widehat{\mathbf{X}}^\ell)$ as above. For almost every $a > 0$, there is a set $\Omega_\varphi^a \subset \Omega_0$ with $\Pr(\Omega_\varphi^a) = 1$ such that for all $x, y \in \mathbf{X}$,*

$$V_\varphi^{x,a} = V_\varphi^{y,a} =: V_\varphi^a.$$

Proof. For almost every a , there is at least one $x_a \in \mathbf{X}$ such that $\Pr(\Omega_\varphi^{x_a,a}) = 1$. We can apply Proposition (10.12) with arbitrary $y \in \mathbf{X}$, $b = a$ and $\tau = 0$. Then we are allowed to take any $\varepsilon > 0$ and get $V_\varphi^{x_a,a} = V_\varphi^{y,a}$ on $\Omega_\varphi^{x_a,a} \cap \Omega_\varphi^{x_a,a}$. ■

(10.14) PROPOSITION. *Suppose that the Standard Assumptions (7.4) as well as (10.8) hold, and that the extended SDS is conservative. Let $\varphi \in$*

$\mathcal{C}_c^+(\widehat{X}^\ell)$ as above. Then for almost every $a > 0$, the random variable V_φ^a is almost surely constant (depending on φ and—so far—on a).

Proof. Let a be such that $\Pr(\Omega_\varphi^a) = 1$, and choose $x = x_a$ as in the proof of Corollary (10.13).

For $s \in \mathbb{N}$, let $\varepsilon_s = 1/s$ and $\delta_s = \delta(\varepsilon_s, \varphi)$ according to Proposition (10.12). By our assumptions, $(A_{0,n})_{n \geq 1}$ is a topologically recurrent random walk on \mathbb{R}_*^+ , starting at 1. Choose $m \in \mathbb{N}$ and let $\tau_{m,s}$ be the m th return time to the interval $(e^{-\delta_s}, e^{\delta_s})$. For every m and s , this is an almost surely finite stopping time, and we can find $\bar{\Omega}_\varphi^a \subset \Omega_\varphi^a \cap \Omega_\Phi^{x,a}$ with $\Pr(\bar{\Omega}_\varphi^a) = 1$ such that all $\tau_{m,s}$ are finite on that set.

We now apply Proposition (10.12) with $(y, b) = (x, a)$ and $\tau = \tau_{m,s}$. Then

$$\limsup_{n \rightarrow \infty} \left| V^a \varphi - \underbrace{\frac{S_{\tau,n}^{x,a} \varphi}{S_{\tau,n}^{x,a} \Psi}}_{=: U_{n,m,s}} \right| \leq \frac{1}{s} W^{x,a}.$$

Since our stopping time satisfies $\tau \geq m$, the random variable $U_{n,m,s}$ (depending also on φ and (x, a)) is independent of the basic random mappings F_1, \dots, F_m . (Recall that the F_k that appear in $S_{\tau,n}^{x,a}$ are such that $k \geq \tau + 1$.) We get

$$\lim_{s \rightarrow \infty} \limsup_{n \rightarrow \infty} |V^a \varphi - U_{n,m,s}| = 0$$

on $\bar{\Omega}_\varphi^a$. Therefore also $V^a \varphi$ is independent of F_1, \dots, F_m . This holds for every m . By Kolmogorov's 0-1-law, $V^a \varphi$ is almost surely constant.

Note that in the lattice case, the proof simplifies, because we can just take the first return times of $A_{0,n}$ to 1. ■

(10.15) THEOREM. *Given the random i.i.d. Lipschitz mappings F_n , let A_n and B_n be as in (6.2). Suppose that besides the Standard Assumptions (7.4) also (10.8) holds, and that $\Pr[\hat{d}(\widehat{X}_n^{x,a}, \hat{\delta}) \rightarrow \infty] = 0$. Then the SDS induced by the F_n on X has an invariant Radon measure ν that is unique up to multiplication with constants.*

Also, the shift \widehat{T} on $(\widehat{X}^{\mathbb{N}_0}, \mathfrak{B}(\widehat{X}^{\mathbb{N}_0}), \Pr_\lambda)$ is ergodic, where λ is the extension of ν to \widehat{X} and \Pr_λ the associated measure on $\widehat{X}^{\mathbb{N}_0}$.

Proof. Let $\varphi \in \mathcal{C}_c^+(\widehat{X}^\ell)$. Recall that the function $\mathbf{v}_\varphi = \mathbb{E}_\lambda(\varphi | \mathfrak{J}) / \mathbb{E}_\lambda(\Psi | \mathfrak{J})$ on $\widehat{X}^{\mathbb{N}_0}$ is \widehat{T} -invariant. For the random variables $V_\varphi^{x,a} = V_\varphi^a$, this means that for almost every $a > 0$,

$$V_\varphi^a = V_\varphi^{A_{0,n} a} \quad \text{Pr-almost surely for all } n.$$

By Proposition (10.14), these random variables are constant on a set $\bar{\Omega}_\varphi^a \subset \Omega_\varphi^a$ with $\Pr(\bar{\Omega}_\varphi^a) = 1$. Fix one $a_0 > 0$ for which this holds.

In the lattice case, since we have chosen the maximal κ for which $\log A_n \in \kappa\mathbb{Z}$ a.s., the associated centered random walk $\log A_{0,n}$ is recurrent on $\kappa\mathbb{Z}$: for every starting point $a \in \exp(\kappa\mathbb{Z})$, $(A_{0,n}a)_{n \geq 0}$ visits a_0 almost surely. We infer that $V_\varphi^a = V_\varphi^{a_0}$ \Pr -almost surely for every $a \in \exp(\kappa\mathbb{Z})$.

In the non-lattice case, the multiplicative random walk $(A_{0,n}a)_{n \geq 0}$ starting at any $a > 0$ is topologically recurrent on \mathbb{R}_*^+ . This means that for every $a > 0$, with probability 1 there is a random sequence $(n_k)_{k \geq 0}$ such that $A_{0,n_k}a \rightarrow a_0$ as $k \rightarrow \infty$. Proposition (10.12) implies that $V_\varphi^a = V_\varphi^{a_0}$ on a set $\tilde{\Omega}_\varphi^a \subset \Omega_\varphi^{a_0}$ with probability 1.

Now let $\{a_k : k \in \mathbb{N}\}$ be dense in \mathbb{R}_*^+ and such that $\Pr(\tilde{\Omega}_\varphi^{a_k}) = 1$ for all \mathbb{N} . Using Proposition (10.12) once more, we find that for every $a > 0$, $V_\varphi^a = V_\varphi^{a_k} = V_\varphi^{a_0}$ on $\bigcap_k \tilde{\Omega}_\varphi^{a_k}$.

We conclude that \mathbf{v}_φ is constant \Pr_λ -almost surely.

This is true for any $\varphi \in \mathcal{C}_c^+(\hat{\mathcal{X}}^\ell)$. Therefore \hat{T} is ergodic. It follows that up to multiplication with constants, λ is the unique invariant measure on $\hat{\mathcal{X}}$ for the extended SDS, so that ν is the unique invariant measure on \mathcal{X} for the original SDS. By Lemma (7.8)(b), $\text{supp}(\nu) = \mathbf{L}$. ■

We remark that by projecting, also the shift T on $(\mathcal{X}^{\mathbb{N}_0}, \mathfrak{B}(\mathcal{X}^{\mathbb{N}_0}), \Pr_\nu)$ is ergodic.

11. The reflected affine stochastic recursion. We finally consider in detail the SDS of (6.10). Thus, $F_n(x) = |A_n x - B_n|$, so that $l(F_n) = A_n$ and $d(F_n(0), 0) = |B_n|$.

In the case when $\mathbf{E}(\log A_n) < 0$, we can apply Corollary (7.2).

(11.1) COROLLARY. *If $\mathbf{E}(\log^+ A_n) < \infty$ and $-\infty \leq \mathbf{E}(\log A_n) < 0$ then the reflected affine stochastic recursion is strongly contractive on \mathbb{R}^+ . If in addition $\mathbf{E}(\log^+ |B_n|) < \infty$ then it has a unique invariant probability measure ν on \mathbb{R}^+ , and it is (positive) recurrent on $\mathbf{L} = \text{supp}(\nu)$.*

From now on, we shall again focus on the case when $\log A_n$ is centered.

For the time being, we shall only deal with the case when $B_n > 0$. The reflected affine stochastic recursion is topologically irreducible on the set \mathbf{L} given by Proposition (7.3). Here, we shall not investigate the nature of \mathbf{L} in detail. It may be unbounded or compact.

Since $\mathcal{X} = \mathbb{R}^+$, the extended space $\hat{\mathcal{X}}$ is just the first quadrant with hyperbolic metric, and if $f(x) = |ax - b|$ then $\hat{f}(x, y) = (|ax - b|, ay)$. We can apply Corollary (8.6) to the extended process.

(11.2) PROPOSITION. *Assume that (7.4)(i)+(iii) hold, $B_n > 0$ almost surely, and $\mathbf{E}(\log^+ B_n) < \infty$. If the extended process $(\hat{X}_n^{x,a})$ is conservative, then the*

normalized distances $D_n(x, y)$ of (10.7) satisfy (10.8), that is, $\Pr[d(Z_n^x, Z_n^y) \rightarrow 0] = 1$ for all $x, y \in \mathbf{X}$, where $Z_n^x = X_n/A_{0,n}$.

Proof. We have the recursion $Z_0^x = x$ and $Z_n^x = |Z_{n-1}^x - B_n/A_{0,n}|$. We start with a simple exercise whose proof we omit. Let $c_j > 0$ and $f_j(x) = |x - c_j|$, $j = 1, \dots, s$. Then

$$(11.3) \quad f_s \circ \dots \circ f_1(x) \leq \max\{c_1, \dots, c_s\} \quad \text{for all } x \in [0, c_1 + \dots + c_s].$$

We prove that for every $\varepsilon > 0$ and $M > 0$ there is N such that $\Pr(\Gamma_{M,N,\varepsilon}) > 0$, where

$$\Gamma_{M,N,\varepsilon} = [D_N(x, y) < \varepsilon \text{ for all } x, y \text{ with } 0 \leq x, y \leq M].$$

To show this, let μ be the probability measure on $\mathbb{R}_*^+ \times \mathbb{R}_*^+$ governing our SDS, that is, $\Pr[(A_k, B_k) \in U] = \mu(U)$ for any Borel set $U \subset \mathbb{R}_*^+ \times \mathbb{R}_*^+$. By our assumptions, there are $(a_1, b_1), (a_2, b_2) \in \text{supp}(\mu)$ such that $0 < a_1 < 1 < a_2$ and $b_1, b_2 > 0$. We choose $\Delta > 1$ such that $a_1\Delta < 1 < a_2/\Delta$, and we set $b_* = \min\{b_1, b_2\}/\Delta$ and $b^* = \max\{b_1, b_2\}\Delta$.

Let $r, s \in \mathbb{N}$. For $k = r + 1, \dots, r + s$, we recursively define indices $i(k) \in \{1, 2\}$ by

$$i(r + 1) = 1, \quad i(k + 1) = \begin{cases} 1 & \text{if } a_{i(r+1)} \cdots a_{i(k)} \geq 1, \\ 2 & \text{if } a_{i(r+1)} \cdots a_{i(k)} < 1. \end{cases}$$

Therefore $a_1 \leq a_{i(r+1)} \cdots a_{i(k)} \leq a_2$ for all $k > r$. We have

$$\begin{aligned} \Pr[a_2/\Delta^{1/r} \leq A_k \leq a_2\Delta^{1/r} \text{ and } b_* \leq B_k \leq b^*] &> 0, \quad k = 1, \dots, r, \\ \Pr[a_{i(k)}/\Delta^{1/s} \leq A_k \leq a_{i(k)}\Delta^{1/s} \text{ and } b_* \leq B_k \leq b^*] &> 0, \\ &k = r + 1, \dots, r + s. \end{aligned}$$

Since the (A_k, B_k) are i.i.d., we also find that with positive probability,

$$\begin{aligned} a_2^k/\Delta \leq A_{0,k} \leq a_2^k\Delta &\quad \text{for } k = 1, \dots, r, \\ a_1/\Delta \leq A_{r,r+j} \leq a_2\Delta &\quad \text{for } j = 1, \dots, s, \\ b_* \leq B_k \leq b^* &\quad \text{for } k = 1, \dots, r + s, \end{aligned}$$

and thus, again with positive probability,

$$(11.4) \quad \begin{aligned} \frac{B_k}{A_{0,k}} &\leq \frac{b^*\Delta^2}{a_2} \quad \text{for } k = 1, \dots, r, \\ \frac{b_*}{a_2^{r+1}\Delta^2} &\leq \underbrace{\frac{B_{r+j}}{A_{0,r+j}}}_{=: c_j} \leq \frac{b^*\Delta^2}{a_1a_2^r} \quad \text{for } j = 1, \dots, s. \end{aligned}$$

We now set $M' = b^*\Delta^2/a_2$ and then choose r and s sufficiently large such that

$$\frac{b^*\Delta^2}{a_1a_2^r} < \varepsilon \quad \text{and} \quad s \frac{b_*}{a_2^{r+1}\Delta^2} \geq M + M'.$$

We set $N = r + s$ and let $\Gamma'_{M,N,\varepsilon}$ be the event on which the inequalities (11.4) hold. On $\Gamma'_{M,N,\varepsilon}$, we can use (11.3) to get $Z_r^0 \leq M'$. Since $D_n(x, y)$ is decreasing in n , we see for $x \in [0, M]$ that $|Z_r^x - Z_r^0| \leq x \leq M$ and thus $\xi = Z_r^x \in [0, M + M']$. Now we can apply (11.3) with c_j as in (11.4) and obtain $\max_j c_j < \varepsilon$ and $c_1 + \dots + c_s \geq M + M'$. But for the associated mappings f_1, \dots, f_s according to (11.3), we have $Z_N^x = f_s \circ \dots \circ f_1(\xi)$. We see that on the event $\Gamma'_{M,N,\varepsilon}$, one has $Z_N^x < \varepsilon$ for all $x \in [0, M]$, whence $D_N(x, y) < \varepsilon$ for all $x, y \in [0, M]$. So $\Gamma'_{M,N,\varepsilon} \subset \Gamma_{M,N,\varepsilon}$, whence $\Pr(\Gamma_{M,N,\varepsilon}) > 0$.

We can use Lemma (10.10) to conclude. ■

Combining the last proposition with Theorems (9.4) and (10.15), we obtain the main result of this section.

(11.5) THEOREM. *Consider the reflected affine stochastic recursion (6.10) with $A_n, B_n > 0$. Suppose that the Standard Assumptions (7.4) hold. Then the SDS has a unique invariant Radon measure ν on \mathbb{R}^+ , and it is topologically recurrent on $\mathbf{L} = \text{supp}(\nu)$. The time shift on the trajectory space $((\mathbb{R}^+)^{\mathbb{N}_0}, \text{Pr}_\nu)$ is ergodic.*

We now answer the additional question when there is an invariant probability measure, i.e., when $\nu(\mathbf{L}) < \infty$.

(11.6) THEOREM. *In the situation of Theorem (11.5), suppose also that $\mathbb{E}(|\log A_n|^{2+\varepsilon}) < \infty$ and $\Pr[B_n \geq b] = 1$ for some $b > 0$. Then $\nu(\mathbf{L}) < \infty$ if and only if the set \mathbf{L} is bounded.*

The proof will be based on the next proposition, which may be of interest in its own right.

(11.7) PROPOSITION. *For any $x, t \geq 0$, let*

$$\tau_x^{[0,t)} = \inf\{n \geq 1 : X_n^x < t\}$$

be the time of the first visit in the interval $[0, t)$. Under the assumptions of Theorem (11.6), there is $x(t) > 0$ such that for all $x \geq x(t)$,

$$\mathbb{E}(\tau_x^{[0,t)}) = \infty.$$

Proof. Consider the affine recursion without reflection $Y_n^x = A_n Y_{n-1}^x - B_n$. If $Y_k^x \geq t$ for $k = 1, \dots, n$ then $X_k^x = Y_k^x$ for those k , and then $\tau_x^{[0,t)} > n$. That is,

$$\Pr[\tau_x^{[0,t)} > n] \geq \Pr[Y_k^x \geq t, k = 1, \dots, n].$$

We have

$$(11.8) \quad Y_k^x \geq t \Rightarrow \underbrace{\sum_{j=1}^k \frac{B_j}{A_{0,j}}}_{\check{R}_k^0} + \frac{t}{A_{0,k}} \leq x.$$

Now consider the affine stochastic recursion generated by the inverses of the affine mappings $F_n(x) = A_n x - B_n$. These are

$$\check{F}_n(y) = \check{A}_n y + \check{B}_n, \quad \text{where} \quad \check{A}_n = 1/A_n \quad \text{and} \quad \check{B}_n = B_n/A_n.$$

They satisfy moment conditions of the same order as A_n , resp. B_n , so that the associated affine recursion (\check{Y}_n^y) is recurrent on the support of its unique invariant measure. Thus, there is $u > 0$ (sufficiently large) such that $\Pr[\check{Y}_n^y \leq u \text{ infinitely often}] = 1$ for any starting point y . The right process induced by the \check{F}_n is $\check{R}_n^y = \check{F}_1 \circ \dots \circ \check{F}_n(y)$. It is not a Markov chain, but \check{R}_n^y has the same distribution as \check{Y}_n^y . In particular, \check{R}_k^0 appears above in (11.8), and

$$\sum_n \Pr[\check{R}_n^0 \leq u] = \sum_n \Pr[\check{Y}_n^0 \leq u] = \infty.$$

Now, if $\check{R}_n^0 \leq u$, then for $k = 1, \dots, n$,

$$\check{R}_k^0 + \frac{t}{A_{0,k}} \leq \check{R}_n^0 + \underbrace{\frac{B_k}{A_{0,k}} \frac{t}{B_k}}_{\leq \check{R}_n^0} \leq u(1 + t/b) =: x(t).$$

If $x \geq x(t)$ then we see that

$$\Pr[\check{R}_n^0 \leq u] \leq \Pr[Y_k^x \geq t, k = 1, \dots, n].$$

Therefore

$$\sum_n \Pr[\tau_x^{[0,t]} > n] \geq \sum_n \Pr[\check{R}_n^0 \leq u],$$

and the statement follows. ■

Proof of Theorem (11.6). Since ν is a Radon measure, one has $\nu(\mathbf{L}) < \infty$ when \mathbf{L} is bounded.

Conversely, suppose that \mathbf{L} is unbounded. We use the distinction between positive and null recurrence as in [9, Corollary (2.19)]. We fix a suitable $t > 0$ such that the interval $[0, t]$ intersects \mathbf{L} . We consider the probability measure $\nu_t = (1/\nu([0, t]))\nu|_{[0,t]}$ and the SDS $(X_n^{\nu_t})$ with initial distribution ν_t . We shall show that its return time $\tau^{[0,t]}$ to $[0, t]$ has infinite expectation. Then ν cannot be finite.

We know that there is $u \in \mathbf{L}$ with $u > x(t)$, with $x(t)$ as in Proposition (11.7). We let U be an open interval that contains u and does not intersect $[0, t]$. We apply Theorem (7.6) to the starting point $x_0 \in [0, t] \cap \mathbf{L}$. There is an m such that $\Pr[X_m^{x_0} \in U] > 0$. This means that there are $f_1, \dots, f_m \in \text{supp}(\tilde{\mu})$ with $f_m \circ \dots \circ f_1(x_0) \in U$. (Each f_j is of the form $f_j(x) = |a_j x - b_j|$.) There must be a maximal $k < m$ for which $x_k = f_k \circ \dots \circ f_1(0) \in [0, t]$. Note that $x_j \in \mathbf{L}$ for all j by (6.8), which is valid by Proposition (7.3).

We now may assume without loss of generality that $k = 0$. Therefore we can find neighborhoods (open intervals) $U_0, U_1, \dots, U_{m-1}, U_m = U$ of the

respective x_j such that $U_0 \subset [0, t]$, while $U_j \cap [0, t] = \emptyset$ for $j > 0$, and

$$\tilde{\mu}(\{f : f(U_{j-1}) \subset U_j\}) > 0, \quad j = k + 1, \dots, m.$$

This translates into

$$\Pr(A_x) \geq \alpha > 0 \quad \text{for all } x \in U_0, \quad \text{where } A_x = [X_j^x \in U_j, j = 1, \dots, m].$$

So we can now consider the SDS starting at $x \in U_0$, leaving $(0, t]$ at the first step, and reaching some $y \in U$ in m steps. After that, it takes $\tau_y^{[0,t]}$ steps to return to $(0, t]$. We formalize this, and remember that $U_j \cap \mathbf{L} \neq \emptyset$ for every j . Just for the purpose of the next lines, we consider the measure $\sigma_x(B) = \Pr(A_x \cap [X_m^x \in B])$, where $x \in U_0$. It is concentrated on U with $\sigma_x(U) \geq \alpha$, and

$$\mathbf{E}(\tau^{[0,t]}) \geq \int_{U_0} \mathbf{E}(\tau_x^{[0,t]} \cdot \mathbf{1}_{A_x}) d\nu_t(x) \geq \int_{U_0} \underbrace{\left(\int_U (m + \mathbf{E}(\tau_y^{[0,t]})) d\sigma_x(y) \right)}_{=\infty \text{ by Proposition (11.7)}} d\nu_t(x) = \infty.$$

Therefore ν must have infinite mass. ■

We now discuss an example.

(11.9) EXAMPLE. We let $0 < p < 1$ and

$$A_n = \begin{cases} 2 & \text{with probability } p, \\ 1/2 & \text{with probability } q = 1 - p, \end{cases} \quad B_n = 1 \text{ always.}$$

Thus, we randomly iterate the transformations $f_1(x) = |2x - 1|$ and $f_{-1}(x) = |x/2 - 1|$. In other words, $F_n(x) = |2^{\varepsilon_n} x - 1|$, where $(\varepsilon_n)_{n \geq 1}$ is a sequence of i.i.d. ± 1 -valued random variables with $\Pr[\varepsilon_n = 1] = p$ and $\Pr[\varepsilon_n = -1] = q$.

Keeping in mind Remark (7.5)(b), we now determine \mathbf{L} as the smallest non-empty closed set which satisfies $f_{\pm 1}(\mathbf{L}) \subset \mathbf{L}$. First of all, we see that each of the two functions maps the interval $[0, 1]$ into itself. Thus, we must have $\mathbf{L} \subset [0, 1]$.

Let $\alpha = \max \mathbf{L}$. Then $\alpha \geq 2/3$, because $2/3 \in \mathbf{L}$ as the attracting fixed point of f_{-1} . We must have $(1 + \alpha)/2 = f_{-1} \circ f_1 \circ f_{-1}(\alpha) \in \mathbf{L}$, whence it is $\leq \alpha$. Therefore $\alpha = 1$. We get $1 \in \mathbf{L}$. The set of all iterates of 1 under $f_{\pm 1}$ is

$$\{f_{i_1} \circ \dots \circ f_{i_n}(1) : n \geq 0, i_j = \pm 1\} = \mathbb{D}, \quad \text{where } \mathbb{D} = \mathbb{Z}[1/2] \cap [0, 1],$$

and $\mathbb{Z}[1/2]$ stands for the dyadic rationals, i.e., rationals whose denominator is a power of 2. Since \mathbb{D} is dense, $\mathbf{L} = [0, 1]$.

Contractive case ($p < 1/2$). We can apply Corollary (11.1) and get a unique invariant probability measure ν , which is supported on $[0, 1]$.

Log-centered case ($p = 1/2$). Since \mathbf{L} is compact, the extended SDS is clearly conservative. In particular, $D_n(x, y) \rightarrow 0$ almost surely for all x, y . We now undertake an additional effort to clarify that the SDS is *not* locally contractive.

For the symmetric random walk $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ on \mathbb{Z} , let $M_n = \max\{0, S_1, \dots, S_n\}$. Now consider our SDS $(X_n^x)_{n \geq 0}$ with $x \in [0, 1]$. It is an instructive exercise to prove the following by induction on n .

(11.10) LEMMA. *The map $x \mapsto X_n^x$ is continuous and piecewise affine on $[0, 1]$, and there are random variables $\delta \in \{-1, 1\}$ and $C_j = C_{j, M_n} \in \mathbb{Z}[1/2]$ such that*

$$\begin{aligned} X_n^x &= (-1)^j \delta 2^{S_n} x + C_j \quad \text{on } I_{j, M_n}, \quad \text{where} \\ I_{j, k} &= [(j-1)2^{-k}, j2^{-k}], \quad j = 1, \dots, 2^k. \end{aligned}$$

In particular, the images of each of the intervals I_{j, M_n} under $x \mapsto X_n^x$ coincide and have the form

$$[(L_n - 1)/2^{M_n - S_n}, L_n/2^{M_n - S_n}],$$

where L_n is an integer random variable with $1 \leq L_n \leq 2^{M_n - S_n}$.

Recall the *strictly ascending ladder epochs* of the random walk (S_n) ,

$$\mathbf{t}(0) = 0 \quad \text{and} \quad \mathbf{t}(k+1) = \inf\{n > \mathbf{t}(k) : S_n > S_{\mathbf{t}(k)}\}.$$

They are all a.s. finite, and $S_{\mathbf{t}(k)} = M_{\mathbf{t}(k)} = k$. By Lemma (11.10), the image of each interval $I_{j, k}$ is the whole of $[0, 1]$. From this and the specific form that $x \mapsto X_n^x$ has to take, one sees that the only two choices for the mapping $x \mapsto X_{\mathbf{t}(k)}^x$ are

$$X_{\mathbf{t}(k)}^x = f_1^{(k)}(x) \quad \text{or} \quad X_{\mathbf{t}(k)}^x = 1 - f_1^{(k)}(x),$$

where $f^{(k)}$ denotes the k th iterate of the function f . Therefore, considering the fixed points $x_0 = 1$ and $y_0 = 1/3$ of f_1 , we get

$$|X_{\mathbf{t}(k)}^{x_0} - X_{\mathbf{t}(k)}^{y_0}| = 2/3 \quad \text{for all } k.$$

Thus, we do not have local contractivity.

Expanding case ($p > 1/2$). Since \mathbb{L} is compact, the SDS is conservative for any value of p , so that there are always invariant probability measures. We show that in the expanding case, there are infinitely many mutually singular ones. Fix r , an odd prime or $r = 1$, and define

$$\mathbb{D}_r = \left\{ \frac{k}{r2^n} : k, n \in \mathbb{N}_0, k \leq r2^n, \gcd(k, r2^n) = 1 \right\}.$$

(Note that we must have $0 < k < r2^n$ unless $r = 1$ and $n = 0$.) Then it is easy to verify that $f_{\pm 1}(\mathbb{D}_r) \subset \mathbb{D}_r$. Thus, when we start at a point $x \in \mathbb{D}_r$, then (X_n^x) can be seen as a Markov chain on the denumerable state space \mathbb{D}_r . Let $p(x, y) = \Pr[X_1^x = y]$ denote its transition matrix. It is not hard to verify that it is irreducible (all states communicate), although we do not really need this. We partition $\mathbb{D}_r = \bigcup_n \mathbb{D}_{r, n}$, where $\mathbb{D}_{r, n}$ consists of all

$k/(r2^n)$ as above with the specific value of n . If $n \geq 1$, then we see that for each $x \in \mathbb{D}_{r,n}$,

$$p(x, \mathbb{D}_{r,m}) = \sum_{y \in \mathbb{D}_{r,m}} p(x, y) = \begin{cases} p & \text{if } m = n - 1, \\ q & \text{if } m = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

A similar identity for $x \in \mathbb{D}_{r,0}$ does not hold, so that we cannot define the factor chain on \mathbb{N}_0 . Nevertheless, since each $\mathbb{D}_{r,n}$ is finite, we can use comparison with the birth-and-death chain on \mathbb{N}_0 with transition probabilities $\bar{p}(n, n+1) = q$ and $\bar{p}(n, n-1) = p$ for $n \geq 1$. (We do not need to specify the outgoing probabilities at 0.) Thus, our Markov chain on \mathbb{D}_r is positive recurrent when $p > 1/2$, null recurrent when $p = 1/2$, and transient when $p < 1/2$. In particular, when $p > 1/2$, it has a unique invariant probability measure ν_r on the countable set \mathbb{D}_r . Since it is a probability measure, we can lift it to a Borel measure on $[0, 1]$ by setting $\nu_r(B) = \sum_{x \in \mathbb{D}_r \cap B} \nu_r(x)$. Thus, each ν_r is also an invariant probability measure for the (“topological”) SDS on $[0, 1]$, and all the ν_r are pairwise mutually singular.

(11.11) REMARK. Regarding the last example, we underline that the respective discrete, denumerable Markov chains on \mathbb{D}_r have precisely the opposite behavior of the SDS on $[0, 1]$: the Markov chain is transient precisely when the SDS is strongly contractive (and positive recurrent), and it is null recurrent precisely when the SDS is weakly, but not strongly contractive (and null-recurrent). But this fact should not be surprising. Indeed, let us compare this with the affine stochastic recursion $Y_n^x = 2^{L_n}x + B_n$, where (L_n, B_n) are 2-dimensional i.i.d. random variables with $L_n \in \mathbb{Z}$ and $B_n \in \mathbb{Z}[1/2]$. If the starting point x is also a dyadic rational, then we can consider (Y_n^x) as an SDS both on \mathbb{R} with Euclidean distance and on the field \mathbb{Q}_2 of dyadic numbers with the distance induced by the dyadic norm. Under the usual moment conditions, this SDS is transient on \mathbb{R} precisely when it is strongly contractive on \mathbb{Q}_2 , and weakly (but not strongly) contractive on \mathbb{R} precisely when it has the same property on \mathbb{Q}_2 .

In conclusion, we briefly touch on another example, considering only the log-centered case.

(11.12) EXAMPLE. We let $0 < p < 1$ and

$$A_n = \begin{cases} 3 & \text{with probability } 1/2, \\ 1/3 & \text{with probability } 1/2, \end{cases} \quad B_n = 1 \text{ always.}$$

This time, we randomly iterate $g_1(x) = |3x - 1|$ and $g_{-1}(x) = |x/3 - 1|$. A brief discussion shows that the limit set must be unbounded: suppose that $\alpha = \sup \mathbb{L} < \infty$. Then we must have $g_{i_n} \circ \dots \circ g_{i_1}(\alpha) \in \mathbb{L}$ for any choice of n and $i_j \in \{-1, 1\}$ ($j = 1, \dots, n$). But for any α we can find some choice

where $g_{i_n} \circ \cdots \circ g_{i_1}(\alpha) > \alpha$, a contradiction.

Thus, the invariant Radon measure has infinite mass.

A more detailed study of these and similar classes of reflected affine stochastic recursions are planned to be the subject of future work.

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