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ON SOLUTIONS OF FUNCTIONAL EQUATIONS DETERMINING SUBSEMIGROUPS OF L_4^1

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Abstract. Let L_4^1 be the group of 4-jets at zero of diffeomorphisms f of $\mathbb{R}$ with $f(0)=0$. Identifying jets with sequences of derivatives, we determine all subsemigroups of L_4^1 consisting of quadruples $(x_1, f(x_1,x_4), g(x_1,x_4), x_4) \in (\mathbb{R}\setminus\{0\})\times\mathbb{R}^3$ with continuous functions $f,g\colon (\mathbb{R}\setminus\{0\})\times\mathbb{R}\to\mathbb{R}$. This amounts to solving a set of functional equations.

1. Introduction. The groups L_s^n arise when studying jets of local diffeomorphisms of $\mathbb{R}^n$ in the following way. Let $j^s f$ be the s-jet of a diffeomorphism f defined in a neighborhood of $0\in\mathbb{R}^n$ and satisfying $f(0)=0$. We consider the set L_s^n of all such jets equipped with the group operation

$$(j^s f)\circ(j^s g) = j^s(f\circ g), \quad \text{where} \quad (f\circ g)(x) = f(g(x)).$$

Any jet $j^s f$ can be identified with the sequence of partial derivatives at 0 of f of orders $1,\dots,s$. Therefore, L_s^n can be identified with a set of real sequences (see [3]). In those terms, the group L_s^1 can be given the following algebraic description. As a set, we have $L_s^1 = \mathbb{R}_0\times\mathbb{R}^{s-1}$, where $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$. The product is defined as

$$(x_1,\dots,x_s)\circ(y_1,\dots,y_s) = (z_1,\dots,z_s)$$

where for $m=1,\dots,s$ we have

$$z_m = \sum_{k=1}^m x_k \sum\Big\{A_u\cdot y_1^{u_1}\cdots y_m^{u_m} : u_i\in\mathbb{N}\cup\{0\}, \sum_{i=1}^m u_i = k, \sum_{i=1}^m iu_i = m\Big\}$$

and $A_u = m!/\prod_{i=1}^m u_i!(i!)^{u_i}$ (Faà di Bruno's formula).

In particular, multiplication in L_4^1 is given by the following formula:

$$(1)\qquad (x_1,x_2,x_3,x_4)\circ(y_1,y_2,y_3,y_4) = (z_1,z_2,z_3,z_4),$$
$$z_1 = x_1y_1,\quad z_2 = x_1y_2 + x_2y_1^2,\quad z_3 = x_1y_3 + 3x_2y_1y_2 + x_3y_1^3,$$
$$z_4 = x_1y_4 + 4x_2y_1y_3 + 3x_2y_2^2 + 6x_3y_1^2y_2 + x_4y_1^4.$$

Papers [3]–[11] describe certain subsemigroups of L_s^1 for $2\le s\le 5$, consisting of tuples for which one of the coordinates is a function of the

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others. In [4, Section 4], subsemigroups of L_4^1 consisting of elements of the form $(x_1, f(x_1, x_4), f(x_1, x_4), x_4)$ were described in terms of a certain system of functional equations. In [10] subsemigroups of L_4^1 consisting of elements of the form $(x_1, f(x_1, x_4), g(x_1, x_4), x_4)$ were studied with some additional restrictions on f and g. In all solutions, the functions f and g depended on x_1 only.

In this paper we generalize those results. In particular, we show that there do exist solutions depending on both variables x_1, x_4.

MAIN THEOREM 1. *All subsemigroups of L_4^1, consisting of quadruples $(x_1, f(x_1, x_4), g(x_1, x_4), x_4) \in \mathbb{R}_0 \times \mathbb{R}^3$ with continuous functions $f, g\colon \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ belong to one of the families*

$$P_{a,b} = \{(x_1, f_{ab}(x_1, x_4), g_{ab}(x_1, x_4), x_4)\}, \quad a, b \in \mathbb{R},$$
$$Q_{c,d} = \{(x_1, f_{cd}(x_1, x_4), g_{cd}(x_1, x_4), x_4)\}, \quad c \in [0, +\infty),\ d \in \mathbb{R},$$

where

$$f_{ab}(x_1, x_4) = a(x_1 - x_1^2), \quad g_{ab}(x_1, x_4) = \frac{3}{2}\, a^2 x_1 (1 - x_1)^2 + b(x_1 - x_1^3),$$
$$f_{cd}(x_1, x_4) = x_1 \sqrt[3]{q + \sqrt{q^2 + p^3}} + x_1 \sqrt[3]{q - \sqrt{q^2 + p^3}},$$
$$g_{cd}(x_1, x_4) = \frac{3}{2}\, x_1 \sqrt[3]{2q^2 + p^3 + 2q\sqrt{q^2 + p^3}}$$
$$+ \frac{3}{2}\, x_1 \sqrt[3]{2q^2 + p^3 - 2q\sqrt{q^2 + p^3}} + c(4 - 3x_1 - 6x_1^2 + 3x_1^3),$$

with $p(x_1) = \frac{2}{3} c(3x_1 - 2x_1^{-1})$ and $q(x_1, x_4) = \frac{1}{6} x_1^{-1} x_4 + d(1 - x_1^3)$.

2. Auxiliary results

LEMMA 1. *Let $\Phi\colon \mathbb{R}_0 \times \mathbb{R} \times \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ be any function. If $F\colon \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ satisfies*

$$F(x_1 \cdot y_1, \Phi(x_1, x_2, y_1, y_2)) = x_1^k F(y_1, y_2) + y_1^l F(x_1, x_2)$$

for some $k \neq l$ and $F(1, x_2) \equiv 0$ then $F(x_1, x_2) = a(x_1^k - x_1^l)$ for some constant a.

Proof. The substitution $x_1 \mapsto y_1^{-1}$ gives

$$0 = y_1^{-k} F(y_1, y_2) + y_1^l F(y_1^{-1}, x_2),$$

hence

$$F(y_1^{-1}, x_2) = -y_1^{-k-l} F(y_1, y_2) \quad \text{for all } y_1 \in \mathbb{R}_0,\ x_2, y_2 \in \mathbb{R}. \tag{2}$$

When we switch $y_1 \leftrightarrow y_1^{-1}$ and rename $x_2 \mapsto z_2$, $y_2 \mapsto x_2$, we get

$$F(y_1, z_2) = -y_1^{k+l} F(y_1^{-1}, x_2). \tag{3}$$

Substituting (2) into (3), we obtain $F(y_1, z_2) = -y_1^{k+l}[-y_1^{-k-l}F(y_1, y_2)] = F(y_1, y_2)$, i.e., F does not depend on the second variable: $F(x_1, x_2) = \phi(x_1)$. The original equation reduces to $\phi(x_1 \cdot y_1) = x_1^k\phi(y_1) + y_1^l\phi(x_1)$. The interchange $x_1 \leftrightarrow y_1$ gives the equality $x_1^k\phi(y_1) + y_1^l\phi(x_1) = y_1^k\phi(x_1) + x_1^l\phi(y_1)$, hence

$$\frac{\phi(y_1)}{y_1^k - y_1^l} = \frac{\phi(x_1)}{x_1^k - x_1^l} = a, \qquad \text{a constant.}$$

Therefore $F(x_1, x_2) = \phi(x_1) = a(x_1^k - x_1^l)$. ∎

For a fixed linear transformation of the real plane, one can investigate (see [1], [2]), for which functions F the graph $\{(x, y) : y = F(x)\}$ remains invariant under this transformation. This question easily translates into the functional equation $F(F(t)) = p \cdot F(t) - q \cdot t$ for some $p, q \in \mathbb{R}$. We will be interested in continuous solutions of such equations. For example, we have

LEMMA 2. *All continuous solutions of the equation* $F(F(t)) = 2F(t) - t$ *are of the form* $F(t) = t + c$ *for some* $c \in \mathbb{R}$.

Proof. Let us write $F(t) = t + h(t)$. Then $h(t + h(t)) = h(F(t)) = F(F(t)) - F(t) = F(t) - t = h(t)$, i.e., h satisfies Euler's equation. From [1, Thm. 14.5] it follows that continuous solutions of this equation are constant. Hence $F(t) = t + c$ for some constant $c \in \mathbb{R}$. ∎

In the proof of the main result we will consider continuous functions F satisfying

$$F(F(t)) = p \cdot F(t) - q \cdot t \tag{4}$$

where $p > 0$, $q > 0$ and the equation $\lambda^2 - p\lambda + q = 0$ has real roots λ_1, λ_2 satisfying $1 \le \lambda_1 < \lambda_2$.

LEMMA 3. *Let a continuous function* $F\colon \mathbb{R} \to \mathbb{R}$ *satisfy equation* (4) *and* $F(0) = 0$. *Then*

(i) *For any* $t_2 > t_1$ *we have* $F(t_2) - F(t_1) \ge \lambda_1(t_2 - t_1)$.
(ii) *F is a homeomorphism of the real line onto itself.*
(iii) *Let* $\varepsilon \in \{-1, +1\}$. *If* $F(\varepsilon t) \not\equiv \lambda_1\varepsilon t$ *for* $t \ge 0$ *then for any* $\beta \in (\lambda_1, \lambda_2)$ *there exists a sequence* $0 < t_n \to +\infty$ *such that* $\varepsilon F(\varepsilon t_n) > \beta t_n$ *for* $n \ge 1$.

Proof. Notice that F is 1-1. In fact, if $F(t_1) = F(t_2)$ then $qt_1 = pF(t_1) - F(F(t_1)) = pF(t_2) - F(F(t_2)) = qt_2$, i.e., $t_1 = t_2$.

From the continuity it follows that F is a monotonic function, vanishing at 0 only. Hence, for positive t we have either $F(t) < 0$ or $F(t) > 0$. In the first case we would have $F(t) > 0$ for negative t and hence for any $t > 0$ we get $0 < F(F(t)) = pF(t) - qt < 0$, a contradiction. Therefore F is increasing. Take any $t_2 > t_1$. Then $p(F(t_2) - F(t_1)) - q(t_2 - t_1) = F(F(t_2)) - F(F(t_1)) > 0$, hence $F(t_2) - F(t_1) > (q/p)(t_2 - t_1)$. Suppose that for some $\alpha > 0$ the inequality $F(t_2) - F(t_1) > \alpha(t_2 - t_1)$ holds for all $t_2 > t_1$. Then

$$\begin{aligned} p(F(t_2) - F(t_1)) - q(t_2 - t_1) &= F(F(t_2)) - F(F(t_1)) \\ &> \alpha(F(t_2) - F(t_1)) > \alpha^2(t_2 - t_1), \end{aligned}$$

hence $F(t_2) - F(t_1) > \frac{\alpha^2+q}{p}(t_2 - t_1)$. Define $\alpha_1 = q/p$ and $\alpha_{n+1} = (\alpha_n^2 + q)/p$. By induction it follows that $F(t_2) - F(t_1) > \alpha_n(t_2 - t_1)$ for all $t_2 > t_1$ and $n \geq 1$. It is easy to see that $\alpha_n < \lambda_1$ for all $n \geq 1$. It follows that the sequence (α_n) is increasing and bounded, hence convergent and $\lim \alpha_n = \lambda_1$. Therefore $F(t_2) - F(t_1) \geq \lambda_1(t_2 - t_1)$ for all $t_2 > t_1$, which proves (i).

By setting $t_1 = 0$ we get $F(t) \geq \lambda_1 t$ for all $t \geq 0$. By setting $t_2 = 0$ we obtain $F(t) \leq \lambda_1 t$ for all $t \leq 0$. Consequently, we obtain (ii): F is a homeomorphism of $\mathbb{R}$ onto itself.

To prove (iii), fix $\varepsilon = \pm 1$. Notice that the above inequality can be written as $\varepsilon F(\varepsilon t) \geq \lambda_1 t$ for $t \geq 0$. Suppose for some $\lambda_1 < \beta < \lambda_2$ the desired sequence (t_n) does not exist. Then there exists $t_0 \geq 0$ such that $\varepsilon F(\varepsilon t) \leq \beta t$ for all $t \geq t_0$. Define $\beta_1 = \beta$ and $\beta_{n+1} = q \cdot (p - \beta_n)^{-1}$. It is easy to check that $\lambda_1 < \beta_n < \lambda_2 < p$. We show by induction that $\varepsilon F(\varepsilon t) \leq \beta_n t$ for all $t \geq t_0$ and $n \in \mathbb{N}$. For $n = 1$ this is clear. Suppose it is true for n. Notice that for $t \geq t_0$ we have $\varepsilon F(\varepsilon t) \geq \lambda_1 t > t \geq t_0$, hence $\varepsilon F(F(\varepsilon t)) \leq \beta_n \varepsilon F(\varepsilon t)$ for $t \geq t_0$. But then

$$p\varepsilon F(\varepsilon t) - q\varepsilon^2 t = \varepsilon F(F(\varepsilon t)) \leq \beta_n \varepsilon F(\varepsilon t) \quad \text{implies} \quad \varepsilon F(\varepsilon t) \leq \beta_{n+1} t.$$

The sequence (β_n) is decreasing and hence convergent to λ_1. It follows that $\varepsilon F(\varepsilon t) \leq \lambda_1 t$ for all $t \geq t_0$. This implies that $\varepsilon F(\varepsilon t) = \lambda_1 t$ for $t \geq t_0$.

Let $t_1 = \inf\{t > 0 : \varepsilon F(\varepsilon t) = \lambda_1 t\}$. If $t_1 > 0$ then pick any $\gamma \in (\lambda_1, \lambda_2)$. By the continuity of F, we can find $t_2 \in (0, t_1)$ such that $\varepsilon F(\varepsilon t) \leq \gamma t$ for $t \geq t_2$. But then $\varepsilon F(\varepsilon t) = \lambda_1 t$ for $t \geq t_2$ by the previous paragraph applied to $\beta = \gamma$, contradicting the definition of t_1. Hence $t_1 = 0$ and $F(\varepsilon t) \equiv \lambda_1 \varepsilon t$ for $t \geq 0$, a contradiction. ∎

LEMMA 4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$ satisfying the equation*

$$\begin{aligned} f(\varDelta(x,y)) &= f(y) + f(x), \\ \varDelta(x,y) &= x + y + 9f(x)f(y)^2 + 9f(x)^2 f(y). \end{aligned} \tag{5}$$

Then either $f \equiv 0$, or f is a homeomorphism of the real line and $f^{-1}(t) = 3t^3 + at$ for some $a \geq 0$.

Proof. Obviously the constant function $f \equiv 0$ satisfies the equation. Now suppose that f is not constant. Notice that if $f(t) = 0$ for all $t > 0$ then for such t we have $\varDelta(-t,t) = 0$ and $f(-t) = f(-t) + f(t) = f(\varDelta(-t,t)) =$

$f(0) = 0$, so $f \equiv 0$. Thus $f(t) \neq 0$ for some $t > 0$. Analogously one can show that $f(t) \neq 0$ for some $t < 0$.

Set $F(t) = \Delta(t, t)$. Then we have $f(F(t)) = 2f(t)$ and

$$\begin{aligned} F(F(t)) &= 2F(t) + 18f(F(t))^3 = 2F(t) + 18 \cdot 8f(t)^3 \\ &= 2F(t) + 8(F(t) - 2t) = 10F(t) - 16t. \end{aligned}$$

Thus F is a continuous solution of the equation $F(F(x)) = 10F(x) - 16x$ and $F(0) = 0$. Moreover, $\lambda^2 - 10\lambda + 16$ has roots 2 and 8. By Lemma 3(i), for any $t_2 > t_1$ we have $F(t_2) - F(t_1) \geq 2(t_2 - t_1)$, hence $2t_2 + 18f(t_2)^3 - 2t_1 - 18f(t_1)^3 \geq 2(t_2 - t_1)$. It follows that $f(t_2)^3 \geq f(t_1)^3$ and $f(t_2) \geq f(t_1)$.

We prove that f is not bounded from above. Suppose the contrary and take any $M > 18 \sup\{f(t)\}^3$. Then $F(t) < 2t + M$ for all $t > 0$. Pick any $\beta \in (2, 8)$. There exists $t_0 > 0$ such that $\beta t > 2t + M$ for $t > t_0$. Because $F(t) \not\equiv 2t$ for $t > 0$, Lemma 3(iii) yields a sequence $t_n \to +\infty$ with $F(t_n) > \beta t_n$. Pick $t_N > t_0$. Then $2t_N + M > F(t_N) > \beta t_N > 2t_N + M$, a contradiction. In the same way, using Lemma 3(iii) with $\varepsilon = -1$, one proves that f is not bounded from below. Hence f maps the real line onto itself.

It follows that $f(t) = 0$ for $t = 0$ only. For suppose that $f(z) = 0$ for some $z \neq 0$. Then for any x we have $\Delta(x, z) = x + z$ and $f(x + z) = f(x)$. Therefore $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, hence bounded, a contradiction.

It follows that f is an odd function. In fact, for any x we can find y so that $f(y) = -f(x)$, as f is onto. Then $\Delta(x, y) = x + y$ and $f(x + y) = f(\Delta(x, y)) = f(x) + f(y) = 0$. It follows that $x + y = 0$, hence $y = -x$ and so $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Now we can prove that f is strictly increasing, in particular it is a homeomorphism of the real line. If not, f has a constant value c on some interval (r, s). Then f also has the constant value $-c$ on the interval $(-s, -r)$. For any $x \in (r, s)$, $y \in (-s, -r)$ we have $\Delta(x, y) = x + y + 9c^2 \cdot (-c) + 9c \cdot (-c)^2 = x + y$ and $f(x + y) = f(\Delta(x, y)) = f(x) + f(y) = c + (-c) = 0$. It follows that f has infinitely many zeros, a contradiction. This proves that f is 1-1, hence a homeomorphism.

Let $g = f^{-1}$. Substitute $x = g(u)$, $y = g(v)$ in equations (5). We get

$$f(\Delta(g(u), g(v))) = u + v, \qquad \Delta(g(u), g(v)) = g(u) + g(v) + 9uv^2 + 9u^2v,$$

hence

$$g(u + v) = gf(\Delta(g(u), g(v)) = \Delta(g(u), g(v)) = g(u) + g(v) + 9uv^2 + 9u^2v.$$

Substitute $g(t) = 3t^3 + h(t)$. Then $3(u + v)^3 + h(u + v) = 3u^3 + h(u) + 3v^3 + h(v) + 9uv^2 + 9u^2v$. It follows that h is a continuous solution of the Cauchy equation $h(u + v) = h(u) + h(v)$, hence $h(t) = at$ and $f^{-1}(t) = 3t^3 + at$. It must be $a \geq 0$, as f is a homeomorphism. It is easy to verify that any such function solves our equation. ∎

3. Proof of the Main Theorem. A subset $P = \{(x_1, f(x_1, x_4), g(x_1, x_4), x_4) : x_1 \in \mathbb{R}_0, x_4 \in \mathbb{R}\} \subset L_4^1$ is a subsemigroup iff for any $x_1, y_1 \in \mathbb{R}_0$ and $x_4, y_4 \in \mathbb{R}$,

$$(x_1, f(x_1, x_4), g(x_1, x_4), x_4) \circ (y_1, f(y_1, y_4), g(y_1, y_4), y_4) \in P,$$

which, using (1), translates to the following system of functional equations:

$$f(x_1 y_1, \Delta) = x_1 f(y_1, y_4) + y_1^2 f(x_1, x_4), \tag{6}$$

$$g(x_1 y_1, \Delta) = x_1 g(y_1, y_4) + 3y_1 f(x_1, x_4) f(y_1, y_4) + y_1^3 g(x_1, x_4), \tag{7}$$

$$\begin{aligned}\Delta = \Delta(x_1, x_4, y_1, y_4) &= x_1 y_4 + 4y_1 f(x_1, x_4) g(y_1, y_4) \\ &\quad + 3f(x_1, x_4) f(y_1, y_4)^2 + 6y_1^2 g(x_1, x_4) f(y_1, y_4) + x_4 y_1^4.\end{aligned} \tag{8}$$

Elementary (but rather tedious) calculations show that the subsemigroups $P_{a,b}$ and $Q_{c,d}$, defined in the formulation of the Main Theorem, satisfy this system of equations. We shall prove that they exhaust the list of subsemigroups of L_4^1 of the desired form.

We can make this system more symmetric, by substituting

$$g(u, v) = h(u, v) + \frac{3}{2u} f(u, v)^2,$$

where $h\colon \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ is a new unknown function. This leads to a new system

$$f(x_1 y_1, \Delta'(x_1, x_4, y_1, y_4)) = x_1 f(y_1, y_4) + y_1^2 f(x_1, x_4), \tag{9}$$

$$h(x_1 y_1, \Delta'(x_1, x_4, y_1, y_4)) = x_1 h(y_1, y_4) + y_1^3 h(x_1, x_4), \tag{10}$$

$$\begin{aligned}\Delta'(x_1, x_4, y_1, y_4) &= x_1 y_4 + x_4 y_1^4 + y_1 f(x_1, x_4) h(y_1, y_4) \\ &\quad + 6y_1^2 f(y_1, y_4) h(x_1, x_4) + 9f(x_1, x_4) f(y_1, y_4)^2 \\ &\quad + \frac{9y_1^2}{x_1} f(x_1, x_4)^2 f(y_1, y_4).\end{aligned} \tag{11}$$

Let us write $\widetilde{f}(u) = f(1, u)$, $\widetilde{h}(u) = h(1, u)$, $\widetilde{\Delta}(u, v) = \Delta'(1, u, 1, v)$. When we plug $x_1 = 1$, $y_1 = 1$ into equations (9)–(11), we get

$$\widetilde{f}(\widetilde{\Delta}(x_4, y_4)) = \widetilde{f}(y_4) + \widetilde{f}(x_4), \tag{12}$$

$$\widetilde{h}(\widetilde{\Delta}(x_4, y_4)) = \widetilde{h}(y_4) + \widetilde{h}(x_4), \tag{13}$$

$$\begin{aligned}\widetilde{\Delta}(x_4, y_4) &= y_4 + x_4 + 4\widetilde{f}(x_4)\widetilde{h}(y_4) + 6\widetilde{f}(y_4)\widetilde{h}(x_4) \\ &\quad + 9\widetilde{f}(x_4)\widetilde{f}(y_4)^2 + 9\widetilde{f}(x_4)^2\widetilde{f}(y_4).\end{aligned} \tag{14}$$

Now the proof splits into two cases: Case I: $\widetilde{f}$ is constant and Case II: $\widetilde{f}$ is not constant.

4. Case I: $\widetilde{f}$ is constant. From (12) it follows that $\widetilde{f} \equiv 0$. Lemma 1 applied to (6) implies that $f(x_1, x_4) = a(x_1 - x_1^2)$.

Now we determine h. Equation (14) reduces to $\widetilde{\Delta}(x_4, y_4) = x_4 + y_4$, and from (13) and the continuity of $\widetilde{h}$ it follows that $\widetilde{h}(u) = Cu$ for some constant $C \in \mathbb{R}$. We will show that $C = 0$. To this end, fix $\mu \in \mathbb{R}_0$ and substitute in (10)–(11) the values $x_1 = \mu$, $y_1 = \mu^{-1}$:

$$\begin{aligned} C \cdot \Delta'(\mu, x_4, \mu^{-1}, y_4) &= \mu h(\mu^{-1}, y_4) + \mu^{-3} h(\mu, x_4), \\ \Delta'(\mu, x_4, \mu^{-1}, y_4) &= \mu y_4 + \mu^{-4} x_4 + 4a(1-\mu) h(\mu^{-1}, y_4) \\ &\quad + 6a(\mu^{-3} - \mu^{-4}) h(\mu, x_4). \end{aligned}$$

Consequently,

$$\begin{aligned} &[4aC - (4aC+1)\mu] h(\mu^{-1}, y_4) + C\mu y_4 \\ &\qquad = [6aC - (6aC-1)\mu]\mu^{-4} h(\mu, x_4) - C\mu^{-4} x_4. \end{aligned}$$

It follows that the left hand side does not depend on y_4 and the right hand side does not depend on x_4. If $aC \neq 0$, then for $\mu = 4aC/(4aC+1)$ or for $\mu = 6aC/(6aC-1)$ (at least one of these numbers is well defined) it would not be the case. Therefore $aC = 0$ and

$$-\mu h(\mu^{-1}, y_4) + C\mu y_4 = \mu^{-3} h(\mu, x_4) - C\mu^{-4} x_4. \tag{15}$$

We switch the sides, $x_4 \leftrightarrow y_4$ and $\mu \leftrightarrow \mu^{-1}$ in (15):

$$\mu^3 h(\mu^{-1}, y_4) - C\mu^4 y_4 = -\mu^{-1} h(\mu, x_4) + C\mu^{-1} x_4. \tag{16}$$

When we add (15) multiplied by μ^2 to (16), we get $C(\mu^3 - \mu^4) y_4 = C(\mu^{-1} - \mu^{-2}) x_4$ for all $x_4, y_4 \in \mathbb{R}$ and $\mu \in \mathbb{R}_0$. It follows that $C = 0$.

We have just proved that $h(1, v) \equiv 0$. From Lemma 1 it follows that $h(x_1, x_4) = b(x_1 - x_1^3)$ for some constant b. In particular, in Case I, we have proved that

$$\begin{aligned} f(x_1, x_4) &= a(x_1 - x_1^2), \\ g(x_1, x_4) &= \frac{3}{2x_1} f(x_1, x_4)^2 + h(x_1, x_4) = \frac{3}{2} a^2 x_1 (1 - x_1)^2 + b(x_1 - x_1^3), \end{aligned}$$

as desired.

5. Case II: $\widetilde{f}$ is not constant. Let us notice that the function $\widetilde{f}$ attains value 0. To see this, we substitute $x_1 = -1$, $y_1 = -1$, $x_4 = t$, $y_4 = t$ into (9) and (11), where t is a new variable. We obtain $f(1, \Delta'(-1, t, -1, t)) = 0$, $\Delta'(-1, t, -1, t) = 2f(-1, t)h(-1, t)$, hence

$$\widetilde{f}(2f(-1, t) \cdot h(-1, t)) = 0 \qquad \text{for all } t \in \mathbb{R}. \tag{17}$$

In particular, there exists $z \in \mathbb{R}$ such that $\widetilde{f}(z) = 0$. We plug $y_4 = z$ in (12)–(14) to get

$$\widetilde{f}(\widetilde{\Delta}(x_4,z)) = \widetilde{f}(x_4), \tag{18}$$
$$\widetilde{h}(\widetilde{\Delta}(x_4,z)) = \widetilde{h}(z) + \widetilde{h}(x_4), \tag{19}$$
$$\widetilde{\Delta}(x_4,z) = z + x_4 + 4\widetilde{f}(x_4)\widetilde{h}(z). \tag{20}$$

Consider $G(u) = \widetilde{\Delta}(u,z)$. From (18) and (20) we get $G(G(u)) = z + G(u) + 4\widetilde{f}(G(u))\widetilde{h}(z) = z+G(u)+4\widetilde{f}(u)\widetilde{h}(z) = 2G(u)-u$. From Lemma 2 it follows that $G(u) = u+c$ for some $c \in \mathbb{R}$. Therefore $z+x_4+4\widetilde{f}(x_4)\widetilde{h}(z) = \widetilde{\Delta}(x_4,z) = G(x_4) = x_4+c$, i.e., $\widetilde{f}(x_4)\widetilde{h}(z)$ is a constant. Because by assumption $\widetilde{f}$ is not constant, it follows that $\widetilde{h}(z) = 0$. Hence equations (18) and (20) reduce to

$$\widetilde{f}(x_4 + z) = \widetilde{f}(x_4), \tag{21}$$

which holds for any $x_4 \in \mathbb{R}$ and any $z \in \mathbb{R}$ such that $\widetilde{f}(z) = 0$.

Our objective is to prove that $\widetilde{h} \equiv 0$. To this end, consider $\phi(t) = f(-1,t)h(-1,t)$. Then equation (17) reads $\widetilde{f}(2\phi(t)) = 0$ for all $t \in \mathbb{R}$. We check that the (continuous) function ϕ is constant. In fact, otherwise the image of 2ϕ contains an open interval I which, by (17), is contained in the zero set of $\widetilde{f}$. Let r be the middle point of this interval and let 2ε be its length. Then for $|x_4| < \varepsilon$ we have $r + x_4 \in I$, hence by (21): $\widetilde{f}(x_4) = \widetilde{f}(x_4 + r) = 0$. This means that $\widetilde{f}$ vanishes in the ε-neighborhood of 0. Equation (21) then says that $\widetilde{f}$ is locally constant, hence $\widetilde{f} \equiv 0$. This contradicts our standing assumption that $\widetilde{f}$ is not constant. Therefore $\phi(t)$ is a constant function, equal to, say, m.

Now we use equations (9)–(10). First we plug in $x_1 = -1$, $y_1 = 1$, to obtain

$$f(-1, \Delta'(-1,x_4,1,y_4)) = -\widetilde{f}(y_4) + f(-1,x_4), \tag{22}$$
$$h(-1, \Delta'(-1,x_4,1,y_4)) = -\widetilde{h}(y_4) + h(-1,x_4). \tag{23}$$

We multiply (22) and (23) and use the relation $f(-1,t)h(-1,t) = m$:

$$\begin{gathered} m = \widetilde{f}(y_4)\widetilde{h}(y_4) - \widetilde{f}(y_4)h(-1,x_4) - f(-1,x_4)\widetilde{h}(y_4) + m, \\ \widetilde{f}(y_4)\widetilde{h}(y_4) = \widetilde{f}(y_4)h(-1,x_4) + f(-1,x_4)\widetilde{h}(y_4). \end{gathered} \tag{24}$$

Now we apply the same trick, but this time we plug in $x_1 = 1$, $y_1 = -1$, to obtain

$$f(-1, \Delta'(1,x_4,-1,y_4)) = f(-1,y_4) + \widetilde{f}(x_4), \tag{25}$$
$$h(-1, \Delta'(1,x_4,-1,y_4)) = h(-1,y_4) - \widetilde{h}(x_4). \tag{26}$$

As before, we multiply (25) and (26):

$$\begin{gathered} m = m - f(-1,y_4)\widetilde{h}(x_4) + \widetilde{f}(x_4)h(-1,y_4) - \widetilde{f}(x_4)\widetilde{h}(x_4), \\ \widetilde{f}(x_4)h(-1,y_4) = f(-1,y_4)\widetilde{h}(x_4) + \widetilde{f}(x_4)\widetilde{h}(x_4). \end{gathered} \tag{27}$$

Let us switch x_4 with y_4 in (27):

$$\widetilde{f}(y_4)h(-1,x_4) = f(-1,x_4)\widetilde{h}(y_4) + \widetilde{f}(y_4)\widetilde{h}(y_4). \tag{28}$$

When we add equations (24) and (28), cancellations occur and we are left with

$$f(-1,x_4)\widetilde{h}(y_4) = 0 \quad \text{for all } x_4, y_4 \in \mathbb{R}.$$

If $\widetilde{h} \not\equiv 0$ then $f(-1,x_4) \equiv 0$ and we see from (28) that

$$\widetilde{f}(y_4) \cdot (\widetilde{h}(y_4) - h(-1,x_4)) \equiv 0. \tag{29}$$

By assumption, $\widetilde{f}(y_4)$ is not constant; therefore we can find $y_4 = \xi$ so that $\widetilde{f}(\xi) \neq 0$. Then (29) implies that $h(-1,x_4) = \widetilde{h}(\xi)$, i.e., $h(-1,x_4)$ is a constant function. Moreover, its constant value is equal to $\widetilde{h}(\xi)$ for any ξ such that $\widetilde{f}(\xi) \neq 0$. Hence $\widetilde{h}$ is constant on the set $\{\xi : \widetilde{f}(\xi) \neq 0\}$. However, we have earlier observed that $\widetilde{h}(z) = 0$ whenever $\widetilde{f}(z) = 0$. From the continuity of $\widetilde{h}$ it then follows that $\widetilde{h} \equiv 0$. From Lemma 1 we get $h(x_1,x_4) = b(x_1 - x_1^3)$ for some constant $b \in \mathbb{R}$.

It remains to determine f. A substitution $\widetilde{h} = 0$ in (14) yields

$$\widetilde{f}(\widetilde{\Delta}(x_4,y_4)) = \widetilde{f}(y_4) + \widetilde{f}(x_4),$$
$$\widetilde{\Delta}(x_4,y_4) = y_4 + x_4 + 9\widetilde{f}(x_4)\widetilde{f}(y_4)^2 + 9\widetilde{f}(x_4)^2\widetilde{f}(y_4).$$

Moreover, $h(-1,0) = 0$, hence from (17) it follows that $\widetilde{f}(0) = 0$, i.e., $\widetilde{f}$ satisfies the assumptions of Lemma 4. As $\widetilde{f}$ is not constant, it follows that it is a homeomorphism with $\widetilde{f}^{-1}(t) = 3t^3 + 6ct$ for some $c \geq 0$.

Now we return to equations (9) and (11) and use the formula $h(x_1,x_4) = b(x_1 - x_1^3)$:

$$f(x_1y_1, \Delta'(x_1,x_4,y_1,y_4)) = x_1 f(y_1,y_4) + y_1^2 f(x_1,x_4), \tag{30}$$

$$\begin{aligned}\Delta'(x_1,x_4,y_1,y_4) = {} & x_1y_4 + x_4y_1^4 + 4b(y_1^2 - y_1^4)f(x_1,x_4) \\ & + 6by_1^2(x_1 - x_1^3)f(y_1,y_4) \\ & + 9f(x_1,x_4)f(y_1,y_4)^2 + \frac{9y_1^2}{x_1}f(x_1,x_4)^2 f(y_1,y_4).\end{aligned} \tag{31}$$

When we set $x_1 = y_1^{-1}$, we get

$$\widetilde{f}(\Delta'(y_1^{-1},x_4,y_1,y_4)) = y_1^{-1}f(y_1,y_4) + y_1^2 f(y_1^{-1},x_4),$$
$$\begin{aligned}\Delta'(y_1^{-1},x_4,y_1,y_4) = {} & y_1^{-1}y_4 + x_4y_1^4 + 4b(y_1^2 - y_1^4)f(y_1^{-1},x_4) \\ & + 6b(y_1 - y_1^{-1})f(y_1,y_4) \\ & + 9f(y_1^{-1},x_4)f(y_1,y_4)^2 + 9y_1^3 f(y_1^{-1},x_4)^2 f(y_1,y_4).\end{aligned}$$

We apply $\widetilde{f}^{-1}$ to the first equation:

$$\Delta'(y_1^{-1}, x_4, y_1, y_4) = 3[y_1^{-1}f(y_1, y_4) + y_1^2 f(y_1^{-1}, x_4)]^3 \\ + 6c[y_1^{-1}f(y_1, y_4) + y_1^2 f(y_1^{-1}, x_4)].$$

After using the formula for Δ', expanding the third power and some cancellations, we get

$$y_1^4 x_4 + [4b(y_1^2 - y_1^4) - 6cy_1^2]f(y_1^{-1}, x_4) - 3y_1^6 f(y_1^{-1}, x_4)^3 = \\ - y_1^{-1}y_4 + [6cy_1^{-1} - 6b(y_1 - y_1^{-1})]f(y_1, y_4) + 3y_1^{-3}f(y_1, y_4)^3.$$

Notice that the left hand side does not depend on y_4, while the right hand side does not depend on x_4. Hence both sides depend on y_1 only:

$$l(y_1^{-1}) = y_1^4 x_4 + [4b(y_1^2 - y_1^4) - 6cy_1^2]f(y_1^{-1}, x_4) - 3y_1^6 f(y_1^{-1}, x_4)^3, \tag{32}$$

$$r(y_1) = -y_1^{-1}y_4 + [6cy_1^{-1} - 6b(y_1 - y_1^{-1})]f(y_1, y_4) + 3y_1^{-3}f(y_1, y_4)^3. \tag{33}$$

Let us plug $y_1 = u^{-1}$, $x_4 = v$ in (32) and $y_1 = u$, $y_4 = v$ in (33):

$$l(u) = u^{-4}v + [4b(u^{-2} - u^{-4}) - 6cu^{-2}]f(u, v) - 3u^{-6}f(u, v)^3,$$

$$r(u) = -u^{-1}v + [6cu^{-1} - 6b(u - u^{-1})]f(u, v) + 3u^{-3}f(u, v)^3.$$

Notice that $u^3 l(u) + r(u) = (-2b - 6c)(u - u^{-1})f(u, v)$, which implies that the expression $(b + 3c)(u - u^{-1})f(u, v)$ does not depend on v. If $b + 3c \neq 0$ then for each fixed $u \neq \pm 1$ the function $f(u, \cdot)$ is constant. But then for any $v_1 \neq v_2$ we get $\widetilde{f}(v_1) = f(1, v_1) = \lim f(1 + 1/n, v_1) = \lim f(1 + 1/n, v_2) = f(1, v_2) = \widetilde{f}(v_2)$, a contradiction, as $\widetilde{f}$ is a bijection. Therefore $b = -3c$ and

$$r(u) = -u^{-1}v + 6c(3u - 2u^{-1})f(u, v) + 3u^{-3}f(u, v)^3.$$

Let $p(u) = \frac{2}{3}c(3u - 2u^{-1})$ and $q(u, v) = \frac{1}{6}(u^{-1}v + r(u))$. Then

$$u^{-3}f(u, v)^3 = -3p(u)f(u, v) + 2q(u, v). \tag{34}$$

We use (34) in the cube of equation (30):

$$\begin{aligned}(x_1y_1)^{-3}(x_1 f(y_1, y_4) + y_1^2 f(x_1, x_4))^3 &= (x_1y_1)^{-3}f(x_1y_1, \Delta')^3 \\ &= -3p(x_1y_1)f(x_1y_1, \Delta') + 2q(x_1y_1, \Delta') \\ &= -3p(x_1y_1)[x_1 f(y_1, y_4) + y_1^2 f(x_1, x_4)] \\ &\quad + \frac{1}{3}r(x_1y_1) + \frac{1}{3}(x_1y_1)^{-1}\Delta'(x_1, x_4, y_1, y_4).\end{aligned}$$

When we expand the left hand side and use (31), we get, after reductions,

$$\begin{aligned}&(x_1y_1)^{-3}(x_1^3 f(y_1, y_4)^3 + y_1^6 f(x_1, x_4)^3) \\ &\quad = -3p(x_1y_1)(x_1 f(y_1, y_4) + y_1^2 f(x_1, x_4)) + \frac{1}{3}r(x_1y_1) + \frac{1}{3}(x_1y_1)^{-1} \\ &\qquad \times [x_1y_4 + x_4y_1^4 - 12c(y_1^2 - y_1^4)f(x_1, x_4) - 18cy_1^2(x_1 - x_1^3)f(y_1, y_4)].\end{aligned}$$

After applying (34) to the left hand side, we get, after reductions,

$$\begin{aligned}\frac{1}{3}r(y_1) + \frac{1}{3}y_1^3 r(x_1) - \frac{1}{3}r(x_1y_1) \\ &= \big(3p(x_1)y_1^3 - 3p(x_1y_1)y_1^2 - 4cx_1^{-1}(y_1 - y_1^3)\big)f(x_1,x_4) \\ &\quad + \big(3p(y_1) - 3p(x_1y_1)x_1 - 6cy_1(1-x_1^2)\big)f(y_1,y_4) \equiv 0,\end{aligned}$$

hence we are left with

$$r(x_1y_1) = y_1^3 r(x_1) + r(y_1).$$

By Lemma 1, we have $r(x_1) = 6d(1-x_1^3)$ for some constant d.

Because

$$f(x_1,x_4)^3 + 3x_1^3 p(x_1) f(x_1,x_4) - 2x_1^3 q(x_1,x_4) = 0$$

where $p(x_1) = \frac{2}{3}c(3x_1 - 2x_1^{-1})$ and $q(x_1,x_4) = \frac{1}{6}x_1^{-1}x_4 + d(1-x_1^3)$, by the Cardano formulas we get

$$f(x_1,x_4) = x_1\sqrt[3]{q+\sqrt{q^2+p^3}} + x_1\sqrt[3]{q-\sqrt{q^2+p^3}}.$$

Finally,

$$\begin{aligned}g(x_1,x_4) &= \frac{3}{2x_1}f(x_1,x_4)^2 - 3c(x_1-x_1^3) \\ &= \frac{3}{2}x_1\Big[\sqrt[3]{2q^2+p^3+2q\sqrt{q^2+p^3}} + \sqrt[3]{2q^2+p^3-2q\sqrt{q^2+p^3}}\Big] \\ &\quad + c(4 - 3x_1 - 6x_1^2 + 3x_1^3). \ \blacksquare\end{aligned}$$

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