COLLOQUIUM MATHEMATICUM

VOL. 119 2010 NO. 2

LIMITING BEHAVIORS OF THE BROWNIAN MOTIONS ON HYPERBOLIC SPACES

BY

H. MATSUMOTO (Nagoya)

Abstract. Using explicit representations of the Brownian motions on hyperbolic spaces, we show that their almost sure convergence and the central limit theorems for the radial components as time tends to infinity can be easily obtained. We also give a straightforward strategy to obtain explicit expressions for the limit distributions or Poisson kernels.

1. Introduction. Hyperbolic spaces are non-compact Riemannian symmetric spaces of rank one. They are classified into four types: the real one $\mathbb{H}_r^n = SO_0(1,n)/SO(n)$, the complex one $\mathbb{H}_c^n = SU(1,n)/SU(n)$, the quaternionic one $\mathbb{H}_q^n = Sp(1,n)/(Sp(1)\times Sp(n))$ and the Cayley hyperbolic plane. In this article we consider the limiting behaviors of the Brownian motions, that is, the diffusion processes generated by the Laplace–Beltrami operators, on the first three types of hyperbolic spaces.

Hyperbolic spaces have negative bounded curvature. Brownian motions on negatively curved manifolds have been studied by many authors in connection with the so-called Liouville property, and it is well known that they tend to infinity almost surely as time tends to infinity. See, e.g., Kifer [18]. Needless to say, the limit distributions are given by the Poisson kernels.

On the other hand, the Brownian motions on Riemannian symmetric spaces of non-compact type have also been studied by several authors since the work by Malliavin–Malliavin [22]. Among them, we refer to Babillot [3], where a central limit theorem for the radial components of the Brownian motions has been shown.

The purpose of this article is to show that the well-known properties of the Brownian motions on the manifolds mentioned above are easily and directly obtained for hyperbolic spaces if we adopt the upper half space realizations of the spaces instead of the ball models. Moreover, we obtain explicit expressions for the Poisson kernels, which are known in harmonic analysis, in a probabilistic manner as the densities of the limit distributions.

2010 *Mathematics Subject Classification*: 58J65, 60J60.

Key words and phrases: Brownian motion, hyperbolic space, Laplace–Beltrami operator, geometric Brownian motion.

DOI: 10.4064/cm119-2-3

We will see that the same procedure may be applied to the three types of spaces, including $\mathbb{H}_q^n$ whose geometry is complicated and has not been studied in detail yet. First, by solving the corresponding stochastic differential equations, we represent the Brownian motions in closed forms as Wiener functionals. Then their almost sure convergence is readily seen from the representations. Moreover, by inserting the representations into the formulae for the distance functions, we can also show the central limit theorems for the radial components.

For the computations of the limiting distributions or Poisson kernels, we need some results on the distributions of the random variables defined by perpetual (infinite) integrals over time of the usual geometric Brownian motions with negative drifts. These auxiliary results are given in the Appendix; by using them, we compute the Fourier transforms of the limiting distributions and the inverse transforms in direct ways.

2. Real hyperbolic spaces. For $n \geq 1$, let $\mathbb{H}_r^{n+1}$ be the upper half space in $\mathbb{R}^{n+1}$,

$$\mathbb{H}_r^{n+1} = \{z = (x, y) = (x_1, \ldots, x_n, y) : x \in \mathbb{R}^n,\ y > 0\},$$

endowed with the Riemannian metric $ds^2 = y^{-2}(dx^2 + dy^2)$. The volume element is given by $y^{-n-1}dxdy$, and the distance function $d(z, z')$ is given by

$$\cosh(d(z, z')) = \frac{|x - x'|^2 + y^2 + (y')^2}{2yy'}, \tag{2.1}$$

in an obvious notation, where $|x|$ is the Euclidean norm. The Laplace–Beltrami operator is written as

$$\Delta_r = y^2 \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + y^2 \frac{\partial^2}{\partial y^2} - (n-1)y \frac{\partial}{\partial y}.$$

For details on the fundamental objects on $\mathbb{H}_r^{n+1}$, see, e.g., Davies [7].

We first explicitly express the Brownian motion on $\mathbb{H}_r^{n+1}$ as a Wiener functional by solving the corresponding stochastic differential equation. Let $(W^{(n+1)}, \mathcal{B}^{(n+1)}, P^{(n+1)})$ be the $(n+1)$-dimensional standard Wiener space with the canonical filtration $\{\mathcal{B}_s^{(n+1)}\}_{t\geq 0}$. Corresponding to the rectangular coordinates, we denote an element of $W^{(n+1)}$ by

$$(w(\cdot), B(\cdot)) \quad \text{or} \quad (w_1(\cdot), \ldots, w_n(\cdot), B(\cdot)),$$

which is an $\mathbb{R}^{n+1}$-valued continuous function on $[0, \infty)$ with $w_i(0) = B(0) = 0$. Then the *Brownian motion* on $\mathbb{H}_r^{n+1}$, the diffusion process with infinitesimal generator $\Delta_r/2$, is obtained by solving the following system of stochastic differential equations defined on $(W^{(n+1)}, \mathcal{B}^{(n+1)}, P^{(n+1)})$ (see [17]):

$$dX_i(t) = Y(t)dw_i(t), \qquad i = 1, \dots, n,$$
$$dY(t) = Y(t)dB(t) - \frac{n-1}{2} Y(t)dt.$$

The unique solution $Z_z = \{(X(t,z), Y(t,z))\}_{t\geq 0}$, $z = (x,y)$, satisfying $X(0) = x$ and $Y(0) = y$ is given by

$$X_i(t,z) = x_i + \int_0^t y \exp(B_s^{(-\mu)})dw_i(s), \qquad Y(t,z) = y\exp(B_t^{(-\mu)}),$$

where $B_s^{(-\mu)} = B(s) - \mu s$ and $\mu = n/2$. $\{Y(t,z)\}$ is the usual geometric Brownian motion with negative drift and it is easy to see that $Z_z(t)$ converges to the boundary as $t \to \infty$ almost surely.

Now we consider the exponential functional $A_t^{(-\mu)}$ given by

$$A_t^{(-\mu)} = \int_0^t \exp(2B_s^{(-\mu)})\, ds.$$

Then it is easy to obtain the identity in law

$$(X(t,z), Y(t,z)) \overset{\text{(law)}}{=} (x + yw\big(A_t^{(-\mu)}\big), y\exp(B_t^{(-\mu)}))$$

for fixed $t > 0$.

An explicit expression for the density of the distribution of $(A_t^{(-\mu)}, B_t^{(-\mu)})$ was given by Yor [29]. By using it, Gruet [14] has shown an expression for the heat kernel of the semigroup generated by Δ_r. For the classical expression, see Davies [7]. We also refer to [2, 16, 23, 25] for related topics.

We combine the identity in law with formula (2.1) to get

$$\cosh(d(Z(t,z),z)) \overset{\text{(law)}}{=} \frac{1}{2}\{|w(A_t^{(-\mu)})|^2 + 1\}\exp(-B_t^{(-\mu)}) + \frac{1}{2}\exp(B_t^{(-\mu)}).$$

Since $A_t^{(-\mu)}$ converges as $t \to \infty$ and $\log(\cosh(u)) = u \cdot (1 + o(1))$ as $u \to \infty$, we readily get the following central limit theorem.

THEOREM 2.1. *The probability distribution of* $t^{-1/2}(d(Z(t,z),z) - nt/2)$ *converges weakly as* $t \to \infty$ *to the standard normal distribution.*

Recall the formula $\Delta_r d(z_0, \cdot) = n \coth d(z_0, \cdot)$. Then, by Itô's formula,

$$\begin{aligned} d(Z_z(t), z) = {} & \sum_{i=1}^n \int_0^t \frac{1}{\sinh d(z, Z_z(s))} \frac{X_z^i(s) - x}{y}\, dw_s^i \\ & + \int_0^t \frac{1}{\sinh d(z, Z_z(s))} \left(\frac{Y_z(s)}{y} - \cosh d(z, Z_z(s)) \right) dB(s) \\ & + \frac{n}{2} \int \coth d(z, Z_z(s))\, ds, \end{aligned}$$

from which the theorem may also be deduced.

Next we recall Dufresne's identity (Theorem A.1 in the Appendix), $A_\infty^{(-\mu)} \stackrel{\text{(law)}}{=} (2\gamma_\mu)^{-1}$ for a gamma random variable γ_μ with parameter μ. Then, for a bounded continuous function φ on $\mathbb{R}^n$, we obtain

$$\begin{aligned} E[\varphi(X(t,z))] &= E[\varphi(x + yw(A_t^{(-n/2)}))] \\ &\to \int_0^\infty \frac{1}{\Gamma(n/2)}\, t^{(n/2)-1} e^{-t}\, dt \int_{\mathbb{R}^n} \varphi(x+\eta)\, \frac{1}{(2\pi y^2/2t)^{n/2}} \exp\Big(-\frac{|\eta|^2}{2y^2/2t}\Big) d\eta \\ &= \int_{\mathbb{R}^n} \varphi(\xi)\, d\xi \int_0^\infty \frac{1}{\Gamma(n/2)}\, \frac{1}{\pi^{n/2} y^n}\, t^{n-1} \exp\Big(-\frac{y^2 + |\xi - x|^2}{y^2}\, t\Big)\, dt \\ &= \int_{\mathbb{R}^n} \varphi(\xi) p_{n+1}(\xi - x, y)\, d\xi, \end{aligned}$$

where

$$p_{n+1}(\xi, y) = \frac{2^{n-1}\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y^n}{(y^2 + |\xi|^2)^n}, \qquad \xi \in \mathbb{R}^n,$$

and we have used the duplication formula for the gamma function.

Hence we have proved the following.

Theorem 2.2. *For any $(x,y) \in \mathbb{H}_r^{n+1}$, $X(t,z)$ converges almost surely as $t \to \infty$, and the density of the limit distribution is the Poisson kernel $p_{n+1}(\xi - x, y)$. In particular, when $n = 1$, the limit distribution is Cauchy.*

We end this section by a remark on the Poisson kernel in Euclidean spaces and on Fourier transforms. The Poisson kernel on $\mathbb{R}^{n+1}$ of the hyperplane $\{y = 0\}$ is given by

$$q_{n+1}(\xi, y) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y}{(y^2 + |\xi|^2)^{(n+1)/2}},$$

which is different from $p_{n+1}(\xi, y)$ for $n \geq 2$. The Brownian motion on the hyperbolic plane $\mathbb{H}^2$ is a time change of the 2-dimensional standard Brownian motion, and the Poisson kernels coincide.

It is well known that the Fourier transform of $q_{n+1}(\xi, y)$ in ξ is the simple exponential function,

$$\int_{\mathbb{R}^n} e^{\sqrt{-1}\,\langle \lambda, \xi \rangle} q_{n+1}(\xi, y)\, d\xi = e^{-y|\lambda|}.$$

For hyperbolic spaces, we can show, for example,

$$\begin{aligned} \varphi_3(\lambda; y) &\equiv \int_{\mathbb{R}^2} e^{\sqrt{-1}\,\langle \lambda, \xi \rangle} p_3(\xi, y)\, d\xi = y|\lambda| K_1(y|\lambda|), \\ \varphi_4(\lambda; y) &\equiv \int_{\mathbb{R}^3} e^{\sqrt{-1}\,\langle \lambda, \xi \rangle} p_4(\xi, y)\, d\xi = (y|\lambda| + 1) e^{-y|\lambda|}, \end{aligned}$$

where K_1 is the modified Bessel function. By the strong Markov property, we can easily show that the distribution of $X(\tau_a)$ for the first hitting time τ_a of the Brownian motion $\{Z_z(t,z)\}$ at the level $y=a$, $a>0$, is determined by the characteristic function

$$E[\exp(\sqrt{-1}\,\langle\lambda, X(\tau_a)\rangle)] = e^{\sqrt{-1}\,\langle\lambda,x\rangle}\,\frac{\varphi_n(\lambda;y)}{\varphi_n(\lambda;a)}, \qquad \lambda\in\mathbb{R}^n.$$

See [4] for related topics. See also the recent paper by Byczkowski–Małecki [6] and the references cited therein for the Poisson kernels of balls in $\mathbb{H}_r^n$.

3. Complex hyperbolic spaces. Let $\mathbb{H}_c^n$, $n\ge 2$, be the upper half space of $\mathbb{C}^n$ given by

$$\{z=(z_1,z_2,\dots,z_n)=(z_1,\widetilde{z})\in\mathbb{C}^n : h(z)\equiv \mathrm{Im}(z_1)-|\widetilde{z}|^2>0\},$$

endowed with the Bergmann metric

$$ds^2 = -\sum_{j,k=1}^n \partial_{z_j}\partial_{\overline{z_k}}\log(h)dz_j d\overline{z_k}.$$

The unit ball $\{|z|<1\}$ in $\mathbb{C}^n$ with the Bergmann metric

$$-\sum_{j,k=1}^n \partial_{z_j}\partial_{\overline{z_k}}\log(1-|z|^2)dz_j d\overline{z_k}$$

is isometric with $\mathbb{H}_c^n$. For details, we refer to [8, 9, 11, 26]. We should be aware of different conventions. The curvatures of these manifolds are bounded and negative, but are not constant (cf. [11, p. 190]). See also the recent works by Graczyk and Żak [12, 31] on the Brownian motions on $\mathbb{H}_c^n$, where the unit ball model has been adopted.

We stick to the upper half space model and change the first coordinate to $x_1=\mathrm{Re}(z_1)/2$ and $y=h(z)^{1/2}$. Then we have the same realization of the complex hyperbolic space $SU(1,n)/SU(n)$ as in Venkov [26]: if we write $z_k=x_k+\sqrt{-1}\,y_k$, $k=2,\dots,n$, the Riemannian metric is written as

$$ds^2=\frac{1}{y^2}\,dy^2+\frac{1}{y^2}\sum_{k=2}^n(dx_k^2+dy_k^2)+\frac{1}{y^4}\Big(dx_1+\sum_{k=2}^n(x_k dy_k-y_k dx_k)\Big)^2,$$

for $y>0$, $x_i,y_i\in\mathbb{R}$, and the distance function $d(z,z')$ is given by

$$(\cosh(d(z,z')))^2=\frac{((y')^2+\Phi)^2+4\varphi^2}{4y^2(y')^2},$$

where

$$\Phi=y^2+|\widetilde{z}'-\widetilde{z}|^2 \quad\text{and}\quad \varphi=x_1'-x_1+\sum_{k=2}^n(y_k'x_k-x_k'y_k). \tag{3.1}$$

The Laplace–Beltrami operator is given by

$$\Delta_c = y^4 \frac{\partial^2}{\partial x_1^2} + y^2 \frac{\partial^2}{\partial y^2} - (2n-1) y \frac{\partial}{\partial y} + y^2 \sum_{k=2}^{n} \left\{ \left(\frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial y_k} - x_k \frac{\partial}{\partial x_1} \right)^2 \right\}. \tag{3.2}$$

Let $(W^{(2n)}, \mathcal{B}^{(2n)}, P^{(2n)})$ be the $2n$-dimensional standard Wiener space. Denoting an element of $W^{(2n)}$ by

$$(B(\cdot), w_2(\cdot), w_3(\cdot), \ldots, w_{2n}(\cdot)) \quad \text{or} \quad (B(\cdot), w_2(\cdot), \widetilde{w}(\cdot)),$$

we can check that the Brownian motion $\{Z(t)\}$ on $\mathbb{H}_c^n$ with $Z(0) = (x_1, y, z_2, \ldots, z_n)$, z_k being identified with (x_k, y_k), is given by

$$\begin{aligned} X_1(t) &= x_1 + \int_0^t Y(s)^2 \, dw_2(s) + 2 \sum_{k=2}^{n} S_k(t) \\ Y(t) &= y \exp(B(t) - nt), \\ X_k(t) &= x_k + \int_0^t Y(s) \, dw_{2k-1}(s), \\ Y_k(t) &= y_k + \int_0^t Y(s) \, dw_{2k}(s), \qquad k = 2, \ldots, n, \end{aligned} \tag{3.3}$$

where we have used the obvious notations $X_1(t), Y(t), X_k(t), Y_k(t)$ for the components of $Z(t)$, and $S_k(t)$ is the stochastic area enclosed by the curve $\{(X_k(s), Y_k(s))\}_{0 \le s \le t}$ and its chord,

$$S_k(t) = \frac{1}{2} \int_0^t (Y_k(s) \, dX_k(s) - X_k(s) \, dY_k(s)).$$

$\{Y(t)\}$ is again the usual geometric Brownian motion with negative drift and $Z(t)$ converges as $t \to \infty$. Hence we easily obtain the following central limit theorem.

THEOREM 3.1. *For the Brownian motion $\{Z(t)\}$ on the n-dimensional complex hyperbolic space, the probability law of $t^{-1/2}(d(Z(t), Z(0)) - nt)$ converges weakly as $t \to \infty$ to the standard normal distribution.*

Next we compute the limiting distribution of $(X_1(t), \widetilde{Z}(t))$ as $t \to \infty$, where

$$\widetilde{Z}(t) = (X_2(t), Y_2(t), \ldots, X_n(t), Y_n(t)).$$

If we consider the ball model, we obtain the Poisson kernels as the densities of the image measures of the uniform measure on the sphere by the isometries. However, since the same strategy works in the more complicated case of the quaternionic hyperbolic space whose geometry is not well understood yet (see a recent work by Kim–Parker [19] and references cited therein), we give the following straightforward computations.

We first fix t and consider the characteristic function. As in the previous section, we set

$$A_t^{(-\mu)} = \int_0^t e^{2B_s^{(-\mu)}}\,ds, \qquad \widetilde{A}_t^{(-\mu)} = \int_0^t e^{4B_s^{(-\mu)}}\,ds,$$

$B_s^{(-\mu)} = B(s) - \mu s$ and $\mu = n$. For the stochastic analysis on $\mathbb{H}_c^n$ and $\mathbb{H}_q^n$, we need to consider these two exponential functionals. By (3.3), it is easy to see that, for fixed $t > 0$, $(X_1(t), \widetilde{Z}(t))$ is identical in law with

$$(x_1 + y^2 w_2(\widetilde{A}_t^{(-\mu)}) + y\phi(A_t^{(-\mu)}) + 2y^2 \sum \widetilde{S}_k(A_t^{(-\mu)}), \widetilde{z} + y\widetilde{w}(A_t^{(-\mu)})),$$

where $\sum$ denotes the sum over $k = 2, \dots, n$, $\phi(t) = \sum(y_k w_{2k-1}(t) - x_k w_{2k}(t))$ and

$$\widetilde{S}_k(t) = \frac{1}{2}\int_0^t (w_{2k}(s)\,dw_{2k-1}(s) - w_{2k-1}(s)\,dw_{2k}(s)).$$

Hence we may write, for any bounded continuous function g on $\mathbb{R}^{2(n-1)}$,

$$E[e^{\sqrt{-1}\,pX_1(t)} g(\widetilde{Z}(t))] = E\Big[e^{\sqrt{-1}\,p(x_1 + y^2 w_2(\widetilde{A}_t^{(-\mu)}) + y\phi(A_t^{(-\mu)}))} g(\widetilde{z} + y\widetilde{w}(A_t^{(-\mu)})) \times E\Big[\prod_{k=2}^n e^{2\sqrt{-1}\,py^2 \widetilde{S}_k(A_t^{(-\mu)})} \,\Big|\, \{B(s)\}_{s\geq 0}, \widetilde{w}(A_t^{(-\mu)})\Big]\Big].$$

Then, applying the Lévy formula for the characteristic function of the stochastic area (cf. [17, p. 473]), we get

$$E[e^{\sqrt{-1}\,pX_1(t)} g(\widetilde{Z}(t))] = E\Big[e^{\sqrt{-1}\,p(x_1 + y^2 w_2(\widetilde{A}_t^{(-\mu)}) + y\phi(A_t^{(-\mu)}))} g(\widetilde{z} + y\widetilde{w}(A_t^{(-\mu)})) \times \Big(\frac{py^2 A_t^{(-\mu)}}{\sinh(py^2 A_t^{(-\mu)})}\Big)^{n-1} \exp\Big((1 - py^2 A_t^{(-\mu)} \coth(py^2 A_t^{(-\mu)})) \frac{|\widetilde{w}(A_t^{(-\mu)})|^2}{2A_t^{(-\mu)}}\Big)\Big].$$

Moreover we take the conditional expectation given $\{B(s)\}_{s\geq 0}$ or $\{Y(s)\}_{s\geq 0}$ to obtain

$$E[e^{\sqrt{-1}\,pX_1(t)} g(\widetilde{Z}(t))] = e^{\sqrt{-1}\,px_1} E\Big[e^{-p^2 y^4 \widetilde{A}_t^{(-\mu)}/2} \int_{\mathbb{R}^{2(n-1)}} e^{\sqrt{-1}\,p\sum(y_k\xi_k - x_k\eta_k)} \times g(\widetilde{z} + \zeta)\Big(\frac{p}{2\pi\sinh(py^2 A_t^{(-\mu)})}\Big)^{n-1} e^{-p\coth(py^2 A_t^{(-\mu)})|\zeta|^2/2}\,d\xi\,d\eta\Big],$$

where $\zeta = (\xi, \eta) = (\xi_2, \eta_2, \dots, \xi_n, \eta_n)$.

Now we put, for $\mathbf{q}=(q_2,\dots,q_n),\mathbf{r}=(r_2,\dots,r_n)\in\mathbb{R}^{n-1}$,

$$g(\zeta)=\exp(\sqrt{-1}\,(\langle\mathbf{q},\xi\rangle+\langle\mathbf{r},\eta\rangle)).$$

Then, carrying out the Gaussian integration with respect to ξ and η, we get

$$\begin{aligned}&E\Big[\exp\Big\{\sqrt{-1}\,(pX_1(t)+\sum(q_kX_k(t)+r_kY_k(t)))\Big\}\Big]\\&=e^{\sqrt{-1}\,f}E\bigg[e^{-p^2y^4\widetilde{A}_t^{(-\mu)}/2}\bigg(\frac{1}{\cosh(py^2A_t^{(-\mu)})}\bigg)^{n-1}e^{-F\tanh(py^2A_t^{(-\mu)})}\bigg],\end{aligned}$$

where

$$\begin{aligned}f&=f(p,\mathbf{q},\mathbf{r})=px_1+\sum(q_kx_k+r_ky_k),\\F&=F(p,\mathbf{q},\mathbf{r})=\sum\frac{(q_k+py_k)^2+(r_k-px_k)^2}{2p}.\end{aligned}$$

Now, letting $t\to\infty$, we obtain the following.

PROPOSITION 3.2. *For any $p\in\mathbb{R}$ and $\mathbf{q},\mathbf{r}\in\mathbb{R}^{n-1}$,*

$$\begin{aligned}(3.4)\qquad&\lim_{t\to\infty}E\Big[\exp\Big\{\sqrt{-1}\,(pX_1(t)+\sum(q_kX_k(t)+r_kY_k(t)))\Big\}\Big]\\&=e^{\sqrt{-1}\,f}E\bigg[e^{-p^2y^4\widetilde{A}_\infty^{(-n)}/2}\bigg(\frac{1}{\cosh(py^2A_\infty^{(-n)})}\bigg)^{n-1}e^{-F\tanh(py^2A_\infty^{(-n)})}\bigg].\end{aligned}$$

Denote the right hand side of (3.4) by $I(p,\mathbf{q},\mathbf{r})$. By using the joint Laplace transform of $A_\infty^{(-n)}$ and $\widetilde{A}_\infty^{(-n)}$ given by Corollary A.7 in the Appendix, we obtain

$$\begin{aligned}I(p,\mathbf{q},\mathbf{r})&=e^{\sqrt{-1}\,f}\int_0^\infty E[e^{-p^2y^4\widetilde{A}_\infty^{(-n)}/2}\,|\,A_\infty^{(-n)}=u]\bigg(\frac{1}{\cosh(py^2u)}\bigg)^{n-1}\\&\qquad\qquad\times e^{-F\tanh(py^2u)}P(A_\infty^{(-n)}\in du)\\&=e^{\sqrt{-1}\,f}\int_0^\infty\frac{1}{2^n\Gamma(n)}\bigg(\frac{py^2}{\sinh(py^2u)}\bigg)^{n+1}e^{-py^2\coth(py^2u)/2}\\&\qquad\qquad\times\bigg(\frac{1}{\cosh(py^2u)}\bigg)^{n-1}e^{-F\tanh(py^2u)}\,du.\end{aligned}$$

Then, changing the variable, we see that $I(p,\mathbf{q},\mathbf{r})$ is equal to

$$\begin{aligned}&e^{\sqrt{-1}\,f}\,\frac{(py^2)^n}{2^n\Gamma(n)}\int_0^\infty\bigg(\frac{1}{\sinh(u)}\bigg)^{n+1}\bigg(\frac{1}{\cosh(u)}\bigg)^{n-1}e^{-py^2\coth(u)/2}\\&\qquad\times\exp\bigg(-\sum\frac{(q_k+py_k)^2+(r_k-px_k)^2}{2p}\tanh(u)\bigg)\,du\end{aligned}$$

if $p > 0$, and to

$$e^{\sqrt{-1}f}\,\frac{(-py^2)^n}{2^n\Gamma(n)}\int_0^\infty\left(\frac{1}{\sinh(u)}\right)^{n+1}\left(\frac{1}{\cosh(u)}\right)^{n-1}e^{py^2\coth(u)/2}$$
$$\times\exp\left(\sum\frac{(q_k+py_k)^2+(r_k-px_k)^2}{2p}\tanh(u)\right)du$$

if $p < 0$. From these expressions, we can take the Fourier inversion

$$f_n(x_1',\tilde{z}';z)\equiv\frac{1}{(2\pi)^{2n-1}}\int_{\mathbb{R}^{2n-1}}I(p,\mathbf{q},\mathbf{r})e^{-\sqrt{-1}\,(px_1'+\sum(q_kx_k'+r_ky_k'))}\,dp\,dq_2\cdots dr_n.$$

For the integral with respect to q_k when $p > 0$, we note as usual

$$-\frac{(q_k+py_k)^2}{2p}\tanh(u)+\sqrt{-1}\,q_k(x_k-x_k')$$
$$=-\frac{\tanh(u)}{2p}(q_k+py_k-\sqrt{-1}\,p(x_k-x_k')\coth(u))^2$$
$$-\sqrt{-1}\,py_k(x_k-x_k')-\frac{p}{2}(x_k-x_k')^2\coth(u).$$

We do the same computations also for the other variables and for $p < 0$. After some manipulations, we obtain

$$f_n(x_1',\tilde{z}';z)=\frac{y^{2n}}{(4\pi)^n\Gamma(n)}\int_{\mathbb{R}}|p|^{2n-1}e^{\sqrt{-1}\,\varphi p}dp\int_0^\infty\left(\frac{1}{\sinh(u)}\right)^{2n}e^{-\Phi|p|\coth(u)/2}\,du$$
$$=\frac{2y^{2n}}{\pi^n\Gamma(n)}\int_0^\infty p^{2n-1}\cos(2\varphi p)dp\int_0^\infty\left(\frac{1}{\sinh(u)}\right)^{2n}e^{-\Phi p\coth(u)}\,du,$$

where we have made a simple change of variable for the second equality, and φ and Φ are given by (3.1).

In the last integral, we change the variable to $k = \coth(u)$ to obtain

$$f_n(x_1',\tilde{z}';z)=\frac{2y^{2n}}{\pi^n\Gamma(n)}\int_0^\infty p^{2n-1}\cos(2\varphi p)\,dp\int_1^\infty e^{-\Phi pk}(k^2-1)^{n-1}\,dk.$$

Now we recall the following integral representation of the modified Bessel function (cf. Formula (5.10.24) in [20] or 3.387.3 in [13]):

$$K_\nu(z)=\frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{z}{2}\right)^\nu\int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2}\,dt,\qquad \nu>0. \tag{3.5}$$

Then we obtain

$$f_n(x_1',\tilde{z}';z)=\frac{2^{n+1/2}y^{2n}}{\pi^{n+1/2}\Phi^{n-1/2}}\int_0^\infty p^{n-1/2}\cos(2\varphi p)K_{n-1/2}(\Phi p)\,dp.$$

For the last integral, we may apply the formula

$$\int_0^\infty x^\lambda K_\mu(ax)\cos(bx)\,dx = 2^{\lambda-1}a^{-\lambda-1}\Gamma\Big(\frac{\mu+\lambda+1}{2}\Big)\Gamma\Big(\frac{1+\lambda-\mu}{2}\Big) \times {}_2F_1\Big(\frac{\mu+\lambda+1}{2},\frac{1+\lambda-\mu}{2};\frac{1}{2};-\frac{b^2}{a^2}\Big).$$

(cf. Formula 6.699.4 in [13]) and ${}_2F_1(n,a,a;z)=(1-z)^{-n}$, where ${}_2F_1$ is the Gauss hypergeometric function. Then we obtain

$$f_n(x_1',\widetilde{z}';z) = \frac{2^{2n-1}\Gamma(n)y^{2n}}{\pi^n\Phi^{2n}}\ {}_1F_0\Big(n;-\frac{4\varphi^2}{\Phi^2}\Big) = \frac{2^{2n-1}\Gamma(n)y^{2n}}{\pi^n(4\varphi^2+\Phi^2)^n}. \tag{3.6}$$

THEOREM 3.3 (cf. [8]). *For any $z\in\mathbb{H}_c^n$, $(X_1(t),\widetilde{Z}(t))$ converges almost surely as $t\to\infty$, and the density of the limit distribution on $\mathbb{R}^{2n-1}$ is the Poisson kernel given by* (3.6).

4. Quaternionic hyperbolic spaces. For the quaternionic hyperbolic space $Sp(1,n)/(Sp(1)\times Sp(n))$, $n\ge 2$, we follow the conventions in Venkov [26]. See also Helgason [15], Lohoué–Rychener [21], and Kim–Parker [19] for the basic properties. For $n\ge 2$, let $\mathbb{H}_q^n$ be the upper half space in $\mathbb{C}^{2n}$,

$$\mathbb{H}_q^n = \{z=(z_1,z_2,\dots,z_{2n})=(z_1,\widetilde{z})\in\mathbb{C}^{2n} : \mathrm{Im}(z_1)>0\},$$

with the Riemannian metric

$$\begin{aligned} ds^2 &= \frac{dy^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^n (dz_k d\overline{z_k} + dz_{n+k}d\overline{z_{n+k}}) \\ &\quad + \frac{1}{y^4}\Big(dx_1 + \mathrm{Im}\sum_{k=2}^n(\overline{z_k}dz_k + \overline{z_{n+k}}dz_{n+k})\Big)^2 \\ &\quad + \frac{1}{y^4}\Big|dz_{n+1} + \sum_{k=2}^n (z_{n+k}dz_k - z_k dz_{n+k})\Big|^2, \end{aligned}$$

where $z_1 = x_1+\sqrt{-1}\,y$. We will write $z_k = x_k+\sqrt{-1}\,y_k$ for $k=2,\dots,2n$. Note that the first and $(n+1)$th components, z_1 and z_{n+1}, play special roles.

The volume element is $y^{-4n-3}dx_1dy\prod_{k=2}^{2n}dx_kdy_k$ and the distance function $d(z,z')$ is given by

$$(\cosh(d(z,z')))^2 = \frac{((y')^2+\Phi)^2+4(\varphi_1^2+\varphi_2^2+\varphi_3^2)}{4y^2(y')^2}, \tag{4.1}$$

where

$$
(4.2)\quad
\begin{aligned}
\Phi &= y^2 + \sum_{k=2}^{n} (|z'_k - z_k|^2 + |z'_{n+k} - z_{n+k}|^2),\\
\varphi_1 &= x'_1 - x_1 + \sum_{k=2}^{n} ((y'_k x_k - x'_k y_k) + (y'_{n+k} x_{n+k} - x'_{n+k} y_{n+k})),\\
\varphi_2 &= x'_{n+1} - x_{n+1} + \sum_{k=2}^{n} ((x'_k x_{n+k} - x'_{n+k} x_k) + (y'_{n+k} y_k - y'_k y_{n+k})),\\
\varphi_3 &= y'_{n+1} - y_{n+1} + \sum_{k=2}^{n} ((x'_k y_{n+k} - y'_{n+k} x_k) + (y'_k x_{n+k} - x'_{n+k} y_k)).
\end{aligned}
$$

Note that φ_i's do not depend on y.

The Laplace–Beltrami operator Δ_q may be written in a convenient way as

$$
(4.3)\quad
\begin{aligned}
\Delta_q = {} & y^4 \frac{\partial^2}{\partial x_1^2} + y^2 \frac{\partial^2}{\partial y^2} - (4n+1) y \frac{\partial}{\partial y} + y^4 \Big(\frac{\partial^2}{\partial x_{n+1}^2} + \frac{\partial^2}{\partial y_{n+1}^2} \Big) \\
& + y^2 \sum_{k=2}^{n} \Big[\Big(\frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial x_1} - x_{n+k} \frac{\partial}{\partial x_{n+1}} - y_{n+k} \frac{\partial}{\partial y_{n+1}} \Big)^2 \\
& \qquad + \Big(\frac{\partial}{\partial y_k} - x_k \frac{\partial}{\partial x_1} + y_{n+k} \frac{\partial}{\partial x_{n+1}} - x_{n+k} \frac{\partial}{\partial y_{n+1}} \Big)^2 \\
& \qquad + \Big(\frac{\partial}{\partial x_{n+k}} + y_{n+k} \frac{\partial}{\partial x_1} + x_k \frac{\partial}{\partial x_{n+1}} + y_k \frac{\partial}{\partial y_{n+1}} \Big)^2 \\
& \qquad + \Big(\frac{\partial}{\partial y_{n+k}} - x_{n+k} \frac{\partial}{\partial x_1} - y_k \frac{\partial}{\partial x_{n+1}} + x_k \frac{\partial}{\partial y_{n+1}} \Big)^2 \Big].
\end{aligned}
$$

Note that the coefficients of $\partial^2/\partial x_1 \partial x_{n+1}, \partial^2/\partial x_1 \partial y_{n+1}, \partial^2/\partial x_{n+1} \partial y_{n+1}$ are zero. We can use the same procedure as for the complex hyperbolic space if we consider a 4×4 skew-symmetric matrix instead of 2-dimensional one.

First we give an explicit expression for the Brownian motion, the diffusion process with generator $\Delta_q/2$, on $\mathbb{H}_q^n$. Let $(W^{(4n)}, \mathcal{B}^{(4n)}, P^{(4n)})$ be the $4n$-dimensional Wiener space and denote an element in $W^{(4n)}$ by

$$
\begin{aligned}
&(B_1(\cdot), B(\cdot), w_{2,1}(\cdot), w_{2,2}(\cdot), \ldots, w_{n,1}(\cdot), w_{n,2}(\cdot), \\
&\qquad B_2(\cdot), B_3(\cdot), w_{n+2,1}(\cdot), w_{n+2,2}(\cdot), \ldots, w_{2n,1}(\cdot), w_{2n,2}(\cdot)).
\end{aligned}
$$

Then we can check that the Brownian motion $(X_1(t), Y(t), \widetilde{Z}(t))$ starting from $(x_1, y, \widetilde{z})$ is given by

$$
\begin{aligned}
X_1(t) &= x_1 + \int_0^t Y(s)^2\,dB_1(s) + \sum_{k=2}^n \int_0^t \{Y_k(s)\,dX_k(s) - X_k(s)\,dY_k(s) \\
&\qquad + Y_{n+k}(s)\,dX_{n+k}(s) - X_{n+k}(s)\,dY_{n+k}(s)\}, \\
Y(t) &= y\exp(B(t) - (2n+1)t), \\
X_k(t) &= x_k + \int_0^t Y(s)\,dw_{k,1}(s), \\
Y_k(t) &= y_k + \int_0^t Y(s)\,dw_{k,2}(s), \qquad k = 2, \dots, n, \\
X_{n+1}(t) &= x_{n+1} + \int_0^t Y(s)^2\,dB_2(s) \\
&\quad + \sum_{k=2}^n \int_0^t \{-X_{n+k}(s)\,dX_k(s) + X_k(s)\,dX_{n+k}(s) \\
&\qquad + Y_{n+k}(s)\,dY_k(s) - Y_k(s)\,dY_{n+k}(s)\}, \\
Y_{n+1}(t) &= y_{n+1} + \int_0^t Y(s)^2\,dB_3(s) \\
&\quad + \sum_{k=2}^n \int_0^t \{-Y_{n+k}(s)\,dX_k(s) + X_k(s)\,dY_{n+k}(s) \\
&\qquad - X_{n+k}(s)\,dY_k(s) + Y_k(s)\,dX_{n+k}(s)\}, \\
X_{n+k}(t) &= x_{n+k} + \int_0^t Y(s)\,dw_{n+k,1}(s), \\
Y_{n+k}(t) &= y_{n+k} + \int_0^t Y(s)\,dw_{n+k,2}(s), \qquad k = 2, \dots, n.
\end{aligned}
$$

Then, from (4.1), it is easy to show the following central limit theorem.

THEOREM 4.1. *The probability law of* $(d(Z(t), Z(0)) - (2n+1)t)/\sqrt{t}$ *converges weakly as* $t \to \infty$ *to the standard normal distribution.*

Next we show that $(X_1(t), \widetilde{Z}(t))$ converges in law as $t \to \infty$. To identify the limit distribution, we set

$$
f_n(x_1', \widetilde{z}'; z) = \frac{2^{4n+1}\Gamma(2n)}{\pi^{2n}} \frac{y^{2(2n+1)}}{(\Phi^2 + 4(\varphi_1^2 + \varphi_2^2 + \varphi_3^2))^{2n+1}},
$$

where Φ and φ_i's are given by (4.2). Then f_n is the Poisson kernel of the boundary $\partial\mathbb{H}_q^n = \{y = 0\}$.

THEOREM 4.2. *$(X_1(t), \widetilde{Z}(t))$, with values in $\mathbb{R} \times \mathbb{C}^{2n-1}$, converges almost surely as $t \to \infty$, and the density of the limit distribution is given by $f_n(x_1', \widetilde{z}'; z)$.*

In the following we give a proof of Theorem 4.2. First we consider the characteristic function of $(X_1(t), \widetilde{Z}(t))$ for fixed t. For convenience we put, for $k \neq 1, n+1$,

$$X_k^0(t) = X_k(t) - x_k = \int_0^t Y(s)\, dw_{k,1}(s),$$

$$Y_k^0(t) = Y_k(t) - y_k = \int_0^t Y(s)\, dw_{k,2}(s),$$

and

$$\theta_k = \begin{pmatrix} x_k \\ y_k \\ x_{n+k} \\ y_{n+k} \end{pmatrix}, \quad \Theta_k(t) = \begin{pmatrix} X_k(t) \\ Y_k(t) \\ X_{n+k}(t) \\ Y_{n+k}(t) \end{pmatrix}, \quad \Theta_k^0(t) = \begin{pmatrix} X_k^0(t) \\ Y_k^0 \\ X_{n+k}^0 \\ Y_{n+k}^0(t) \end{pmatrix}.$$

Moreover, for $\xi = {}^t(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $w_k = {}^t(u_k, v_k, u_{n+k}, v_{n+k}) \in \mathbb{R}^4$, and $w = (w_2, \ldots, w_n)$, we set

$$\Psi(t) = \xi_1 X_1(t) + \xi_2 X_{n+1}(t) + \xi_3 Y_{n+1}(t),$$
$$U_k(t) = \langle w_k, \Theta_k(t)\rangle, \qquad U_k^0(t) = \langle w_k, \Theta_k^0(t)\rangle.$$

Throughout, ${}^tQ$ denotes the transpose of a matrix Q. Then the characteristic function is

$$\begin{aligned}\varphi(t) = E\Big[\exp\Big\{&\sqrt{-1}\,(\xi_1 X_1(t) + \xi_2 X_{n+1}(t) + \xi_3 Y_{n+1}(t)) \\ &+ \sqrt{-1}\sum_{k=2}^{n}(u_k X_k(t) + v_k Y_k(t) + u_{n+k} X_{n+k}(t) + v_{n+k} Y_{n+k}(t))\Big\}\Big] \\ = E\Big[&\exp\Big(\sqrt{-1}\Big(\Psi(t) + \sum_{k=2}^{n} U_k(t)\Big)\Big].\end{aligned}$$

To compute it, we introduce the 4×4 skew symmetric matrix

$$\Xi = \begin{pmatrix} 0 & \xi_1 & -\xi_2 & -\xi_3 \\ -\xi_1 & 0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & 0 & \xi_1 \\ \xi_3 & -\xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

Then

$$\Psi(t)+\sum_{k=2}^{n}U_k(t)=\psi+\int_0^t Y(s)^2(\xi_1\,dB_1(s)+\xi_2\,dB_2(s)+\xi_3\,dB_3(s))$$
$$+\sum_{k=2}^{n}\langle \Xi\theta_k+w_k,\Theta_k^0(t)\rangle+\sum_{k=2}^{n}\int_0^t\langle \Xi\Theta_k^0(s),d\Theta_k^0(s)\rangle,$$

where $\psi=\xi_1x_1+\xi_2x_{n+1}+\xi_3y_{n+1}+\sum_{k=2}^n\langle w_k,\theta_k\rangle$. Note that $\{\sum_{j=1}^3\xi_jB_j(s)\}$ is identical in law with $\{|\xi|B_1(s)\}$, $|\xi|=(\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}$.

The eigenvalues of Ξ are $\pm\sqrt{-1}\,|\xi|$, each of multiplicity two. Moreover there exists an orthogonal matrix Q such that ${}^tQ\Xi Q=K$ is of the standard form. We take

$$Q=\begin{pmatrix}0 & \frac{\xi_1}{|\xi|} & \frac{\sqrt{\xi_2^2+\xi_3^2}}{|\xi|} & 0\\ 1 & 0 & 0 & 0\\ 0 & \frac{\xi_3}{|\xi|} & \frac{-\xi_1\xi_3}{\sqrt{\xi_2^2+\xi_3^2}|\xi|} & \frac{\xi_2}{\sqrt{\xi_2^2+\xi_3^2}}\\ 0 & \frac{-\xi_2}{|\xi|} & \frac{\xi_1\xi_2}{\sqrt{\xi_2^2+\xi_3^2}|\xi|} & \frac{\xi_3}{\sqrt{\xi_2^2+\xi_3^2}}\end{pmatrix}\quad\text{and}\quad K=\begin{pmatrix}0 & -|\xi| & 0 & 0\\ |\xi| & 0 & 0 & 0\\ 0 & 0 & 0 & -|\xi|\\ 0 & 0 & |\xi| & 0\end{pmatrix}.$$

We also put $\widehat{w}_k={}^tQw_k$, $\widehat{\theta}_k={}^tQ\theta_k$ and

$$\widehat{\Theta}_k^0(t)=(\widehat{X}_k^0(t),\widehat{Y}_k^0(t),\widehat{X}_{n+k}^0(t),\widehat{Y}_{n+k}^0(t))={}^tQ\Theta_k^0(t).$$

By the rotation invariance of the probability law of Brownian motions, we see that $\{\widehat{\Theta}_k^0(s)\}$ is a simple time change of a 4-dimensional standard Brownian motion.

Under these notations, we have

$$\langle \Xi\theta_k+w_k,\Theta_k^0(t)\rangle=\langle K\widehat{\theta}_k+\widehat{w}_k,\widehat{\Theta}_k^0(t)\rangle$$

and

$$(4.4)\qquad \int_0^t\langle \Xi\Theta_k^0(s),d\Theta_k^0(s)\rangle=\int_0^t\langle K\widehat{\Theta}_k^0(s),d\widehat{\Theta}_k^0(s)\rangle$$
$$=|\xi|\int_0^t\{\widehat{X}_k^0(s)\,d\widehat{Y}_k^0(s)-\widehat{Y}_k^0(s)\,d\widehat{X}_k^0(s)$$
$$+\widehat{X}_{n+k}^0(s)\,d\widehat{Y}_{n+k}^0(s)-\widehat{Y}_{n+k}^0(s)\,d\widehat{X}_{n+k}^0(s)\}.$$

As in the previous sections, we set

$$A_t^{(-\mu)}=\int_0^t\exp(2B_s^{(-\mu)})\,ds\quad\text{and}\quad \widetilde{A}_t^{(-\mu)}=\int_0^t\exp(4B_s^{(-\mu)}(s))\,ds,$$

$B_s^{(-\mu)}=B(s)-\mu s$ and $\mu=2n+1$. Then, by taking the conditional expectation given $\{Y(s)\}_{s\geq0}$ and $\widehat{\Theta}_k^0(t)$, $k=2,\dots,n$, and applying the Lévy formula, we obtain

$$\varphi(t) = e^{\sqrt{-1}\,\psi} E\Big[\exp\Big(-\frac{1}{2}|\xi|^2 y^4 \widetilde{A}_t^{(-\mu)} + \sqrt{-1}\sum_{k=2}^{n}\langle K\widehat{\theta}_k + \widehat{w}_k, \widehat{\Theta}_k(t)\rangle\Big)$$

$$\times \prod_{k=2}^{n} E\Big[\exp\Big(\sqrt{-1}\int_0^t \langle \Xi\Theta_k^0(s), d\Theta_k^0(s)\rangle\Big)\,\Big|\,\{Y(s)\}, \widehat{\Theta}_k(t)\Big]\Big]$$

$$= e^{\sqrt{-1}\,\psi} E\Big[\exp\Big(-\frac{1}{2}|\xi|^2 y^4 \widetilde{A}_t^{(-\mu)} + \sqrt{-1}\sum_{k=2}^{n}\langle K\widehat{\theta}_k + \widehat{w}_k, y\widehat{W}_k(A_t^{(-\mu)})\rangle\Big)$$

$$\times \Big(\frac{|\xi| y^2 A_t^{(-\mu)}}{\sinh(|\xi| y^2 A_t^{(-\mu)})}\Big)^{2(n-1)}$$

$$\times \exp\Big((1 - |\xi| y^2 A_t^{(-\mu)} \coth(|\xi| y^2 A_t^{(-\mu)}))\frac{|\widehat{W}(A_t^{(-\mu)})|^2}{2A_t^{(-\mu)}}\Big)\Big]$$

$$= e^{\sqrt{-1}\,\psi} E\Big[\exp\Big(-\frac{1}{2}|\xi|^2 y^4 \widetilde{A}_t^{(-\mu)}\Big)\int_{\mathbb{R}^{4(n-1)}}\Big(\frac{|\xi|}{2\pi\sinh(|\xi| y^2 A_t^{(-\mu)})}\Big)^{2(n-1)}$$

$$\times \exp\Big(\sqrt{-1}\sum_{k=2}^{n}\Big(\langle K\widehat{\theta}_k + \widehat{w}_k, \zeta_k\rangle - \frac{1}{2}|\xi|\coth(|\xi| y^2 A_t^{(-\mu)})|\zeta_k|^2\Big)\Big)$$

$$d\zeta_2\cdots d\zeta_n\Big],$$

where $\{(\widehat{W}_2(s), \ldots, \widehat{W}_n(s))\}$ is a $4(n-1)$-dimensional standard Brownian motion, independent of $\{Y(s)\}$ or $\{B(s)\}$. We carry out the Gaussian integration over $\mathbb{R}^{4(n-1)}$ to obtain

$$\varphi(t)$$
$$= e^{\sqrt{-1}\,\psi} E\Big[\Big(\frac{1}{\cosh(|\xi| y^2 A_t^{(-\mu)})}\Big)^{2(n-1)} e^{-|\xi|^2 y^4 \widetilde{A}_t^{(-\mu)}/2 - F\tanh(|\xi| y^2 A_t^{(-\mu)})/2|\xi|}\Big],$$

where $F = \sum_{k=2}^{n} |K\widehat{\theta}_k + \widehat{w}_k|^2$. Hence, letting t tend to ∞, we obtain

$$\lim_{t\to\infty}\varphi(t)$$
$$= e^{\sqrt{-1}\,\psi} E\Big[\Big(\frac{1}{\cosh(|\xi| y^2 A_\infty^{(-\mu)})}\Big)^{2(n-1)} e^{-|\xi|^2 y^4 \widetilde{A}_\infty^{(-\mu)}/2 - F\tanh(|\xi| y^2 A_\infty^{(-\mu)})/2|\xi|}\Big].$$

Now, applying (A.9), we obtain an explicit expression for the Fourier transform of the Poisson kernel f_n.

PROPOSITION 4.3. *Under the notations above, the Fourier transform of the limit distribution of* $(X_1(t), \widetilde{Z}(t))$ *as* $t \to \infty$ *is given by*

$$\varPhi(\xi,w)=\frac{e^{\sqrt{-1}\,\psi}(|\xi|y^2)^{2n+1}}{2^{2n+1}\varGamma(2n+1)}\int_0^\infty\left(\frac{1}{\cosh(u)}\right)^{2(n-1)}\left(\frac{1}{\sinh(u)}\right)^{2(n+1)}$$
$$\times\exp\left(-\frac{1}{2}|\xi|y^2\coth(u)-\frac{F}{2|\xi|}\tanh(u)\right)du.$$

We invert the Fourier transform. That is, setting $x'=(x_1',x_{n+1}',y_{n+1}')$, $\theta_k'={}^t(x_k',y_k',x_{n+k}',y_{n+k}')$ and

$$\psi'=\langle\xi,x'\rangle+\sum_{k=2}^n\langle w_k,\theta_k'\rangle,$$

we compute

$$\bar{f}_n(x_1',\widetilde{z}';z)\equiv\frac{1}{(2\pi)^{4n-1}}\int_{\mathbb{R}^{4n-1}}e^{\sqrt{-1}\,(\psi-\psi')}\,d\xi\,dw$$
$$\times\int_0^\infty\frac{(|\xi|y^2)^{2n+1}}{2^{2n+1}\varGamma(2n+1)}\left(\frac{1}{\cosh(u)}\right)^{2(n-1)}\left(\frac{1}{\sinh(u)}\right)^{2(n+1)}$$
$$\times\exp\left(-\frac{1}{2}|\xi|y^2\coth(u)-\frac{\tanh(u)}{2|\xi|}\sum_{k=2}^n|K\widehat{\theta}_k\widehat{w}_k|^2\right)du.$$

Recall the definitions $\widehat{w}_k={}^tQw_k$ and $\widehat{\theta}_k={}^tQ\theta_k$. Then, changing the order of the integrations, we have

$$\bar{f}_n(x_1',\widetilde{z}';z)=\frac{y^{2(2n+1)}}{(2\pi)^{4n-1}2^{2n+1}\varGamma(2n+1)}\int_0^\infty\left(\frac{1}{\cosh(u)}\right)^{2(n-1)}\left(\frac{1}{\sinh(u)}\right)^{2(n+1)}du$$
$$\times\int_{\mathbb{R}^3}e^{\sqrt{-1}\,\langle\xi,x-x'\rangle}e^{-|\xi|y^2\coth(u)/2}|\xi|^{2n+1}\,d\xi$$
$$\times\int_{\mathbb{R}^{4(n-1)}}e^{\sqrt{-1}\,\sum_{k=2}^n\langle\widehat{w}_k,\widehat{\theta}_k-\widehat{\theta}_k'\rangle-\tanh(u)\sum_{k=2}^n|K\widehat{\theta}_k+\widehat{w}_k|^2/2|\xi|}\prod_{k=2}^n dw_k.$$

We can easily carry out the third Gaussian integration since $Q\in O(4)$ and we obtain

$$\bar{f}_n(x_1',\widetilde{z}';z)=\frac{y^{2(2n+1)}}{(4\pi)^{2n+1}\varGamma(2n+1)}\int_0^\infty\left(\frac{1}{\sinh(u)}\right)^{4n}du$$
$$\times\int_{\mathbb{R}^3}e^{\sqrt{-1}\,\{\langle\xi,x-x'\rangle+|\xi|\phi(\widehat{\theta},\widehat{\theta}')\}-|\xi|\varPhi\coth(u)/2}|\xi|^{4n-1}d\xi,$$

where $\varPhi=y^2+\sum_{k=2}^n|\widehat{\theta}_k-\widehat{\theta}_k'|^2=y^2+\sum_{k=2}^n|\theta_k-\theta_k'|^2$ and

$$\phi(\widehat{\theta},\widehat{\theta}')=\sum_{k=2}^n\langle K\widehat{\theta}_k,\widehat{\theta}_k'\rangle=\sum_{k=2}^n(\widehat{y}_k'\widehat{x}_k-\widehat{x}_k'\widehat{y}_k+\widehat{y}_{n+k}'\widehat{x}_{n+k}-\widehat{x}_{n+k}'\widehat{y}_{n+k}).$$

Changing the variable to $v = \coth(u)$, we first compute the integral in u. Then, by using formula (3.5) again, we get

$$\int_0^\infty \left(\frac{1}{\sinh(u)}\right)^{4n} e^{-|\xi|\Phi\coth(u)/2}\,du = \int_1^\infty (v^2-1)^{2n-1}e^{-|\xi|\Phi v/2}\,dv = \frac{\Gamma(2n)}{\sqrt{\pi}}\left(\frac{4}{|\xi|\Phi}\right)^{2n-1/2} K_{2n-1/2}\left(\frac{|\xi|\phi}{2}\right).$$

Moreover, by definitions, we see

$$\langle \xi, x'-x\rangle + |\xi|\phi(\widehat{\theta},\widehat{\theta}') = -\langle \xi, \varphi\rangle,$$

where $\varphi = {}^t(\varphi_1,\varphi_2,\varphi_3)$ is given by (4.2).

Combining these identities, we obtain

$$\begin{aligned}\bar{f}_n(x_1',\widetilde{z}';z) &= \frac{y^{2(2n+1)}}{8(2n+1)\pi^{2n+3/2}\Phi^{2n-1/2}} \int_{\mathbb{R}^3} e^{-\sqrt{-1}\,\langle\xi,\varphi\rangle} K_{2n-1/2}\left(\frac{\Phi}{2}\,|\xi|\right)|\xi|^{2n-1/2}\,d\xi \\ &= \frac{y^{2(2n+1)}}{8(2n+1)\pi^{2n+3/2}\Phi^{2n-1/2}} \int_{\mathbb{R}^3} e^{\sqrt{-1}\,|\varphi|\xi_3} K_{2n-1/2}\left(\frac{\Phi}{2}\,|\xi|\right)|\xi|^{2n-1/2}\,d\xi.\end{aligned}$$

Moreover, changing the variables to the spherical coordinates, we obtain

$$\bar{f}_n(x_1',\widetilde{z}';z) = \frac{4\pi y^{2(2n+1)}}{8(2n+1)\pi^{2n+3/2}\Phi^{2n-1/2}|\varphi|} \times \int_0^\infty r^{2n+1/2}K_{2n-1/2}\left(\frac{\Phi}{2}\,r\right)\sin(|\varphi|r)\,dr.$$

For the integral on the right hand side, the following formula is available (cf. Formula 6.699.3 in [13]):

$$\int_0^\infty x^\lambda K_\mu(ax)\sin(bx)\,dx = 2^\lambda a^{-\lambda-2}b\times\Gamma\left(\frac{2+\mu+\lambda}{2}\right)\Gamma\left(\frac{2+\lambda-\mu}{2}\right) \times {}_2F_1\left(\frac{2+\mu+\lambda}{2},\frac{2+\lambda-\mu}{2};\frac{3}{2};-\frac{b^2}{a^2}\right).$$

In our case $\lambda = 2n+1/2$, $\mu = 2n-1/2$ and

$$ {}_2F_1\left(\frac{2+\mu+\lambda}{2},\frac{2+\lambda-\mu}{2};\frac{3}{2};-\frac{b^2}{a^2}\right) = {}_2F_1\left(2n+1,\frac{3}{2};\frac{3}{2};-\frac{b^2}{a^2}\right) = \left(1+\frac{b^2}{a^2}\right)^{-(2n+1)}$$

by the formula ${}_2F_1(-n,\beta;\beta;-z) = (1+z)^\beta$ (cf. Formula 9.121.1 in [13]).

Applying this identity we arrive at our result

$$\bar{f}_n(x_1', \widetilde{z}'; z) = \frac{2^{4n+1}\Gamma(2n)}{\pi^{2n}} \frac{y^{2(2n+1)}}{(\Phi^2 + 4|\varphi|^2)^{2n+1}}.$$

Appendix. Perpetual integrals of geometric Brownian motion. In this appendix we consider two perpetual integrals of geometric Brownian motions. Let $B = \{B(t)\}_{t\geq 0}$ be a one-dimensional Brownian motion with $B_0 = 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For $\mu > 0$, we set $B^{(-\mu)} = \{B_t^{(-\mu)} \equiv B(t) - \mu t\}$, a Brownian motion with negative constant drift $-\mu$. Then Dufresne's perpetual integral is defined by

$$A_\infty^{(-\mu)} = \int_0^\infty \exp(2B_s^{(-\mu)})\, ds. \tag{A.1}$$

We also consider another integral,

$$a_\infty^{(-\mu)} = \int_0^\infty \exp(B_s^{(-\mu)})\, ds.$$

Then the following is known:

THEOREM A.1 (Dufresne [10]). *Let γ_μ be a gamma random variable with density $\Gamma(\mu)^{-1}x^{\mu-1}e^{-x}$. Then $A_\infty^{(-\mu)}$ is distributed as $(2\gamma_\mu)^{-1}$, and accordingly $a_\infty^{(-\mu)} \overset{\text{(law)}}{=} 2(\gamma_{2\mu})^{-1}$.*

REMARK A.2. Several different proofs of this theorem are known. In particular, see Yor [28]. The density of the exponential functional $A_t = \int_0^t \exp(2B_s)\, ds$ for fixed t has been obtained by Yor [29] and the joint distribution of (A_t, a_t) in an obvious notation has been studied in [2]. See also [24, 25, 27] for several results and applications of these perpetual integrals and exponential functionals. Recently Baudoin–O'Connell [5] have shown several formulae, including (A.2) below, for the exponential functionals and discussed their close relation to the theory of quantum Toda lattice.

What we need in Sections 3 and 4 is the following explicit expression for the conditional Laplace transform of $A_\infty^{(-\mu)}$ given $a_\infty^{(-\mu)}$, which was originally obtained by Yor [30]. We set

$$f_1(v) = \frac{2^{2\mu}}{\Gamma(2\mu)} v^{-(2\mu+1)} e^{-2/v}, \qquad v > 0,$$

which is the density of the random variable $a_\infty^{(-\mu)}$ or $2/\gamma_{2\mu}$.

THEOREM A.3. *For $\lambda > 0$ and $v > 0$,*

$$E\bigg[\exp\bigg(-\frac{1}{2}\lambda^2 A_\infty^{(-\mu)}\bigg)\,\bigg|\, a_\infty^{(-\mu)} = v\bigg] f_1(v) = \frac{1}{2\Gamma(2\mu)}\bigg(\frac{\lambda}{\sinh(\lambda v/2)}\bigg)^{2\mu+1}\exp\bigg(-\lambda\coth\bigg(\frac{\lambda v}{2}\bigg)\bigg). \tag{A.2}$$

We have this nice result only for the particular choice of A_t and a_t, when the ratio of the coefficients in the exponential functionals is two.

We give another proof of the theorem for completeness. Note that by letting λ tend to 0 in (A.2), we obtain Theorem A.1.

For this purpose, we consider the Brownian motion $\{B_t^{(\mu)} = B_t + \mu t\}$ with the opposite positive drift and set $X_x(s) = x\exp(B_s^{(\mu)})$, which defines a diffusion process with infinitesimal generator

$$\frac{1}{2}x^2\frac{d^2}{dx^2} + \bigg(\frac{1}{2}+\mu\bigg)x\frac{d}{dx}.$$

Letting τ_z be the first hitting time of $\{X_x(s)\}$ at z, we set, for $\lambda > 0$ and $\kappa \in \mathbb{R}$,

$$v_z(x) = E\bigg[\exp\bigg(-\frac{\lambda^2}{2}\int_0^{\tau_z} X_x(s)^{-2}\,ds + \lambda\kappa\int_0^{\tau_z} X_x(s)^{-1}\,ds\bigg)\bigg].$$

In [24] we have considered the case of $\kappa = 0$ and showed that $v_z(x)$ may be represented by means of the modified Bessel function to give another proof of Theorem A.1. Following the same lines, we first give a representation for $v_z(x)$ by means of Whittaker functions.

Let $W_{\kappa,\mu}$ be a Whittaker function: if $\mu - \kappa + 1/2 > 0$,

$$W_{\kappa,\mu}(z) = \frac{e^{-z/2}z^{\mu+1/2}}{\Gamma(\mu-\kappa+1/2)}\int_0^\infty e^{-zt}t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}\,dt. \tag{A.3}$$

From this expression it is easy to see that $\lim_{z\to\infty} W_{\kappa,\mu}(z) = 0$ when $|\kappa|$ is small. We also recall that $W_{\kappa,\mu}$ solves the equation

$$W''(z) + \bigg(-\frac{1}{4} + \frac{\kappa}{z} - \frac{\mu^2 - 1/4}{z^2}\bigg)W(z) = 0.$$

PROPOSITION A.4. *For $\mu > 0, \lambda > 0$ and $\kappa \in \mathbb{R}$,*

$$v_z(x) = \bigg(\frac{z}{x}\bigg)^{\mu-1/2}\frac{W_{\kappa,\mu}(2\lambda/x)}{W_{\kappa,\mu}(2\lambda/z)}. \tag{A.4}$$

Proof. We only have to consider the case of $\kappa < 0$. The general case can be deduced from this by analytic continuation in κ. Note that if $\kappa < 0$, then $v_z(x)$ is increasing in x $(> z)$.

First we note that $v_z(x)$ is a solution for

$$\frac{1}{2}x^2v''(x) + \left(\frac{1}{2}+\mu\right)xv'(x) = \left(\frac{\lambda^2}{2x^2} - \frac{\lambda\kappa}{x}\right)v(x)$$

and satisfies

$$v_z(x)|_{x=z} = 1 \quad \text{and} \quad \lim_{x\downarrow 0} v_z(x) = 0. \tag{A.5}$$

We now change the variable to $\xi = \lambda/x$ and set

$$v_z(x) = \xi^{\mu-1/2}\phi(\xi).$$

Then, by straightforward computations, we see that ϕ satisfies

$$\phi''(\xi) + \left(-1 + \frac{2\kappa}{\xi} - \frac{\mu^2-(1/4)}{\xi^2}\right)\phi(\xi) = 0.$$

By considering the boundary conditions (A.5), we can easily show

$$\phi(\xi) = \left(\frac{z}{\lambda}\right)^{\mu-1} \frac{W_{\kappa,\mu}(2\xi)}{W_{\kappa,\mu}(2\lambda/z)}$$

and hence the result (A.4). ∎

Proposition A.5. *For* $\mu > 0, \lambda > 0$ *and* $\kappa \in \mathbb{R}$,

$$E\left[\exp\left(-\frac{1}{2}\lambda^2 A_\infty^{(-\mu)} + \lambda\kappa a_\infty^{(-\mu)}\right)\right] = \frac{\Gamma(\mu-\kappa+1/2)}{\Gamma(2\mu)}(2\lambda)^{\mu-1/2}W_{\kappa,\mu}(2\lambda). \tag{A.6}$$

Proof. By the symmetry of the probability law of Brownian motion, $\{-B_t\} \overset{(\text{law})}{=} \{B_t\}$, we have

$$\begin{aligned}\lim_{z\to\infty} v_z(1) &= E\left[\exp\left(-\frac{1}{2}\lambda^2\int_0^\infty e^{-2B_s^{(\mu)}}ds + \lambda\kappa\int_0^\infty e^{-B_s^{(\mu)}}ds\right)\right] \\ &= E\left[\exp\left(-\frac{1}{2}\lambda^2\int_0^\infty e^{2B_s^{(-\mu)}}ds + \lambda\kappa\int_0^\infty e^{B_s^{(-\mu)}}ds\right)\right].\end{aligned}$$

On the other hand, by using the fact that

$$W_{\kappa,\mu}(z) = \frac{\Gamma(2\mu)}{\Gamma(\mu-\kappa+1/2)} z^{-\mu+1/2}(1+o(1)) \quad \text{as } z\downarrow 0, \tag{A.7}$$

we see from the expression (A.4) that

$$\lim_{z\to\infty} v_z(1) = \frac{\Gamma(\mu-\kappa+1/2)}{\Gamma(2\mu)}(2\lambda)^{\mu-1/2}W_{\kappa,\mu}(2\lambda). \ ∎$$

Remark A.6. The asymptotic behavior (A.7) of $W_{\kappa,\mu}$ can be easily deduced from the definition of Whittaker functions.

Now we are in a position to complete our proof of Theorem A.3. By (A.3) and (A.6), we have

$$E\Big[\exp\Big(-\frac{1}{2}\lambda^2 A_\infty^{(-\mu)} + \lambda\kappa a_\infty^{(-\mu)}\Big)\Big] = \frac{e^{-\lambda}}{\Gamma(2\mu)}(2\lambda)^{2\mu}\int_0^\infty e^{-2\lambda t} t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}\,dt.$$

Now we change the variable via $e^{\lambda v} = 1+t^{-1}$ or $t = (e^{\lambda v}-1)^{-1}$. Then some elementary computations show that this integral is equal to

$$\frac{\lambda^{2\mu+1}}{2\Gamma(2\mu)}\int_0^\infty e^{-\lambda\coth(\lambda v/2)}\Big(\frac{1}{\sinh(\lambda v/2)}\Big)^{2\mu+1} e^{\lambda\kappa v}\,dv.$$

This completes our proof since

$$E\Big[\exp\Big(-\frac{1}{2}\lambda^2 A_\infty^{(-\mu)} + \lambda\kappa a_\infty^{(-\mu)}\Big)\Big] = \int_0^\infty E\Big[\exp\Big(-\frac{1}{2}\lambda^2 A_\infty^{(-\mu)}\Big)\,\Big|\, a_\infty^{(-\mu)} = v\Big] f_1(v) e^{\lambda\kappa v}\,dv. \ \blacksquare$$

Corollary A.7. *Define another perpetual integral* $\widetilde{A}_\infty^{(-\mu)}$ *by*

$$\widetilde{A}_\infty^{(-\mu)} = \int_0^\infty \exp(4B_s^{(-\mu)})\,ds \tag{A.8}$$

and let $f_2(v)$ *be the density of* $A_\infty^{(-\mu)}$ *or* $(2\gamma_\mu)^{-1}$. *Then*

$$E\Big[\exp\Big(-\frac{1}{2}\lambda^2 \widetilde{A}_\infty^{(-\mu)}\Big)\,\Big|\, A_\infty^{(-\mu)} = v\Big] f_2(v) = \frac{1}{2^\mu\Gamma(\mu)}\Big(\frac{\lambda}{\sinh(\lambda v)}\Big)^{\mu+1}\exp\Big(-\frac{\lambda}{2}\coth(\lambda v)\Big). \tag{A.9}$$

Acknowledgements. The author sincerely thanks the anonymous referee for careful reading of the manuscript and for many comments.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 19204010), Japan Society for the Promotion of Science.

REFERENCES

[1] L. Alili and J.-C. Gruet, *An explanation of a generalized Bougerol's identity in terms of hyperbolic Brownian motion*, in: Exponential Functionals and Principal Values Related to Brownian Motion, Bibl. Rev. Mat. Iberoamer., Univ. Autónoma de Madrid, 1997, 15–33.

[2] L. Alili, H. Matsumoto and T. Shiraishi, *On a triplet of exponential Brownian functionals*, in: Séminaire de Probabilités XXXV, Lecture Notes in Math. 1755, Springer, 2001, 396–415.

[3] M. Babillot, *A probabilistic approach to heat diffusion on symmetric spaces*, J. Theor. Probab. 7 (1994), 599–607.

[4] P. Baldi, E. Casadio Tarabusi, A. Figà-Talamanca and M. Yor, *Non-symmetric hitting distributions on the hyperbolic half-plane and subordinated perpetuities*, Rev. Mat. Iberoamer. 17 (2001), 587–605.

[5] F. Baudoin and N. O'Connell, *Exponential functionals of Brownian motion and class one Whittaker functions*, preprint, arXiv:0809.2506v2 [math. PR].

[6] T. Byczkowski and J. Małecki, *Poisson kernel and Green function of the ball in real hyperbolic spaces*, Potential Anal. 27 (2007), 1–26.

[7] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, 1989.

[8] A. Debiard, *Comparaison des espaces H^p géométrique et probabilistes au-dessus de l'espace hermitien hyperbolique*, Bull. Sci. Math. 103 (1979), 305–351.

[9] —, *Espaces H^p au dessus de l'espace hermitien hyperbolique de $\mathbb{C}^n$ $(n > 1)$, II*, J. Funct. Anal. 40 (1981), 185–265.

[10] D. Dufresne, *The distribution of a perpetuity, with application to risk theory and pension funding*, Scand. Actuarial J. (1990), 39–79.

[11] W. M. Goldman, *Complex Hyperbolic Geometry*, Oxford Univ. Press, 1999.

[12] P. Graczyk and T. Żak, *Fine structure of the complex hyperbolic Brownian motion and Rudin's question*, Probab. Math. Statist. 28 (2008), 345–357.

[13] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 6th ed., A. Jeffrey (ed.), Academic Press, 1980.

[14] J.-C. Gruet, *Semi-groupe du mouvement Brownien hyperbolique*, Stoch. Stoch. Rep. 56 (1996), 53–61.

[15] S. Helgason, *Geometric Analysis on Symmetric Spaces*, Amer. Math. Soc., 1994.

[16] N. Ikeda and H. Matsumoto, *Brownian motion on the hyperbolic plane and Selberg trace formula*, J. Funct. Anal. 162 (1999), 63–110.

[17] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland/Kodansha, 1989.

[18] Y. Kifer, *Brownian motion and positive harmonic functions on complete manifolds of non-positive curvature*, in: From Local Times to Global Geometry, Control and Physics, D. Elworthy (ed.), Pitman Res. Notes Math. Ser. 150, Longman, 1986, 187–232.

[19] I. Kim and J. R. Parker, *Geometry of quaternionic hyperbolic manifolds*, Math. Proc. Cambridge Philos. Soc. 135 (2003), 291–320.

[20] N. N. Lebedev, *Special Functions and their Applications*, Dover, New York, 1972.

[21] N. Lohoué und T. Rychener, *Die Resolvente von Δ auf symmetrischen Räumen vom nichtkompakten Typ*, Comment. Math. Helv. 57 (1982), 445–468.

[22] M.-P. Malliavin et P. Malliavin, *Factorisations et lois limites de la diffusion horizontale au-dessus d'un espace riemannien symétrique*, in: Théorie du potentiel et analyse harmonique, J. Faraut (ed.), Lecture Notes in Math. 404, Springer, 1974, 164–217.

[23] H. Matsumoto, *Closed form formulae for the heat kernels and the Green functions for the Laplacians on the symmetric spaces of rank one*, Bull. Sci. Math. 125 (2001), 553–581.

[24] H. Matsumoto and M. Yor, *Exponential functionals of Brownian motion, I, Probability laws at fixed time*, Probab. Surveys 2 (2005), 312–347.

[25] —, —, *Exponential functionals of Brownian motion, II, Some related diffusion processes*, ibid. 2 (2005), 348–384.

[26] A. B. Venkov, *Expansion in automorphic eigenfunctions of the Laplace–Beltrami operators in classical symmetric spaces of rank one, and the Selberg trace formula*, Proc. Steklov Inst. Math. 125 (1973), 1–48.

[27] M. Yor, *Exponential Functionals of Brownian Motion and Related Processes*, Springer, 2001.

[28] —, *Sur certaines fonctionnelles exponentielles du mouvement brownien réel*, J. Appl. Probab. 29 (1992), 202–208. (English transl. is found in [27].)

[29] —, *On some exponential functionals of Brownian motion*, Adv. Appl. Probab. 24 (1992), 509–531 (also found in [27]).

[30] —, *Interpretations in terms of Brownian and Bessel meanders of the distribution of a subordinated perpetuity*, in: Lévy Processes, Theory and Applications, O. E. Barndorff-Nielsen et al. (eds.), Birkhäuser, 2001, 361–375.

[31] T. Żak, *Poisson kernel and Green function of balls for complex hyperbolic Brownian motion*, Studia Math. 183 (2007), 161–193.

H. Matsumoto
Graduate School of Information Science
Nagoya University
Chikusa-ku, Nagoya 464-8601, Japan
E-mail: matsu@is.nagoya-u.ac.jp

Current address:
Department of Mathematical Sciences
Faculty of Science
Yamagata University
1-4-12 Kojirakawa-machi
Yamagata 990-8560, Japan
E-mail: matsu@sci.kj.yamagata-u.ac.jp

Received 3 February 2009;
revised 8 July 2009 (5154)