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MODULES FOR WHICH THE NATURAL MAP OF THE MAXIMAL SPECTRUM IS SURJECTIVE

BY

H. ANSARI-TOROGHY and R. OVLYAEE-SARMAZDEH (Rasht)

Abstract. Let R be a commutative ring with identity. The purpose of this paper is to introduce two new classes of modules over R, called Ms modules and fulmaximal modules respectively. The first (resp. second) class contains the family of finitely generated and primeful (resp. finitely generated and multiplication) modules properly. Our concern is to extend some properties of primeful and multiplication modules to these new classes of modules.

1. Introduction. Throughout this paper, R will denote a commutative ring with identity $1 \neq 0$ and all modules are unitary. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote respectively the natural numbers, the ring of integers and the field of quotients of $\mathbb{Z}$. Also "$\subset$" denotes strict inclusion. Further $\mathrm{Spec}(R)$ (resp. $\mathrm{Max}(R)$) will denote the set of all prime (resp. maximal) ideals of R. Moreover, $V(I)$ will denote the set of all prime ideals of R which contain I. Also $(\mathbf{0})$ (resp. (0)) will denote the zero submodule (resp. zero ideal).

Let M be an R-module. Define $(N : M) = \{r \in R : rM \subseteq N\}$ for any submodule N of M. Also for a prime ideal p of R, M_p (resp. R_p) will denote $S^{-1}M$ (resp. $S^{-1}R$), where $S = R \setminus p$. Moreover, the set $\{p \in \mathrm{Spec}(R) : M_p \neq (\mathbf{0})\}$ is called the *support* of M, and denoted by $\mathrm{Supp}(M)$. Further the supremum of the lengths r of all strictly decreasing chains $p_0 \supset p_1 \supset \cdots \supset p_r$ of prime ideals of $\mathrm{Supp}(M)$ is called the *Krull dimension* of M, and denoted by $\mathrm{K.dim}(M)$. The Krull dimension of R, denoted by $\mathrm{K.dim}(R)$, is defined similarly by putting $M = R$. Also for every ideal I of R containing $\mathrm{Ann}(M)$, $\bar{R}$ and $\bar{I}$ will denote respectively $R/\mathrm{Ann}(M)$ and $I/\mathrm{Ann}(M)$.

Let M be an R-module. A proper submodule P of M is said to be *prime* if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P : M)$. If P is a prime (resp. maximal) submodule of M, then $(P : M)$ is a prime (resp. maximal) ideal of R. Now if p is an ideal of R and P is a prime (resp. maximal) submodule of M with $(P : M) = p$, then P is said to be a *p-prime* (resp. *p-maximal*) submodule of M. The set of all prime (resp.

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maximal) submodules of M is denoted by $\mathrm{Spec}(M)$ (resp. $\mathrm{Max}(M)$). Also for a prime (resp. maximal) ideal p of R, the collection of all p-prime (resp. p-maximal) submodules of M is denoted by $\mathrm{Spec}_p(M)$ (resp. $\mathrm{Max}_p(M)$) (see [5, 6, 8]). Note that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and $\mathrm{Spec}(M)$ may be empty for some non-zero module M. For example, the Prüfer group $\mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module has no prime submodule for every prime integer p (see [6, p. 3745]).

Let M be an R-module with $\mathrm{Spec}(M) \neq \emptyset$. Then the map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(\bar{R})$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, will be called the *natural map of* $\mathrm{Spec}(M)$. Also the map $\phi : \mathrm{Max}(M) \to \mathrm{Max}(\bar{R})$ defined by $\phi(N) = \overline{(N : M)}$ for every maximal submodule N of M is called the *natural map of* $\mathrm{Max}(M)$ (see [6, 7]).

In [7], C. P. Lu introduced the class of primeful modules and considered the main properties of this class. An R-module M is said to be *primeful* if either $M = (\mathbf{0})$, or $M \neq (\mathbf{0})$ and the natural map of $\mathrm{Spec}(M)$ is surjective.

Now let M be an R-module. We say M is a *Max-surjective* module, or an Ms module for short, if either $M = (\mathbf{0})$, or $M \neq (\mathbf{0})$ and the natural map of $\mathrm{Max}(M)$ is surjective. Further we say M is a *fulmaximal* module if either $M = (\mathbf{0})$, or $M \neq (\mathbf{0})$ and every prime submodule of M is contained in some maximal submodule. In Section 3, it is shown that the class of Ms modules contains the family of finitely generated and primeful modules properly (see Example 3.2 and Proposition 3.3(c)). Results 3.4 and 3.6 of this section extend the properties of primeful modules to Ms modules. Theorem 3.7 provides some useful characterizations and shows that M is an Ms module if and only if $pM \in \mathrm{Spec}(M)$ for every $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. Also it is shown (see Theorem 3.8) that if M is a flat module or $\mathrm{K.dim}(M) = 0$, then M is an Ms module if and only if M is a primeful module.

In Section 4, it is shown that the class of fulmaximal modules contains the classes of finitely generated, multiplication, and semisimple modules properly (see Theorem 4.2). Also it is shown that if $(M_i)_{i\in I}$ is a family of prime-distributive R-modules, then $M = \bigoplus_{i\in I} M_i$ is fulmaximal if and only if each M_i $(i \in I)$ is fulmaximal (see Theorem 4.6). Further it is proved (see Proposition 4.4) that if M is a fulmaximal R-module, then M_p is a fulmaximal R_p-module for every prime ideal p of R. Finally, Theorem 4.8 provides another characterization for these modules and says that if R is a one-dimensional Noetherian domain, then M is fulmaximal if and only if for every (0)-prime submodule P of M, M/P is not a divisible R-module.

2. Auxiliary results

Definitions 2.1. Let M be an R-module.

(a) A proper submodule N of M is said to be *prime* if for any $r \in R$ and any $m \in M$ with $rm \in N$ we have $m \in N$ or $r \in (N:M)$ (see [5, 6, 8]).

(b) M is called a *primeful* R-module if either $M = (\mathbf{0})$, or $M \neq (\mathbf{0})$ and the natural map of $\mathrm{Spec}(M)$ is surjective (see [7]).
(c) M is called a *multiplication* R-module if for every submodule N of M there exists an ideal I of R such that $N = IM$ (see [2, [3]]).
(d) M is called a *weak-multiplication* R-module if M is primeless (i.e. $\mathrm{Spec}(M) = \emptyset$) or every prime submodule N of M is of the form IM for some ideal I of R (see [1]).
(e) M is said to be *co-semisimple* if each proper submodule of M is an intersection of maximal submodules. Every semisimple module is of course co-semisimple (see [4]).

Remark 2.2. Let M be an R-module. Then M is primeful in each of the following cases (see [7]):

(a) M is a finitely generated R-module.
(b) M is a faithfully flat R-module.
(c) M is a ring S containing R as a subring (with the same identity) such that the spectral map $\theta : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ defined by $P \mapsto P \cap R$ is surjective. (For example, when S is an integral extension of R.)

Remark 2.3 (see [6]). Let M be an R-module.

(a) If K is a submodule of M such that $(K : M)$ is a maximal ideal of R, then K is a prime submodule of M.
(b) If N is a maximal submodule of M, then N is a prime submodule of M and $(N : M)$ is a maximal ideal of R.
(c) If M is a non-zero finitely generated R-module, then every proper submodule of M is contained in some maximal submodule of M. Also the natural map of $\mathrm{Max}(M)$ is surjective so that $\mathrm{Max}(M) \neq \emptyset$.
(d) Let $p \in \mathrm{Spec}(R)$. Then the prime submodules of the R_p-module M_p are in one-to-one correspondence with the prime submodules N of M such that $(N : M) \subseteq p$.

3. Max-surjective modules

Definition 3.1. Let M be an R-module. We say that M is a *Max-surjective* module, or an Ms module for short, if either $M = (\mathbf{0})$, or $M \neq (\mathbf{0})$ and the natural map of $\mathrm{Max}(M)$ is surjective.

Example 3.2. Every finitely generated R-module is an Ms module by 2.3(c). However, the converse is not true in general. To see this, let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$. Then we have $\mathrm{Spec}(M) = \{pM : p \in \mathrm{Max}(\mathbb{Z})\} \cup \{\mathbb{Z}(p^\infty) \oplus (0)\}$ and $\mathrm{Max}(M) = \{pM : p \in \mathrm{Max}(\mathbb{Z})\}$. Clearly M is an Ms $\mathbb{Z}$-module while it is not a finitely generated $\mathbb{Z}$-module.

Proposition 3.3. *Let M be an R-module.*

(a) *If $p \in \mathrm{Max}(R)$, then every p-prime submodule of M is contained in some p-maximal submodule of M.*
(b) $\mathrm{Max}(M) \neq \emptyset$ *if and only if* $\mathrm{Spec}_p(M) \neq \emptyset$ *for some maximal ideal p of R.*
(c) *If M is primeful, then M is an Ms module. But the converse is not true in general.*
(d) *If M is a non-zero primeful module, then* $\mathrm{Max}(M) \neq \emptyset$.

Proof. (a) Let $P \in \mathrm{Spec}_p(M)$ so that $\mathrm{Ann}(M/P) = (P : M) = p$. It is clear that M/P has a structure of a $k = R/p$-vector space. Hence every subset of M/P is an R-module if and only if it is a k-vector space. But the vector space M/P has a maximal submodule, K say. Now by the above arguments, K is also an R-module so that $K = Q/P$ for some submodule Q of M. It turns out that $P \subseteq Q \in \mathrm{Max}(M)$. Also $p = (P : M) \subseteq (Q : M)$ implies that $p = (Q : M)$.

Part (b) and the first statement of (c) are immediate consequences of (a). To see the second statement of (c), let I be the set of all prime integers and $M = \bigoplus_{p\in I}(\mathbb{Z}/p\mathbb{Z})$. Then it is easy to see that M is a faithful $\mathbb{Z}$-module with $\mathrm{Spec}(M) = \mathrm{Max}(M) = \{pM : p \in I\}$. Hence M is an Ms $\mathbb{Z}$-module which is not primeful. Part (d) follows from (c). This completes the proof.

The following lemma is an analogue of Nakayama's Lemma.

Lemma 3.4. *Let M be an Ms R-module. Then M satisfies the following condition* (NAK)*: If I is an ideal of R contained in the Jacobson radical of R, then $IM = M$ implies that $M = (\mathbf{0})$.*

Proof. Since $IM = M$, we have $mM = M$ for every maximal ideal m of R. Now if $M \neq (\mathbf{0})$, then $\mathrm{Ann}(M) \neq R$. Choose $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. Then there exists $P \in \mathrm{Max}(M)$ such that $(P : M) = p$. Hence p is a maximal ideal of R such that $pM \subseteq P \subset M$. Hence $pM \neq M$, a contradiction. This completes the proof.

Remark 3.5. By Proposition 3.3(c), Lemma 3.4 extends [7, Cor. 3.2].

The *radical* of an ideal I of R, denoted by $\sqrt{I}$, is the set

$$\{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Proposition 3.6. *Let M be an Ms R-module and I be an ideal of R.*

(a) *M/IM is an Ms R-module.*
(b) $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$.
(c) $\sqrt{\mathrm{Ann}(M/IM)} = \sqrt{(I + \mathrm{Ann}(M))}$.
(d) *If $M \neq (\mathbf{0})$ and I is a radical ideal, then $(IM : M) = I$ if and only if* $\mathrm{Ann}(M) \subseteq I$.

Proof. (a) The proof is straightforward and we omit it.

(b) If $M = (\mathbf{0})$, there is nothing to prove. If not, let $p \in V(\mathrm{Ann}(M))$. Then there exists $q \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$ such that $p \subseteq q$. Since M is an Ms module, there is $Q \in \mathrm{Max}(M)$ such that $(Q : M) = q$. It follows that $qM \in \mathrm{Spec}(M)$ by 2.3(a). But $qM_q \in \mathrm{Spec}(M_q)$ by 2.3(d). This implies that $M_q \neq (\mathbf{0})$ so that $q \in \mathrm{Supp}(M)$. It turns out that $p \in \mathrm{Supp}(M)$. Thus $V(\mathrm{Ann}(M)) \subseteq \mathrm{Supp}(M)$. Since the reverse inclusion is always true, we have $V(\mathrm{Ann}(M)) = \mathrm{Supp}(M)$ as required.

(c) It is enough to prove that $V(\mathrm{Ann}(M/IM)) = V((I + \mathrm{Ann}(M))$. It is clear that $V(\mathrm{Ann}(M/IM)) \subseteq V(I + \mathrm{Ann}(M))$. So let $p \in V(I + \mathrm{Ann}(M))$. Then by part (b), $p \in V(\mathrm{Ann}(M)) = \mathrm{Supp}(M)$. This implies that $p \in \mathrm{Supp}(M/IM)$ by Lemma 3.4. But $\mathrm{Supp}(M/IM) = V(\mathrm{Ann}(M/IM))$ by (a) and (b). It turns out that $p \in V(\mathrm{Ann}(M/IM))$ as desired.

(d) The necessity is clear. To see the sufficiency, we have

$$(IM : M) \subseteq \sqrt{(IM : M)} = \sqrt{(I + \mathrm{Ann}(M))} = \sqrt{I} = I$$

by (c). This implies that $(IM : M) = I$ and the proof is complete.

Theorem 3.7. *Let M be a non-zero R-module. Then the following statements are equivalent:*

(a) *M is an Ms R-module;*
(b) *$pM \in \mathrm{Spec}(M)$ for every $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$;*
(c) *M_p is a non-zero Ms R_p-module for every $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$;*
(d) *$(pM : M) = p$ for every $p \in V(\mathrm{Ann}(M))$;*
(e) *$pM_p \neq M_p$ for every $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$.*

Proof. (a)⇔(b): Let $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. Then there exists $P \in \mathrm{Max}(M)$ such that $(P : M) = p$. Hence $pM \subseteq P \subset M$, so that $(pM : M) = p \in \mathrm{Max}(R)$. This implies that $pM \in \mathrm{Spec}(M)$ by 2.3(a). The reverse implication is an immediate consequence of 3.3(a).

(a)⇔(c): Let M be an Ms R-module and $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. By 3.6(b), $p \in \mathrm{Supp}(M)$ so $M_p \neq (\mathbf{0})$. Since M is an Ms R-module, there is $P \in \mathrm{Max}(M)$ such that $(P : M) = p$. This implies that $pM \in \mathrm{Spec}(M)$ by 2.3(a). Therefore $pM_p \in \mathrm{Spec}(M_p)$ by 2.3(d). It follows that $\mathrm{Max}(M_p) \neq \emptyset$ by 3.3(a). Conversely, let $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. Then $\overline{pR_p} \in \mathrm{Max}(\overline{R_p})$. Hence there exists $W \in \mathrm{Max}(M_p)$ such that $(W : M_p) = pR_p$. This implies that $pM_p \neq M_p$, so $pM \neq M$. It follows that $pM \in \mathrm{Spec}_p(M)$ by 2.3(a). Now the claim follows from 3.3(a).

(a)⇔(d): The necessity is clear from 3.6(d). Conversely, if $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$, then $pM \neq M$, so $pM \in \mathrm{Spec}_p(M)$ by 2.3(a). It follows that $\mathrm{Max}_p(M) \neq \emptyset$ by 3.3 as required.

(a)⇔(e): The sufficiency follows from (a)⇔(c) and Lemma 3.4. To see the necessity let $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. Then $pM_p \neq M_p$ implies that

$pM \neq M$ so that $pM \in \mathrm{Spec}(M)$ by 2.3(a). Hence the result follows from (a)$\Leftrightarrow$(b). This completes the proof.

THEOREM 3.8. *Let M be an R-module.*

(a) *If M is a flat module, then M is an Ms module if and only if M is a primeful module.*
(b) *If M is a non-zero divisible module over a Noetherian domain R, then M is an Ms module if and only if R is a field.*
(c) *If* $\mathrm{K.dim}(M) = 0$, *then M is an Ms R module if and only if M is a primeful module.*
(d) *If M is a locally free or a locally finitely generated module at every $p \in V(\mathrm{Ann}(M))$, then M is an Ms module if and only if* $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$. (*We recall that for a prime ideal p of R, M is a locally free* (*resp. locally finitely generated*) *module at p if M_p is a free* (*resp. locally finitely generated*) *R_p-module.*)
(e) *If M is a multiplication module, then M is an Ms module if and only if M is a finitely generated module.*

Proof. (a) Let $\bar{p} \in \mathrm{Spec}(\bar{R})$. Then there exists $q \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$ such that $p \subseteq q$. It follows that $q = (Q : M)$ for some maximal submodule Q of M. Therefore $pM \subseteq qM \subset M$. By [5, Theorem 3], this implies that $pM \in \mathrm{Spec}_p(M)$ as desired. The reverse implication follows from 3.3(c).

(b) If R is a field, then M is faithfully flat, so M is primeful by 2.2. By 3.3(c), this implies that M is an Ms R-module. Conversely, since M is an Ms R-module, $pM \neq M$ for every $p \in V(\mathrm{Ann}(M))$ by 3.7. Further since M is a divisible R-module and R is a Noetherian domain, it follows that $pM = M$ for every non-zero prime ideal of R. Now since $V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R) \neq \emptyset$, we have $V(\mathrm{Ann}(M)) = \{(0)\}$. Hence $\mathrm{Ann}(M) = (0) \in \mathrm{Max}(R)$. Thus R is a field as desired.

(c) Let M be an Ms R-module and $\bar{p} \in \mathrm{Spec}(\bar{R})$. By 3.6, $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$. It follows that $\mathrm{K.dim}(\bar{R}) = 0$. Therefore $\bar{p} \in \mathrm{Max}(\bar{R})$ so that $p = (P : M)$ for some $P \in \mathrm{Max}(M)$. Thus $P \in \mathrm{Spec}_p(M)$ as required. The converse follows from 3.3(c).

(d) The necessity follows from 3.6(b). To prove the converse, let $p \in V(\mathrm{Ann}(M))$. Then M_p is a non-zero primeful R_p-module by 2.2. Therefore M_p is a non-zero Ms R_p-module by 3.3(c). Hence M is an Ms R-module by 3.7((a)$\Leftrightarrow$(c)).

(e) Let M be an Ms module. Then by 3.7, $pM \neq M$ for every $p \in V(\mathrm{Ann}(M)) \cap \mathrm{Max}(R)$. This implies M is a finitely generated $R/\mathrm{Ann}(M)$-module by [3, 3.1]. Thus M is a finitely generated R-module and the proof is complete.

Let M be an R-module. Then the *Jacobson radical* of M, denoted by $\mathrm{Rad}(M)$, is defined to be the intersection of M and all maximal submodules of M.

COROLLARY 3.9. *Let M be a non-zero Ms R-module. Then*

$$\mathrm{Rad}(M) = \bigcap_{\bar{p}\in \mathrm{Max}(\bar{R})} pM.$$

Proof. Since for every maximal submodule P of M, $(P : M) = p$ with $\bar{p} \in \mathrm{Max}(\bar{R})$, we have $\mathrm{Rad}(M) = \bigcap_{P\in\mathrm{Max}(M)} P \supseteq \bigcap_{\bar{p}\in\mathrm{Max}(\bar{R})} pM$. Conversely, if $\bar{p} \in \mathrm{Max}(\bar{R})$, then $pM \neq M$ by 3.7. Thus M/pM is a non-zero vector space over the field R/p. By the corollary to [5, Theorem 3], this implies that

$$pM = \bigcap_{pM\subseteq P\in\mathrm{Max}(M)} P \supseteq \bigcap_{P\in\mathrm{Max}(M)} P = \mathrm{Rad}(M).$$

Therefore $\bigcap_{\bar{p}\in\mathrm{Max}(\bar{R})} pM \supseteq \mathrm{Rad}(M)$, which completes the proof.

4. Fulmaximal modules. It is well known that every prime ideal of R is contained in some maximal ideal. However, this is not true for a module in general. For example, take $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $\mathrm{Spec}(\mathbb{Q}) = \{(\mathbf{0})\}$ and $\mathrm{Max}(\mathbb{Q}) = \emptyset$, so the prime submodule $(\mathbf{0})$ is not contained in any maximal submodule. Further let $\{p_i\}_{i\in\mathbb{N}}$ be the set of all prime integers and let $M = \mathbb{Q} \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}$. Then

$$\mathrm{Spec}(M) = \{p_iM : i \in \mathbb{N}\} \cup \Big\{(\mathbf{0}) \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}\Big\},$$
$$\mathrm{Max}(M) = \{p_iM : i \in \mathbb{N}\}.$$

This shows that prime submodules are not necessarily contained in maximal submodules of M even in the case where $\mathrm{Max}(M) \neq \emptyset$.

In this section, we will investigate those modules in which every prime submodule is contained in some maximal submodule.

DEFINITION 4.1. Let M be an R-module. We say that M is a *fulmaximal* module if either $M = (\mathbf{0})$, or $M \neq (\mathbf{0})$ and every prime submodule of M is contained in some maximal submodule.

THEOREM 4.2. *The class of fulmaximal R-modules contains the following families of R-modules properly:*

(a) *Multiplication R-modules.*
(b) *Finitely generated R-modules.*
(c) *Co-semisimple* (*or semisimple*) *R-modules.*
(d) *Weak multiplication Ms R-modules.*
(e) *Zero-dimensional Ms R-modules* ($\mathrm{K.dim}(M) = 0$).

Proof. (a) Use [3, Theorem 2.5(i)]. Further if we take $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$, then by 3.2, M is a fulmaximal $\mathbb{Z}$-module which is not a multiplication module.

(b) This is clear. Now take $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$; then by 3.2, M is a fulmaximal $\mathbb{Z}$-module which is not finitely generated.

(c) Let M be a co-semisimple R-module. Then every proper submodule of M is an intersection of maximal submodules. Hence M is fulmaximal. Further if $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$ then M is a fulmaximal module by 3.2. But M is not a co-semisimple $\mathbb{Z}$-module, because if N is a proper submodule of $\mathbb{Z}(p^\infty)$, then $N \oplus \mathbb{Z}$ is a proper submodule of M which is not an intersection of maximal submodules of M by 3.2.

(d) Let M be a non-zero weak multiplication Ms R-module and $P \in \mathrm{Spec}(M)$. Set $(P : M) = p$. Choose $q \in \mathrm{Max}(R)$ with $p \subseteq q$. Since M is an Ms module, there exists a maximal submodule Q of M such that $(Q : M) = q$. This implies that $P = (P : M)M \subseteq (Q : M)M = Q$ as desired. Now let p be a prime integer and $M = \mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$ with $S = \mathbb{Z}\backslash(p)$. One can see that M is a faithful $\mathbb{Z}$-module such that $\mathrm{Spec}(M) = \{(\mathbf{0}), P\}$ and $\mathrm{Max}(M) = \{P\}$, where $P = p\mathbb{Z}_{(p)}$. Hence M is a fulmaximal $\mathbb{Z}$-module, while it is not an Ms module.

(e) Let M be an Ms R-module with $\mathrm{K.dim}(M) = 0$. Then by 3.6(b), we have $\mathrm{K.dim}(R/\mathrm{Ann}(M)) = \mathrm{K.dim}(M) = 0$. Let P be a p-prime submodule of the R-module M. Then P is a $\bar{p}$-prime submodule of the $\bar{R}$-module M. But $\bar{p} \in \mathrm{Max}(\bar{R})$ by assumption. Thus by Proposition 3.3, there exists a maximal submodule Q of the $\bar{R}$-module M such that $P \subseteq Q$. Clearly Q is a maximal submodule of the R-module M as required. Now let $\{p_i\}_{i\in\mathbb{N}}$ be the set of all prime integers. Let $M = \bigoplus_{k\neq i\in\mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}$ for some $k \in \mathbb{N}$. Then

$$\mathrm{Spec}(M) = \mathrm{Max}(M) = \{p_iM : k \neq i \in \mathbb{N}\}.$$

This implies that M is a faithful fulmaximal $\mathbb{Z}$-module which is not Ms, and the proof is complete.

We need the following lemma.

LEMMA 4.3. *Let M be an R-module, S be a multiplicatively closed subset of R and $\varphi : M \to S^{-1}M$ be the natural map.*

(a) *If L is a submodule of $S^{-1}M$, then $L = L^{ce} = S^{-1}(L \cap M)$, where "e" and "c" represent extension and contraction respectively.*
(b) *If P is a prime submodule of M with $(P : M) \cap S = \emptyset$, then $P^{ec} = P$.*

Proof. This is straightforward.

PROPOSITION 4.4. *Let M be a fulmaximal R-module.*

(a) *Every homomorphic image of M is fulmaximal.*
(b) *M_p is a fulmaximal R_p-module for every prime ideal p of R.*

Proof. (a) Let $L = M/N$ for some submodule N of M. Then the claim is immediate by the fact that

$$\mathrm{Spec}(M/N) = \{P/N : P \in \mathrm{Spec}(M),\ P \supseteq N\}.$$

(b) Let $p \in \mathrm{Spec}(R)$ and let W be a prime submodule of M_p. Then $Q = W \cap M$ is a prime submodule of M with $(Q : M) \subseteq p$ by 2.3(d). Let N be a maximal submodule of M with $Q \subseteq N$. Then $R_pQ \subseteq R_pN$. Now it remains to prove that $R_pN \in \mathrm{Max}(M_p)$. To see this, let L be a submodule of M_p such that $R_pN \subseteq L \subset M_p$. By Lemma 4.3, $L = R_p(L \cap M)$. It follows that $R_pN \cap M \subseteq L \cap M \subset M$. But $R_pN \cap M = N$ by Lemma 4.3. It follows that $R_pN = L$ as desired. This completes the proof.

Notation and Definition 4.5. Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. For $j \in I$ we denote $\bigoplus_{j\neq i\in I} M_i$ by $D_j(M)$. Then we say that $(M_i)_{i\in I}$ is a family of *prime-distributive* R-modules if

$$\forall j \in I\ \forall P \in \mathrm{Spec}(M) : \quad P \subseteq (P \cap M_j) \oplus D_j(M).$$

For example, if M is a distributive R-module, then every family $(M_i)_{i\in I}$ of submodules of M with $M = \bigoplus_{i\in I} M_i$ is a family of prime-distributive R-modules.

Theorem 4.6. *Let $(M_i)_{i\in I}$ be a family of prime-distributive R-modules. Then $M = \bigoplus_{i\in I} M_i$ is fulmaximal if and only if each M_i $(i \in I)$ is fulmaximal.*

Proof. Let P be a prime submodule of M. Since $P \neq M$, there exists $j \in J$ with $M_j \nsubseteq P$. By [8, 1.6], $P \cap M_j \in \mathrm{Spec}_p(M_j)$. Now since M_j is fulmaximal, there exists $Q_j \in \mathrm{Max}(M_j)$ such that $P \cap M_j \subseteq Q_j$. Set $Q = Q_j \oplus D_j(M)$. As M is a prime-distributive R-module, we have $P \subseteq (P \cap M_j) \oplus D_j(M) \subseteq Q$. It is enough to prove that $Q \in \mathrm{Max}(M)$. To see this, let N be a proper submodule of M such that $Q \subseteq N$. Then $(Q : M) \subseteq (N : M) \subset R$. As $Q_j \in \mathrm{Max}(M_j)$, $q = (Q_j : M_j)$ is a maximal ideal of R. On the other hand, $q \subseteq (Q : M)$. It follows that $(N : M) = q$, so N is a prime submodule of M by 2.3(a). This implies that $N = N \cap C = N \cap \bigoplus_{i\in I} M_i \subseteq (N \cap M_j) \oplus D_j(M) = Q \subseteq N$. It follows that $Q \in \mathrm{Max}(M)$ as required. To see the reverse implication, let $j \in I$ and $P_j \in \mathrm{Spec}(M_j)$. Then $P_j \oplus D_j(M) \in \mathrm{Spec}(M)$ by [7, Lemma 4.6]. This implies that $P_j \oplus D_j(M) \subseteq Q \in \mathrm{Max}(M)$. Hence $P_j \subseteq Q \cap M_j$. Now we show that $Q \cap M_j \in \mathrm{Max}(M_j)$. To see this, let $Q \cap M_j \subseteq N \subset M_j$. As $Q \in \mathrm{Spec}(M)$, we have $Q \subseteq (Q \cap M_j) \oplus D_j(M) \subseteq N \oplus D_j(M) \subset M$. Therefore $Q = (Q \cap M_j) \oplus D_j(M) = N \oplus D_j(M)$, so $Q \cap M_j = N$ as required. This completes the proof.

The next example shows that the prime-distributivity condition cannot be omitted in the above theorem.

Example 4.7. Let $M_n = (1/n)\mathbb{Z}$ and $M = \bigoplus_{n\in\mathbb{N}} M_n$. Then M_n's are fulmaximal $\mathbb{Z}$-modules. But M is not fulmaximal, for if it were, then the homomorphic image $\lim_{\rightarrow} M_n$ of M would be fulmaximal by Proposition 4.4(a). (Here $\lim_{\rightarrow} M_n$ denotes the direct limit of the direct system of R-modules $(M_n)_{n\in\mathbb{N}}$.) This contradicts the fact that $\mathrm{Spec}(\mathbb{Q}) = \{(\mathbf{0})\}$ and $\mathrm{Max}(\mathbb{Q}) = \emptyset$.

Theorem 4.8. *Let R be a one-dimensional Noetherian domain and M be an R-module. Then M is fulmaximal if and only if for every (0)-prime submodule P of M, M/P is not a divisible R-module.*

Proof. Let M be a fulmaximal R-module and $P \in \mathrm{Spec}_{(0)}(M)$. Then there exists a maximal submodule Q of M such that $P \subseteq Q$. Set $(Q : M) = q$. If M/P is a divisible R-module, then $q(M/P) = M/P$, so $qM + P = M$. On the other hand, $P \subseteq Q$ and $qM \subseteq Q$. It follows that $Q = M$, a contradiction. To see the reverse implication, let P be a p-prime submodule of M. Then by 3.3(a), we can assume that $p = (0)$. Thus M/P is not a divisible R-module. Therefore there is a non-zero element a in R such that $a(M/P) \neq M/P$. Now since R/Ra is Artinian, $p_1 \dots p_n \subseteq Ra$ for some $n \in \mathbb{N}$ and maximal ideals p_i $(1 \leq i \leq n)$. It follows that $p_j(M/P) \neq M/P$ for some $j = 1, \dots, n$. This implies that $p_j(M/P) \in \mathrm{Spec}_{p_j}(M/P)$ by 2.3(a). By 3.3(a), it follows that $\mathrm{Max}(M/P) \neq \emptyset$, so P is contained in a maximal submodule of M, and the proof is complete.

Corollary 4.9. *Let R be a one-dimensional Noetherian domain and M be a weak multiplication R-module. If M is not divisible, then M is a fulmaximal R-module. The converse is true if M is not primeless.*

Proof. By [1, 2.4(iii)], M is a torsion or torsion-free R-module. If M is torsion, then $\mathrm{Spec}_{(0)} M = \emptyset$ and hence M is a fulmaximal R-module by 3.3(a). If M is torsion-free, then the torsion submodule $T(M) = (\mathbf{0})$ is the only (0)-prime submodule of M. By 4.8, it follows that M is fulmaximal. The converse is a consequence of 4.8 and the fact that every torsion divisible R-module is primeless. This completes the proof.

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H. Ansari-Toroghy, R. Ovlyaee-Sarmazdeh
Department of Mathematics
Faculty of Science
University of Guilan
P.O. Box 1914
Rasht, Iran
E-mail: ansari@guilan.ac.ir
ovlyaee@guilan.ac.ir

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