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A GENERALIZATION OF BATEMAN'S EXPANSION AND FINITE INTEGRALS OF SONINE'S AND FELDHEIM'S TYPE

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Abstract. Let $\{A_k\}_{k=0}^{+\infty}$ be a sequence of arbitrary complex numbers, let $\alpha, \beta > -1$, let $\{P_n^{\alpha,\beta}\}_{n=0}^{+\infty}$ be the Jacobi polynomials and define the functions

$$H_n(\alpha, z) = \sum_{m=n}^{+\infty} \frac{A_m z^m}{\Gamma(\alpha+n+m+1)(m-n)!},$$
$$G(\alpha, \beta, x, y) = \sum_{r,s=0}^{+\infty} \frac{A_{r+s} x^r y^s}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)r!s!}.$$

Then, for any non-negative integer n,

$$\int_0^{\pi/2} G(\alpha, \beta, x^2 \sin^2 \phi, y^2 \cos^2 \phi) P_n^{\alpha,\beta}(\cos 2\phi) \sin^{2\alpha+1} \phi \cos^{2\beta+1} \phi \, d\phi = \frac{1}{2} H_n(\alpha+\beta+1, x^2+y^2) P_n^{\alpha,\beta}\left(\frac{y^2-x^2}{y^2+x^2}\right).$$

When $A_k = (-1/4)^k$, this formula reduces to Bateman's expansion for Bessel functions. For particular values of y and n one obtains generalizations of several formulas already known for Bessel functions, like Sonine's first and second finite integrals and certain Neumann series expansions. Particular choices of $\{A_k\}_{k=0}^{+\infty}$ allow one to write all these type of formulas for specific special functions, like Gegenbauer, Jacobi and Laguerre polynomials, Jacobi functions, or hypergeometric functions.

1. Introduction and main result. The following formula is very well-known (see [16, p. 36] or [20, p. 373]), and is usually called *Sonine's first finite integral*:

$$\int_0^{\pi/2} \frac{J_\alpha(x \sin \phi)}{\sin^\alpha \phi} \, dm_{\alpha,\beta}(\phi) = 2^\beta \Gamma(\beta+1) \frac{J_{\alpha+\beta+1}(x)}{x^{\beta+1}}. \tag{1.1}$$

Here J_α is the Bessel function of first kind and order α,

$$J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma(\alpha+1)} \sum_{k=0}^{+\infty} \frac{(-x^2/4)^k}{(\alpha+1)_k k!}, \tag{1.2}$$

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α and β are real numbers, $\alpha, \beta > -1$, the expression $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ for $k \geq 1$, $(a)_0 = 1$, and

$$dm_{\alpha,\beta}(\phi) = \sin^{2\alpha+1}\phi \cos^{2\beta+1}\phi \, d\phi \tag{1.3}$$

is a finite positive measure on $[0, \pi/2]$ that we will encounter often along the paper.

Sonine's first finite integral allows one to express any Bessel function in terms of an integral involving a Bessel function of a lower order. It can be easily proven by a term by term integration of the power series defining the Bessel function (1.2).

There are two possible generalizations of formula (1.1). The first, known as *Sonine's second finite integral* (see, again, [16, p. 35] or [20, p. 376]), is the formula

$$\int_0^{\pi/2} \frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi} \frac{J_\beta(y\cos\phi)}{\cos^\beta\phi} \, dm_{\alpha,\beta}(\phi) = x^\alpha y^\beta \frac{J_{\alpha+\beta+1}(\sqrt{x^2+y^2})}{(x^2+y^2)^{\frac{\alpha+\beta+1}{2}}}. \tag{1.4}$$

That this is indeed a generalization of (1.1) can be seen by dividing both sides by y^β and letting $y \to 0$.

The second possible generalization goes in a different direction. Let $\mathcal{H}_0$ be the closed one-dimensional subspace of the Hilbert space $L^2([0,\pi/2], dm_{\alpha,\beta})$ formed by the constant functions. Then Sonine's first finite integral gives the projection onto $\mathcal{H}_0$ of the function $f_x(\phi) = J_\alpha(x\sin\phi)\sin^{-\alpha}\phi$. If we denote by $P_n^{\alpha,\beta}(z)$ the Jacobi polynomials, given by (see [18, p. 62])

$$P_n^{\alpha,\beta}(z) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k k!} \left(\frac{1-z}{2}\right)^k, \tag{1.5}$$

then $\{P_n^{\alpha,\beta}(\cos 2\phi)\}_{n=0}^\infty$ is an orthogonal basis for $L^2([0,\pi/2], dm_{\alpha,\beta})$, and therefore Sonine's first finite integral is, in other words, the 0th Fourier–Jacobi coefficient $\widehat{f_x}(0)$ of f_x. One could calculate all the other Fourier–Jacobi coefficients

$$\widehat{f_x}(n) = \int_0^{\pi/2} \frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi} P_n^{\alpha,\beta}(\cos 2\phi) \, dm_{\alpha,\beta}(\phi),$$

obtaining the following identity:

$$\int_0^{\pi/2} \frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi} P_n^{\alpha,\beta}(\cos 2\phi) \, dm_{\alpha,\beta}(\phi) = \frac{2^\beta\Gamma(\beta+n+1)}{\Gamma(n+1)} \frac{J_{\alpha+\beta+2n+1}(x)}{x^{\beta+1}}. \tag{1.6}$$

Of course, formula (1.1) is the particular case $n = 0$.

The regularity of f_x implies that the Fourier–Jacobi series of f_x converges pointwise to f_x. In other words, formula (1.6) can be reformulated as the following Neumann series expansion (see [20, p. 140]; this formula appears, *in nuce*, again in [16, p. 22], and it has been rediscovered several times, see [19], [15])

$$\tag{1.7} \frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi} = \sum_{n=0}^{\infty} \frac{2^{\beta+1}(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} \frac{J_{\alpha+\beta+2n+1}(x)}{x^{\beta+1}} P_n^{\alpha,\beta}(\cos 2\phi).$$

Both Sonine's second finite integral and the Neumann series expansion (1.7) follow from a more general formula: it is known as *Bateman's expansion* (see [3], [4], or [20, p. 370]),

$$\tag{1.8} \frac{J_\alpha(\rho\sin\phi\sin\theta)}{\rho^\alpha\sin^\alpha\phi\sin^\alpha\theta} \frac{J_\beta(\rho\cos\phi\cos\theta)}{\rho^\beta\cos^\beta\phi\cos^\beta\theta} = \sum_{n=0}^{\infty}(-1)^n\rho^{2n} \frac{J_{\alpha+\beta+2n+1}(\rho)}{\rho^{\alpha+\beta+2n+1}} p_n^{\alpha,\beta}(\cos 2\phi)p_n^{\alpha,\beta}(\cos 2\theta).$$

Here $\rho > 0$, $\phi,\theta\in[0,\pi/2]$, and $\{p_n^{\alpha,\beta}(\cos 2\phi)\}$ are the Jacobi polynomials, properly normalized in order to form an orthonormal basis for $L^2([0,\pi/2], dm_{\alpha,\beta}(\phi))$ (see [18, p. 68])

$$\tag{1.9} p_n^{\alpha,\beta}(z) = \left(\frac{2(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right)^{1/2} P_n^{\alpha,\beta}(z).$$

Observe that formula (1.7) follows by taking $\theta=\pi/2$, and recalling that (see [18, p. 59])

$$P_n^{\alpha,\beta}(-1) = (-1)^n \frac{\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\beta+1)}.$$

Again, one can write the integral counterpart of (1.8) by evaluating on both sides the nth Fourier–Jacobi coefficient in the variable ϕ:

$$\tag{1.10} \int_0^{\pi/2} \frac{J_\alpha(\rho\sin\phi\sin\theta)}{\rho^\alpha\sin^\alpha\phi\sin^\alpha\theta} \frac{J_\beta(\rho\cos\phi\cos\theta)}{\rho^\beta\cos^\beta\phi\cos^\beta\theta} p_n^{\alpha,\beta}(\cos 2\phi)\, dm_{\alpha,\beta}(\phi) = (-1)^n\rho^{2n} \frac{J_{\alpha+\beta+2n+1}(\rho)}{\rho^{\alpha+\beta+2n+1}} p_n^{\alpha,\beta}(\cos 2\theta).$$

Sonine's second finite integral is obtained in the case $n=0$ with the change of variables

$$\begin{cases} y = \rho \cos \theta, \\ x = \rho \sin \theta. \end{cases}$$

In the literature there are several formulas analogous to Sonine's first finite integral, where the Bessel functions are replaced by other special functions. For example, for the Gegenbauer polynomials $P_h^{\alpha+1/2}$ (see [18, p. 80]),

$$P_h^{\alpha+1/2}(x) = \frac{\Gamma(2\alpha+h+1)\Gamma(\alpha+1)}{\Gamma(\alpha+h+1)\Gamma(2\alpha+1)} P_h^{\alpha,\alpha}(x) \tag{1.11}$$

the following formula by Feldheim (see [9, p. 278] or [18, p. 95]) holds:

$$\int_0^{\pi/2} (1-\sin^2\psi\cos^2\phi)^{h/2} P_h^{\alpha+1/2}\left(\frac{\cos\psi}{\sqrt{1-\sin^2\psi\cos^2\phi}}\right) dm_{\alpha,\beta}(\phi) = \tag{1.12}$$
$$\frac{\Gamma(\alpha+h/2+1)\Gamma(\alpha+h/2+1/2)\Gamma(\beta+1)\Gamma(\alpha+\beta+3/2)}{2\Gamma(\alpha+1/2)\Gamma(\alpha+\beta+h/2+2)\Gamma(\alpha+\beta+h/2+3/2)} P_h^{\alpha+\beta+3/2}(\cos\psi).$$

The above formula is a particular case of the more general formula for Jacobi polynomials

$$\begin{aligned} \int_0^{\pi/2} [(1+x)+(1-x)\sin^2\phi]^h P_h^{\alpha,\gamma}\left(\frac{(1+x)-(1-x)\sin^2\phi}{(1+x)+(1-x)\sin^2\phi}\right) dm_{\alpha,\beta}(\phi) \\ = \frac{2^{h-1}\Gamma(\beta+1)\Gamma(\alpha+h+1)}{\Gamma(\alpha+\beta+h+2)} P_h^{\alpha+\beta+1,\gamma}(x), \end{aligned} \tag{1.13}$$

proved by Askey and Fitch (see [2, formula (3.7)]). For the Laguerre polynomials L_h^α, defined by (see [18, p. 103])

$$L_h^\alpha(x) = \frac{\Gamma(\alpha+h+1)}{\Gamma(h+1)\Gamma(\alpha+1)} \sum_{k=0}^{h} \frac{(-h)_k x^k}{(\alpha+1)_k k!}, \tag{1.14}$$

the following formula due to Koshlyakov [12] can be found in [14, p. 462, formula 2], [13, p. 94]:

$$\int_0^{\pi/2} L_h^\alpha(x\sin^2\phi)\, dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+h+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+h+2)} L_h^{\alpha+\beta+1}(x). \tag{1.15}$$

Formula (1.7) has an analog for Whittaker functions $M_{\lambda,\alpha/2}$, defined by (see [6, p. 11])

$$z^{-(\alpha+1)/2} e^{-z/2} M_{\lambda,\alpha/2}(z) = \sum_{k=0}^{+\infty} \frac{\left(\frac{\alpha+1}{2}+\lambda\right)_k}{(\alpha+1)_k k!} (-z)^k.$$

It can be written as follows:

$$(1.16)\qquad (z\sin^2\phi)^{-(\alpha+1)/2}e^{-\frac{z}{2}\sin^2\phi}M_{\lambda+(\beta+1)/2,\alpha/2}(z\sin^2\phi)$$
$$=\sum_{n=0}^{+\infty}\frac{\Gamma(\alpha+\beta+n+1)}{(\alpha+1)_n\Gamma(\alpha+\beta+2n+1)}\left(\lambda+\frac{\alpha+\beta+2}{2}\right)_n z^n P_n^{\alpha,\beta}(\cos 2\phi)$$
$$\times\, z^{-(\alpha+\beta+2n+2)/2}e^{-z/2}M_{\lambda,(\alpha+\beta+2n+1)/2}(z)$$

and can be found in [6, p. 139] or [11, p. 736]. Buchholz attributes this formula to Erdélyi, and he uses (1.7) to prove it.

A natural question one could raise in this context is: can these formulas be generalized in a similar way as, in the case of Bessel functions, Bateman's expansion generalizes Sonine's first finite integral or formula (1.7)? In other words, is there a "Bateman expansion" for Gegenbauer, Laguerre, or Jacobi polynomials, Whittaker functions, or other special functions? We could for example look for a formula of this type:

$$(1.17)\qquad F(\alpha,\rho\sin\theta\sin\phi)F(\beta,\rho\cos\theta\cos\phi)$$
$$=\sum_{n=0}^{\infty}c_nF(\alpha+\beta+2n+1,\rho)\rho^{2n}P_n^{\alpha,\beta}(\cos 2\phi)P_n^{\alpha,\beta}(\cos 2\theta).$$

A first negative answer comes from a theorem of Al-Salam and Carlitz [1].

THEOREM 1.1. *The functional equation*

$$F(\alpha,\rho\sin\theta\sin\phi)F(\beta,\rho\cos\theta\cos\phi)$$
$$=\sum_{n=0}^{\infty}(-1)^nF(\alpha+\beta+2n+1,\rho)\rho^{2n}Q_n(\cos 2\phi)Q_n(\cos 2\theta),$$

where Q_n is a polynomial of degree n, $\alpha,\beta>-1$, and $F(\alpha,\cdot)$ is analytic, is satisfied if and only if

$$F(\alpha,z)=a\,\frac{J_\alpha(bz)}{z^\alpha},$$

with a and b arbitrary constants.

Thus, if a generalized Bateman expansion holds, its structure must be subtler than what we expected in our first guess (1.17).

For any complex sequence $\{A_k\}_{k=0}^{+\infty}$, for any non-negative integer n and for any $\alpha,\beta>-1$, define the functions

$$H_n(\alpha,z)=\sum_{k=n}^{+\infty}\frac{A_kz^k}{\Gamma(\alpha+n+k+1)(k-n)!},$$
$$G(\alpha,\beta,x,y)=\sum_{r,s=0}^{+\infty}\frac{A_{r+s}x^ry^s}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)r!s!}.$$

Observe that

$$\Gamma(\beta+1)G(\alpha,\beta,x,0) = H_0(\alpha,x),$$
$$\Gamma(\alpha+1)G(\alpha,\beta,0,y) = H_0(\beta,y),$$

and that if $A_{r+s} = A_r A_s$ for all r, s (that is, if $A_k = q^k$ for some q) then

$$H_n(\alpha,z) = (qz)^n H_0(\alpha+2n,z),$$
$$G(\alpha,\beta,x,y) = H_0(\alpha,x)H_0(\beta,y).$$

The above identities suggest replacing the product $F(\alpha,\cdot)F(\beta,\cdot)$ in (1.17) with the function $G(\alpha,\beta,\cdot,\cdot)$, and the function $F(\alpha+\beta+2n+1,\cdot)$ with $H_n(\alpha+\beta+1,\cdot)$. Indeed, the following generalization of Bateman's expansion holds.

THEOREM 1.2. *Let* $\{A_k\}_{k=0}^{+\infty}$ *be a sequence of arbitrary complex numbers and* H_n *and* G *be as above. Then*

$$\begin{aligned}(1.18)\qquad & G(\alpha,\beta,\rho^2\sin^2\theta\sin^2\phi,\rho^2\cos^2\theta\cos^2\phi) \\ &= \sum_{n=0}^{+\infty}\frac{1}{2}H_n(\alpha+\beta+1,\rho^2)p_n^{\alpha,\beta}(\cos 2\theta)p_n^{\alpha,\beta}(\cos 2\phi),\end{aligned}$$

and for all $n = 0, 1, \ldots,$

$$\begin{aligned}(1.19)\qquad & \int_0^{\pi/2} G(\alpha,\beta,\rho^2\sin^2\theta\sin^2\phi,\rho^2\cos^2\theta\cos^2\phi)P_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi) \\ &= \frac{1}{2}H_n(\alpha+\beta+1,\rho^2)P_n^{\alpha,\beta}(\cos 2\theta)\end{aligned}$$

provided that the series involved are absolutely convergent.

Proof. The two formulas are equivalent, since the first one gives the Fourier–Jacobi expansion of the function

$$\phi \mapsto G(\alpha,\beta,\rho^2\sin^2\theta\sin^2\phi,\rho^2\cos^2\theta\cos^2\phi),$$

while the second gives its Fourier–Jacobi coefficients. Let us therefore prove the second formula. Set $y = \rho\cos\theta$, $x = \rho\sin\theta$, and define

$$R_n^{\alpha,\beta}(z) = \frac{P_n^{\alpha,\beta}(z)}{P_n^{\alpha,\beta}(1)}.$$

Using the identity

$$R_n^{\alpha,\beta}(z) = \frac{1}{2^n}\sum_{l=0}^{n}\frac{(-n)_l(-n-\beta)_l}{(\alpha+1)_l l!}(z-1)^l(z+1)^{n-l}$$

(see [18, p. 68]), along with a beta integral computation, we obtain

$$\int_0^{\pi/2} G(\alpha,\beta,x^2\sin^2\phi,y^2\cos^2\phi)R_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi)$$

$$= \sum_{r=0}^{+\infty}\sum_{s=0}^{+\infty}\frac{A_{r+s}x^{2r}y^{2s}}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)r!s!}\int_0^{\pi/2} R_n^{\alpha,\beta}(\cos 2\phi)\sin^{2r}\phi\cos^{2s}\phi\,dm_{\alpha,\beta}(\phi)$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{A_{r+s}x^{2r}y^{2s}}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)r!s!2^{\alpha+\beta+s+r+2}} \times\int_{-1}^{1} R_n^{\alpha,\beta}(t)(1-t)^r(1+t)^s(1-t)^\alpha(1+t)^\beta\,dt$$

$$= \sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{A_s x^{2s-2k}y^{2k}}{\Gamma(\alpha+s-k+1)\Gamma(\beta+k+1)(s-k)!k!2^{\alpha+\beta+s+2}} \times\int_{-1}^{1} R_n^{\alpha,\beta}(t)(1-t)^{s-k+\alpha}(1+t)^{k+\beta}\,dt$$

$$= \sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{A_s x^{2s-2k}y^{2k}\Gamma(\beta+k+n+1)}{\Gamma(\beta+k+1)(s-k)!k!2\Gamma(\alpha+\beta+n+s+2)} \times {}_3F_2(-n,-\beta-n,\alpha+s-k+1;\alpha+1,-\beta-k-n;1).$$

On the other hand, recalling that by (1.5),

$$R_n^{\alpha,\beta}(z) = \sum_{l=0}^{+\infty}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l l!}\left(\frac{1-z}{2}\right)^l,$$

we have

$$\frac{1}{2}H_n(\alpha+\beta+1,\rho^2)R_n^{\alpha,\beta}(\cos 2\theta)$$

$$= \sum_{s=n}^{+\infty}\frac{A_s\rho^{2s}}{2\Gamma(\alpha+\beta+n+2+s)(s-n)!}R_n^{\alpha,\beta}\left(\frac{y^2-x^2}{y^2+x^2}\right)$$

$$= \sum_{s=n}^{+\infty}\frac{A_s}{2\Gamma(\alpha+\beta+n+2+s)(s-n)!} \times\sum_{l=0}^{n}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l l!}x^{2l}(x^2+y^2)^{s-l}$$

$$= \sum_{s=n}^{+\infty}\frac{A_s}{2\Gamma(\alpha+\beta+n+2+s)(s-n)!}$$

$$\times \sum_{l=0}^{n} \frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l l!} \sum_{k=0}^{s-l} \binom{s-l}{k} x^{2s-2k} y^{2k}$$

$$= \sum_{s=n}^{+\infty} \sum_{k=0}^{s} \frac{A_s x^{2s-2k} y^{2k}}{2\Gamma(\alpha+\beta+n+2+s)(s-n)!}$$

$$\times \sum_{l=0}^{\min(s-k,n)} \frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l l!} \binom{s-l}{k}$$

$$= \sum_{s=n}^{+\infty} \sum_{k=0}^{s} \frac{A_s x^{2s-2k} y^{2k}}{2\Gamma(\alpha+\beta+n+2+s)(s-n)!}$$

$$\times \sum_{l=0}^{+\infty} \frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l l!} \binom{s-l}{k}$$

$$= \sum_{s=n}^{+\infty} \sum_{k=0}^{s} \frac{A_s x^{2s-2k} y^{2k}}{2\Gamma(\alpha+\beta+n+2+s)(s-n)!} \frac{s!}{(s-k)!k!}$$

$$\times {}_3F_2(-n, \alpha+\beta+n+1, k-s; \alpha+1, -s; 1).$$

Now, by Thomae's identity (see [21]),

$$\begin{aligned} &{}_3F_2(-n, -\beta-n, \alpha+s-k+1; \alpha+1, -\beta-k-n; 1) \\ &\qquad = \frac{(-1)^n(-s)_n}{(\beta+k+1)_n} {}_3F_2(-n, \alpha+\beta+n+1, k-s; \alpha+1, -s; 1), \end{aligned}$$

and the theorem follows. ∎

Observe that formulas (1.8) and (1.10) follow as a particular case of the above theorem, taking $A_k = (-1/4)^k$, so that

$$H_n(\alpha, z^2) = (-1)^n 2^\alpha \frac{J_{\alpha+2n}(z)}{z^\alpha},$$

$$G(\alpha, \beta, x^2, y^2) = 2^\alpha \frac{J_\alpha(x)}{x^\alpha} 2^\beta \frac{J_\beta(y)}{y^\beta}.$$

Certain particular cases of Theorem 1.2 deserve to be properly emphasized, because they generalize Sonine's integrals of first and second type and formula (1.7) to this new general context.

Corollary 1.3. *For $\theta = \pi/2$ (that is, $y = 0$), Bateman's expansion becomes a generalization of formula* (1.7),

$$\begin{aligned} (1.20)\qquad & H_0(\alpha, x^2 \sin^2\phi) \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)} H_n(\alpha+\beta+1, x^2) P_n^{\alpha,\beta}(\cos 2\phi), \end{aligned}$$

or, equivalently

$$(1.21)\qquad \int_0^{\pi/2} H_0(\alpha, x^2\sin^2\phi)P_n^{\alpha,\beta}(\cos 2\phi)\, dm_{\alpha,\beta}(\phi) = \frac{(-1)^n\Gamma(\beta+n+1)}{2\Gamma(n+1)}\, H_n(\alpha+\beta+1; x^2).$$

For $n=0$, formula (1.19) *becomes a generalized version of Sonine's second finite integral*

$$(1.22)\qquad \int_0^{\pi/2} G(\alpha,\beta,x^2\sin^2\phi, y^2\cos^2\phi)\, dm_{\alpha,\beta}(\phi) = \frac{1}{2}H_0(\alpha+\beta+1; x^2+y^2),$$

while formula (1.21) *becomes a generalized version of Sonine's first finite integral*

$$(1.23)\qquad \int_0^{\pi/2} H_0(\alpha; x^2\sin^2\phi)\, dm_{\alpha,\beta}(\phi) = \frac{1}{2}\Gamma(\beta+1)H_0(\alpha+\beta+1; x^2).$$

The next diagram describes the implications between the above formulas. In it, we use the following notations:

BE: The generalized Bateman Expansion formula (1.18).
BI: The generalized Bateman Integral formula (1.19).
NE: The generalized Neumann series Expansion formula (1.20).
NI: The generalized Neumann Integral formula (1.21).
S2: The generalized Sonine's second finite integral (1.22).
S1: The generalized Sonine's first finite integral (1.23).

$$\begin{tikzcd}
\mathrm{BE} \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow, "\theta=\frac{\pi}{2}"'] & & \mathrm{BI} \arrow[dl, Rightarrow, "y=0"'] \arrow[dr, Rightarrow, "n=0"] & \\
\mathrm{NE} \arrow[r, Leftrightarrow] & \mathrm{NI} \arrow[dr, Rightarrow, "n=0"'] & & \mathrm{S2} \arrow[dl, Rightarrow, "y=0"] \\
& & \mathrm{S1} &
\end{tikzcd}$$

2. Particular cases. The formulas we proved in the last section are very general. Thus, it may be useful to write them in terms of the precise special function we are interested in. In this section we will consider Laguerre polynomials, Jacobi polynomials/functions, Gegenbauer polynomials, and hypergeometric functions. In order to avoid cumbersome repetitions, we will state only the integral formulas of BI (1.19) and NI (1.21) type. Indeed, all other integral formulas are particular cases corresponding to $n=0$, while the expansion formulas follow readily from these two by observing that if

$$\int_0^{\pi/2} f(\theta,\phi)P_n^{\alpha,\beta}(\cos 2\phi)\,dm^{\alpha,\beta}(\phi) = C_n P_n^{\alpha,\beta}(\cos 2\theta)$$

then

$$f(\theta,\phi) = \sum_{n=0}^{+\infty} C_n p_n^{\alpha,\beta}(\cos 2\theta) p_n^{\alpha,\beta}(\cos 2\phi),$$

while if

$$\int_0^{\pi/2} f(\phi)P_n^{\alpha,\beta}(\cos 2\phi)\,dm^{\alpha,\beta}(\phi) = D_n$$

then

$$f(\phi) = \sum_{n=0}^{+\infty} \frac{2(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\, D_n P_n^{\alpha,\beta}(\cos 2\phi).$$

On the other hand, although the NI formulas follow from the corresponding BI formulas by taking $y = 0$ (or $\theta = \pi/2$), this substitution carries a few non-trivial computations, and it is therefore worthwile writing both formulas explicitly.

2.1. Laguerre polynomials. When $A_k = (-h)_k$ for a positive integer h, then the two functions H_n and G become

$$H_n(\alpha,z) = \begin{cases} \displaystyle\sum_{k=0}^{h-n} \frac{(-h)_{k+n} z^{k+n}}{\Gamma(\alpha+2n+k+1)k!} \\ \displaystyle\qquad = \frac{(-z)^n \Gamma(h+1)}{\Gamma(\alpha+n+h+1)}\, L_{h-n}^{\alpha+2n}(z) & \text{for } n \le h, \\ 0 & \text{for } n > h, \end{cases} \tag{2.1}$$

$$\begin{aligned} G(\alpha,\beta,x,y) &= \sum_{r+s\le h} \frac{(-h)_{r+s} x^r y^s}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1) r! s!} \\ &= \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\, \Psi_2(-h;\alpha+1,\beta+1;x,y) \end{aligned} \tag{2.2}$$

(here Ψ_2 is the Humbert function, see [8, p. 28]). The function

$$\Psi_2(-h;\alpha+1,\beta+1;x,y)$$

is a polynomial of degree h in the variables x and y and can therefore be expressed in a more friendly fashion, as a linear combination of $L_n^\alpha(x)L_m^\beta(y)$ with $n+m \le h$. The next propositions deal with this task.

PROPOSITION 2.1. *For any $\theta \in [0,\pi/2]$, the polynomials*

$$\Psi_2(-h;\alpha+1,\beta+1;x\sin^2\theta, y\cos^2\theta)$$

solve the differential equation

$$x\,\frac{\partial^2 u}{\partial x^2}+y\,\frac{\partial^2 u}{\partial y^2}+(\alpha+1-x)\,\frac{\partial u}{\partial x}+(\beta+1-y)\,\frac{\partial u}{\partial y}+hu=0 \tag{2.3}$$

Proof. An absolutely convergent series $u(x,y)=\sum a_{r,s}x^r y^s$ solves (2.3) if and only if

$$(r+1)(r+\alpha+1)a_{r+1,s}+(s+1)(s+\beta+1)a_{r,s+1}=(r+s-h)a_{r,s},$$

and this condition holds when

$$a_{r,s}=\frac{(-h)_{r+s}\sin^{2r}\theta\cos^{2s}\theta}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)r!s!}. \ ■$$

PROPOSITION 2.2. *The following equality holds:*

$$\Psi_2(-h;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta)$$
$$=\sum_{m=0}^{h}\frac{\Gamma(h+1)}{(\alpha+1)_m(\beta+1)_{h-m}}\sin^{2m}\theta\cos^{2h-2m}\theta\,L_m^{\alpha}(x)L_{h-m}^{\beta}(y).$$

Proof. Since $\Psi_2(-h;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta)$ is a polynomial of degree h in the x variable, we have

$$\Psi_2(-h;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta)=\sum_{m=0}^{h}B_{h,m}(\theta,y)L_m^{\alpha}(x),$$

and this must be a solution of (2.3) for all θ. Thus

$$\sum_{m=0}^{h}B_{h,m}(\theta,y)x\,\frac{\partial^2 L_m^{\alpha}}{\partial x^2}(x)+y\,\frac{\partial^2 B_{h,m}}{\partial y^2}(\theta,y)L_m^{\alpha}(x)$$
$$+(\alpha+1-x)B_{h,m}(\theta,y)\,\frac{\partial L_m^{\alpha}}{\partial x}(x)$$
$$+(\beta+1-y)\,\frac{\partial B_{h,m}}{\partial y}(\theta,y)L_m^{\alpha}(x)+hB_{h,m}(\theta,y)L_m^{\alpha}(x)=0.$$

It is well known that $x\frac{\partial^2 L_m^{\alpha}}{\partial x^2}(x)+(\alpha+1-x)\frac{\partial L_m^{\alpha}}{\partial x}(x)=-mL_m^{\alpha}(x)$ (see [18, p. 100]). Thus

$$\sum_{m=0}^{h}-mB_{h,m}(\theta,y)L_m^{\alpha}(x)+y\,\frac{\partial^2 B_{h,m}}{\partial y^2}(\theta,y)L_m^{\alpha}(x)$$
$$+(\beta+1-y)\,\frac{\partial B_{h,m}}{\partial y}(\theta,y)L_m^{\alpha}(x)+hB_{h,m}(\theta,y)L_m^{\alpha}(x)=0,$$

and therefore, for all $m=1,\ldots,h$, it must be

$$y\,\frac{\partial^2 B_{h,m}}{\partial y^2}(\theta,y)+(\beta+1-y)\,\frac{\partial B_{h,m}}{\partial y}(\theta,y)+(h-m)B_{h,m}(\theta,y)=0.$$

It is also well known (see again [18, p. 101]) that the only polynomial solution to the above equation is

$$B_{h,m}(\theta, y) = C_{h,m}(\theta) L^{\beta}_{h-m}(y).$$

Thus

$$\Psi_2(-h; \alpha+1, \beta+1; x\sin^2\theta, y\cos^2\theta) = \sum_{m=0}^{h} C_{h,m}(\theta) L^{\beta}_{h-m}(y) L^{\alpha}_{m}(x).$$

In order to determine $C_{h,m}(\theta)$, observe that the coefficient of the term $x^m y^{h-m}$ is

$$\frac{(-h)_h \sin^{2m}\theta \cos^{2h-2m}\theta}{(\alpha+1)_m(\beta+1)_{h-m} m!(h-m)!}$$

in the left hand side, and

$$C_{h,m}(\theta)\, \frac{(-1)^h}{m!(h-m)!}$$

in the right hand side, and these two must coincide. ■

We are now ready to write Bateman's integrals for Laguerre polynomials, in two different versions: one involving the Humbert function Ψ_2, and the other involving just Laguerre polynomials. Applying formula (1.19) with H_n and G given by (2.1) and (2.2), and Ψ_2 given by Proposition 2.2, one obtains the following BI-type integrals:

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+n+h+2)}\,(-\rho)^n L^{\alpha+\beta+2n+1}_{h-n}(\rho) P^{\alpha,\beta}_n(\cos 2\theta) \qquad (2.4)\\
&= \frac{1}{\Gamma(h+1)} \int_0^{\pi/2} P^{\alpha,\beta}_n(\cos 2\phi)\\
&\qquad \times \Psi_2(-h; \alpha+1, \beta+1; \rho\sin^2\phi\sin^2\theta, \rho\cos^2\phi\cos^2\theta)\, dm_{\alpha,\beta}(\phi)\\
&= \sum_{m=0}^{h} \frac{\sin^{2m}\theta\cos^{2h-2m}\theta}{(\alpha+1)_m(\beta+1)_{h-m}}\\
&\qquad \times \int_0^{\pi/2} P^{\alpha,\beta}_n(\cos 2\phi) L^{\alpha}_m(\rho\sin^2\phi) L^{\beta}_{h-m}(\rho\cos^2\phi)\, dm_{\alpha,\beta}(\phi)\\
&= \sum_{m=0}^{h} \frac{L^{\alpha}_m(\rho\sin^2\theta) L^{\beta}_{h-m}(\rho\cos^2\theta)}{(\alpha+1)_m(\beta+1)_{h-m}}\\
&\qquad \times \int_0^{\pi/2} P^{\alpha,\beta}_n(\cos 2\phi) \sin^{2m}\phi \cos^{2h-2m}\phi\, dm_{\alpha,\beta}(\phi)
\end{aligned}$$

where $0 \le n \le h$. For $\theta = \pi/2$ one obtains the NI-type integrals

$$(2.5)\qquad \frac{\Gamma(\alpha+h+1)\Gamma(\beta+n+1)}{2\Gamma(\alpha+\beta+n+h+2)\Gamma(n+1)}\,\rho^n L_{h-n}^{\alpha+\beta+2n+1}(\rho)$$

$$= \int_0^{\pi/2} P_n^{\alpha,\beta}(\cos 2\phi) L_h^{\alpha}(\rho\sin^2\phi)\, dm_{\alpha,\beta}(\phi)$$

$$= \sum_{m=0}^{h} \frac{\Gamma(\alpha+h+1)L_m^{\alpha}(\rho)}{\Gamma(\alpha+1+m)\Gamma(h-m+1)} \int_0^{\pi/2} P_n^{\alpha,\beta}(\cos 2\phi)\sin^{2m}\phi\cos^{2h-2m}\phi\, dm_{\alpha,\beta}(\phi)$$

where $0 \le n \le h$. It is perhaps worthwhile writing explicitly the cases $n = 0$: formula (2.4) gives analogs of Sonine's second finite integral (S2-type formulas)

$$(2.6)\qquad L_h^{\alpha+\beta+1}(\rho) = \frac{2\Gamma(\alpha+\beta+h+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(h+1)}$$

$$\times \int_0^{\pi/2} \Psi_2(-h;\alpha+1,\beta+1;\rho\sin^2\phi\sin^2\theta,\rho\cos^2\phi\cos^2\theta)\, dm_{\alpha,\beta}(\phi)$$

$$= \sum_{m=0}^{h} \frac{2\Gamma(\alpha+\beta+h+2)}{\Gamma(\alpha+m+1)\Gamma(\beta+h-m+1)} \sin^{2m}\theta\cos^{2h-2m}\theta$$

$$\times \int_0^{\pi/2} L_m^{\alpha}(\rho\sin^2\phi) L_{h-m}^{\beta}(\rho\cos^2\phi)\, dm_{\alpha,\beta}(\phi)$$

$$= \sum_{m=0}^{h} L_m^{\alpha}(\rho\sin^2\theta) L_{h-m}^{\beta}(\rho\cos^2\theta) \qquad \text{(see [13, p. 96])},$$

while (2.5) become two S1-type formulas

$$(2.7)\quad L_h^{\alpha+\beta+1}(x) = \frac{2\Gamma(\alpha+\beta+h+2)}{\Gamma(\alpha+h+1)\Gamma(\beta+1)} \int_0^{\pi/2} L_h^{\alpha}(x\sin^2\phi)\, dm_{\alpha,\beta}(\phi)$$

(Koshlyakov (1.15))

$$= \sum_{m=0}^{h} \frac{\Gamma(\beta+h-m+1)}{\Gamma(\beta+1)\Gamma(h-m+1)} L_m^{\alpha}(x) \qquad \text{(see [13, p. 96])}.$$

2.2. Jacobi functions. If in the definition of Jacobi polynomials we assume that $\mu = n$ is not necessarily an integer, and let the sum go from 0 to $+\infty$, we obtain the so-called *Jacobi functions* (see [10])

$$
\begin{aligned}
P_\mu^{\alpha,\gamma}(x) &= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+1)\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} \frac{(-\mu)_k(\mu+\alpha+\gamma+1)_k}{(\alpha+1)_k k!} \left(\frac{1-x}{2}\right)^k \\
&= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+1)\Gamma(\alpha+1)} \left(\frac{1+x}{2}\right)^\mu \sum_{k=0}^{+\infty} \frac{(-\mu)_k(-\mu-\gamma)_k}{(\alpha+1)_k k!} \left(\frac{x-1}{x+1}\right)^k .
\end{aligned}
$$

Thus, if we let $A_k = (-\mu)_k(-\mu-\gamma)_k$, then the two functions H_n and G become

$$
\begin{aligned}
H_n(\alpha, z) &= \sum_{k=0}^{+\infty} \frac{(-\mu)_{k+n}(-\mu-\gamma)_{k+n} z^{k+n}}{\Gamma(\alpha+2n+k+1)k!} \\
&= \frac{\Gamma(\mu+1)\Gamma(\mu+\gamma+1)}{\Gamma(\mu-n+\gamma+1)\Gamma(\alpha+\mu+n+1)}\, z^n(1-z)^{\mu-n} P_{\mu-n}^{\alpha+2n,\gamma}\left(\frac{1+z}{1-z}\right), \\
G(\alpha,\beta,x,y) &= \sum_{r,s=0}^{+\infty} \frac{(-\mu)_{r+s}(-\mu-\gamma)_{r+s} x^r y^s}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)r!s!} \\
&= \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\, F_4(-\mu, -\mu-\gamma; \alpha+1, \beta+1; x, y)
\end{aligned}
$$

where F_4 denotes the Appell function (see [7, p. 224], or [8, p. 23]).

Applying Theorem 1.2 in this case gives (BI)

(2.8)

$$
\begin{aligned}
&\int_0^{\pi/2} (\rho+1)^\mu F_4\left(-\mu, -\mu-\gamma; \alpha+1, \beta+1; \frac{\rho-1}{\rho+1}\sin^2\theta\sin^2\phi, \frac{\rho-1}{\rho+1}\cos^2\theta\cos^2\phi\right) \\
&\qquad\qquad \times P_n^{\alpha,\beta}(\cos 2\phi)\, dm_{\alpha,\beta}(\phi) \\
&\quad = \frac{2^{\mu-n-1}\Gamma(\alpha+1)\Gamma(\beta+1)(-\mu)_n(-\gamma-\mu)_n\Gamma(\mu-n+1)}{\Gamma(\alpha+\beta+n+\mu+2)} \\
&\qquad\qquad \times (\rho-1)^n P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}(\rho) P_n^{\alpha,\beta}(\cos 2\theta)
\end{aligned}
$$

where $0 \le n \le \mu$ if μ is a non-negative integer. Applying formula (1.21), we obtain a NI-type formula

$$
\begin{aligned}
&\int_0^{\pi/2} P_n^{\alpha,\beta}(\cos 2\phi)[1-\rho\sin^2\phi]^\mu P_\mu^{\alpha,\gamma}\left(\frac{1+\rho\sin^2\phi}{1-\rho\sin^2\phi}\right) dm_{\alpha,\beta}(\phi) \qquad (2.9)\\
&\quad = \frac{(-1)^n\Gamma(\mu+\gamma+1)\Gamma(\beta+n+1)\Gamma(\alpha+\mu+1)}{2\Gamma(\mu+\gamma-n+1)\Gamma(\alpha+\beta+n+\mu+2)\Gamma(n+1)} \\
&\qquad\qquad \times \rho^n(1-\rho)^{\mu-n} P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}\left(\frac{1+\rho}{1-\rho}\right)
\end{aligned}
$$

where $0 \le n \le \mu$ if μ is a non-negative integer; Askey and Fitch's formula (1.13) follows on taking $n = 0$, μ a non-negative integer and $x = (1+\rho)/(1-\rho)$.

2.3. Gegenbauer polynomials. The formulas from this subsection follow from those in the previous subsection, by means of the identities (see formula (1.11) and [18, p. 59])

$$P_h^{\alpha+1/2}(r) = \begin{cases} \dfrac{\sqrt{\pi}\,\Gamma(\alpha+m+1/2)}{\Gamma(\alpha+1/2)\Gamma(m+1/2)}\, P_m^{\alpha,-1/2}(2r^2-1) & \text{if } h = 2m, \\ \dfrac{\sqrt{\pi}\,\Gamma(\alpha+m+3/2)}{\Gamma(\alpha+1/2)\Gamma(m+3/2)}\, rP_m^{\alpha,1/2}(2r^2-1) & \text{if } h = 2m+1. \end{cases}$$

Taking $\gamma = -1/2$ and $h = 2\mu$, or $\gamma = 1/2$ and $h = 2\mu+1$ in formulas (2.8), (2.9) according to whether h is even or odd respectively, and letting $\frac{1+\rho}{1-\rho} = 2r^2 - 1$ one obtains (BI)

(2.10)

$$\int_0^{\pi/2} r^h F_4\left(-\frac{h}{2}, -\frac{h-1}{2}; \alpha+1, \beta+1; \frac{r^2-1}{r^2}\sin^2\theta\sin^2\phi, \frac{r^2-1}{r^2}\cos^2\theta\cos^2\phi\right) \times P_n^{\alpha,\beta}(\cos 2\phi)\, dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)(-h/2)_n\left(-\frac{h-1}{2}\right)_n\Gamma(h/2-n+1)\Gamma(h/2-n+1/2)\Gamma(\alpha+\beta+2n+3/2)}{2\sqrt{\pi}\,\Gamma(\alpha+\beta+n+h/2+2)\Gamma(\alpha+\beta+n+h/2+3/2)} \times (r^2-1)^n P_{h-2n}^{\alpha+\beta+2n+3/2}(r) P_n^{\alpha,\beta}(\cos 2\theta)$$

if $0 \le 2n \le h$; letting $\frac{1+\rho}{1-\rho} = 2\cos^2\psi - 1$ in (2.9), we obtain (NI)

(2.11)

$$\int_0^{\pi/2} P_n^{\alpha,\beta}(\cos 2\phi)(1-\sin^2\psi\cos^2\phi)^{h/2} P_h^{\alpha+1/2}\left(\frac{\cos\psi}{\sqrt{1-\sin^2\psi\cos^2\phi}}\right) dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+\beta+2n+3/2)\Gamma(\beta+n+1)\Gamma(\alpha+h/2+1)\Gamma(\alpha+h/2+1/2)}{2\Gamma(\alpha+\beta+n+h/2+2)\Gamma(\alpha+\beta+n+h/2+3/2)\Gamma(\alpha+1/2)\Gamma(n+1)} \times \sin^{2n}\psi\, P_{h-2n}^{\alpha+\beta+2n+3/2}(\cos\psi)$$

if $0 \le 2n \le h$; for $n = 0$ this is Feldheim's formula (1.12).

2.4. Hypergeometric function. If there exist two non-negative integers j and h, and real numbers $a_1, \ldots, a_j, b_1, \ldots, b_h$, such that

$$A_k = \frac{(a_1)_k \ldots (a_j)_k}{(b_1)_k \ldots (b_h)_k},$$

then the functions F_n and G reduce to hypergeometric functions and Kampé de Fériet functions (see [7], [8], [13] for the definitions). Precisely

$$H_n(\alpha,z)=\frac{(a_1)_n\dots(a_j)_n z^n}{\Gamma(\alpha+2n+1)(b_1)_n\dots(b_h)_n}\\ \times {}_jF_{h+1}(a_1+n,\dots,a_j+n;b_1+n,\dots,b_h+n,\alpha+2n+1;z),$$

$$G(\alpha,\beta,x,y)=\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}F^{j:0}_{h:1}\left(\begin{matrix}a_1,\dots,a_j: & ; & ; \\ b_1,\dots,b_h: & \alpha+1; & \beta+1;\end{matrix}\ x,y\right).$$

Thus, the integral formulas are:

$$(2.12)\quad \int_0^{\pi/2} F^{j:0}_{h:1}\left(\begin{matrix}a_1,\dots,a_j: & ; & ; \\ b_1,\dots,b_h: & \alpha+1; & \beta+1;\end{matrix}\ \rho^2\sin^2\theta\sin^2\phi,\rho^2\cos^2\theta\cos^2\phi\right)\\ \times P_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi)\\ =\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(a_1)_n\dots(a_j)_n}{2\Gamma(\alpha+\beta+2n+2)(b_1)_n\dots(b_h)_n}\rho^{2n}\\ \times {}_jF_{h+1}(a_1+n,\dots,a_j+n;b_1+n,\dots,b_h+n,\alpha+\beta+2n+2;\rho^2)P_n^{\alpha,\beta}(\cos 2\theta)$$

and

$$(2.13)\quad \int_0^{\pi/2} P_n^{\alpha,\beta}(\cos 2\phi)\,{}_jF_{h+1}(a_1,\dots,a_j;b_1,\dots,b_h,\alpha+1;x^2\sin^2\phi)\,dm_{\alpha,\beta}(\phi)\\ =\frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)(a_1)_n\dots(a_j)_n}{2\Gamma(\alpha+\beta+2n+2)\Gamma(n+1)(b_1)_n\dots(b_h)_n}(-x^2)^n\\ \times {}_jF_{h+1}(a_1+n,\dots,a_j+n;b_1+n,\dots,b_h+n,\alpha+\beta+2n+2;x^2).$$

When $n=0$, $j=2$ and $h=0$, formula (2.13) is originally due to Bateman (see [5, p. 184]). It also appears as an exercise in [13, p. 277].

Formula (2.12) with $n=0$ follows as a particular case of formula (1a) in [17], by taking there $h=2$, $n=1$, $s=2\alpha+2$, $\sigma=\beta+1$, $\mu=0$, $\rho=1$, $\delta=\alpha+1$, $\delta'=\beta+1$, $a=\rho^2\sin^2\theta$ and $b=\rho^2\cos^2\theta$. Although the authors study several particular cases of their formula, they seem to miss this particular one.

Finally, when $j=1$ and $h=0$, formula (2.13) is equivalent to (an integral version of) Erdélyi's formula (1.16).

3. Final remarks. In [11, p. 738], P. Henrici proves an interesting formula on the product of two Whittaker functions. An equivalent restatement of his formula in terms of the confluent hypergeometric function ${}_1F_1$ is

$$(3.1)\quad \int_0^{\pi/2} {}_1F_1(A,\alpha+1,\rho\sin^2\phi)\,{}_1F_1(B,\beta+1,\rho\cos^2\phi)P_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi)$$
$$= \frac{\Gamma(\alpha+n+1)\Gamma(\beta+1)(\beta-B+1)_n}{2n!\Gamma(\alpha+\beta+2n+2)}$$
$$\times\, {}_3F_2(-\beta-n,\alpha-A+1,-n;B-\beta-n,\alpha+1;1)$$
$$\times\, (-\rho)^n\,{}_1F_1(A+B+n;\alpha+\beta+2n+2;\rho).$$

Henrici shows several particular cases of (3.1), some of which are Bateman's expansion (1.8) and Erdélyi's formula (1.16). Formula (3.1) should be compared with (2.12) in the case $j=1$, $h=0$. On the left hand side it presents the product of two confluent hypergeometric functions, rather than a double series. On the other hand, (2.12) involves a "free" parameter θ, which in Henrici's formula is set equal to $\pi/4$. Although it does not seem possible to deduce Henrici's formula from (2.12), or more generally from (1.19), it is fairly simple to modify the proof of Theorem 1.2 in order to prove (3.1). One only has to replace A_{r+s} with $(A)_r(B)_s$, set $x=y=\rho$, and observe that the following identity holds:

$$\sum_{k=0}^{N-l}\binom{N-l}{k}\frac{(A)_{N-k}(B)_k}{(A+B)_N} = \frac{(A)_l}{(A+B)_l}.$$

Due to the particularity of the above identity, this proof cannot be used to show a hypothetical generalization of Henrici's formula to, say, ${}_2F_1$ or other hypergeometric functions.

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