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REFLEXIVELY REPRESENTABLE BUT NOT HILBERT REPRESENTABLE COMPACT FLOWS AND SEMITOPOLOGICAL SEMIGROUPS

BY

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Abstract. We show that for many natural topological groups G (including the group $\mathbb{Z}$ of integers) there exist compact metric G-spaces (cascades for $G=\mathbb{Z}$) which are reflexively representable but not Hilbert representable. This answers a question of T. Downarowicz. The proof is based on a classical example of W. Rudin and its generalizations. A crucial step in the proof is our recent result which states that every weakly almost periodic function on a compact G-flow X comes from a G-representation of X on reflexive spaces. We also show that there exists a monothetic compact metrizable semitopological semigroup S which does not admit an embedding into the semitopological compact semigroup $\Theta(H)$ of all contractive linear operators on a Hilbert space H (though S admits an embedding into the compact semigroup $\Theta(V)$ for certain reflexive V).

1. Matrix coefficients and Eberlein groups

1.1. *Preliminaries.* Let X be a topological space and S be a *semitopological semigroup* (that is, the multiplication map $S \times S \to S$ is separately continuous). Let $S \times X \to X$, $(s,x) \mapsto sx$, be a (left) action of S on X. As usual, we say that (S,X), or X (when S is understood), is an *S-space* if the action is at least separately continuous. For topological groups we reserve the symbol G. For group actions *G-space*, *G-system* or a *G-flow* will mean that the action is jointly continuous.

As usual, (G,X) is (*point*) *transitive* means that X has a dense G-orbit.

Right actions (X,S) can be defined analogously. If S^{op} is the *opposite semigroup* of S (with the same topology) then (X,S) can be treated canonically as a left action (S^{op},X) (and vice versa). A mapping $h: S_1 \to S_2$ between semigroups S_1 and S_2 is said to be a *co-homomorphism* if $h(st)=h(t)h(s)$ for every $s,t \in S_1$. This is equivalent to h being a usual homomorphism from S_1 to S_2^{op}. If, in addition, S_1 and S_2 carry some topologies and h is a homeomorphism then we say that h is a *co-isomorphism* and S_1 and S_2 are *topologically co-isomorphic.*

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We say that an S-space X is a *subdirect product* of a class Γ of S-spaces if X is an S-subspace of an S-product of some members from Γ.

All topological spaces are assumed to be Tikhonov, that is, Hausdorff and completely regular. For every topological space X denote by $C(X) = C(X, K)$ the algebra of all bounded continuous K-valued functions on X with respect to the sup-norm, where K is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Recall the following very useful fact.

LEMMA 1.1 (Grothendieck's lemma). *Let X be a compact space. Then a bounded subset A of $C(X)$ is weakly compact* (*for short*, *w-compact*) *iff A is pointwise compact.*

Let X be a compact S-flow. Denote by $E := E(X) \subset X^X$ the corresponding (compact right topological) enveloping semigroup. It is the pointwise closure of the set of translations $\{\widetilde{s} : X \to X\}_{s\in S}$ in the product space X^X.

1.2. *WAP functions and systems.* A function $f \in C(X)$ on an S-space X is *weakly almost periodic* (wap, for short) if the orbit $fS := \{fs\}_{s\in S}$ of f (with respect to the canonical right action $C(X) \times S \to C(X)$, $(\varphi, s) \mapsto \varphi s$, where $(\varphi s)(x) := \varphi(sx)$) is relatively weakly compact in $C(X)$. The set $\mathrm{WAP}(X)$ of all wap functions on X is a closed subalgebra of $C(X)$. In particular, we can consider S as a natural S-space $X := S$. The corresponding algebra of wap functions will be denoted simply by $\mathrm{WAP}(S)$.

REMARK 1.2. A bounded function $f \in C(X) = C(X, K)$ on an S-space X is wap iff it has the *double limit property* (DLP). This follows easily by Grothendieck's classical results (see Lemma 1.1 and Theorem A.4 of [2, Appendix A]). Recall that DLP for the function f means precisely that for every pair of sequences $\{s_n\}_{n\in\mathbb{N}}, \{x_m\}_{m\in\mathbb{N}}$ in S and X respectively,

$$\lim_m \lim_n s_n x_m = \lim_n \lim_m s_n x_m$$

whenever both limits exist.

DEFINITION 1.3 (Ellis and Nerurkar [10] for $S := G$). A compact S-space X is *weakly almost periodic* (wap, for short) if $\mathrm{WAP}(X) = C(X)$.

LEMMA 1.4 (Ellis and Nerurkar [10] for $S := G$). *Let X be a compact S-space. The following conditions are equivalent*:

(1) *(S, X) is wap.*
(2) *The enveloping semigroup $E(X)$ of (S, X) consists of continuous maps.*

The following well known fact easily follows from Lemma 1.4.

FACT 1.5. *If (S, X) is wap then the enveloping semigroup $E(X)$ is a compact semitopological semigroup.*

Every metrizable wap compact G-system X comes from representations of (G, X) on reflexive Banach spaces (see Theorem 2.10(3) below).

For more details about wap functions on S-spaces (including the proof of Lemma 1.4) see [19] or [27].

1.3. *Representations of groups and operator topologies.* Let V be a Banach space. Denote by $\mathrm{Aut}(V)$ the group of all continuous linear automorphisms of V. Its subgroup of all linear surjective isometries $V \to V$ will be denoted by $\mathrm{Is}(V)$. In the present paper we consider only group representations into $\mathrm{Is}(V)$. More precisely, a *representation* (*co-representation*) of a topological group G on a Banach space V is a homomorphism (resp. co-homomorphism) $h\colon G \to \mathrm{Is}(V)$. One can endow $\mathrm{Is}(V)$ with the *strong operator topology* inherited from V^V. Denote by V_w the space V in its weak topology. The corresponding topology on $\mathrm{Is}(V)$ inherited from V_w^V is the *weak operator topology.* Recall that a Banach space V is said to have the *point of continuity property* (PCP) if every bounded weakly closed subset $C \subset V$ admits a point of continuity of the identity map $(C, \mathrm{weak}) \to (C, \mathrm{norm})$ (see for example [18, p. 55]). Every reflexive space has PCP.

Theorem 1.6 ([25]). *Let V be a Banach space with PCP (e.g., reflexive).*

(1) *For every bounded subgroup H of* $\mathrm{Aut}(V)$ *the weak and strong operator topologies coincide on H. Hence every weakly continuous (co)representation $h : G \to \mathrm{Is}(V)$ on V with PCP is strongly continuous.*
(2) *The weak and strong operator topologies coincide on* $\mathrm{Is}(V)$.

1.4. *Matrix coefficients*

Definition 1.7. Let $h : G \to \mathrm{Is}(V)$ be a given co-representation of G on V and

$$V \times G \to V, \qquad (v, g) \mapsto vg := h(g)(v),$$

be the corresponding right action.

For every pair of vectors $v \in V$ and $\psi \in V^*$ there exists a canonically associated *matrix coefficient* defined by

$$m_{v,\psi} : G \to K, \qquad g \mapsto \langle vg, \psi\rangle = \langle v, g\psi\rangle.$$

Denote by $\widetilde{v} : \mathrm{Is}(V) \to V$, $i \mapsto \widetilde{v}(i) = i(v)$, the orbit map. Then the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "m_{v,\psi}"] \arrow[d, "h"'] & K \\ \mathrm{Is}(V) \arrow[r, "\widetilde{v}"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

If $h : G \to \mathrm{Is}(V)$ is a representation (that is, a group homomorphism) then it is natural to define a *matrix coefficient* $m_{v,\psi}$ by

$$m_{v,\psi} : G \to K, \quad g \mapsto \langle gv, \psi \rangle = \langle v, \psi g \rangle.$$

If $h : G \to \mathrm{Is}(H)$ is a continuous group representation into a Hilbert space H and $\psi = v$, then the corresponding map $g \mapsto \langle gv, v \rangle$ is a *positive definite function* (pdf) on G. Denote by $P(G)$ the set of all pdfs on G. The converse is also true: every continuous pdf comes from some continuous Hilbert representation (see for example [3]).

We say that a vector $v \in V$ is *norm* (resp. *weakly*) *G-continuous* if the corresponding orbit map $\widetilde{v} : G \to V$, $\widetilde{v}(g) = vg$, defined through $h : G \to \mathrm{Is}(V)$, is norm (resp., weakly) continuous. Similarly one can define a norm G-continuous vector $\psi \in V^*$ (with respect to the dual representation of G on V^*).

Note that if the co-representation $h : G \to \mathrm{Is}(V)$ is weakly continuous (that is, each $v \in V$ is weakly continuous) then $m_{v,\psi} \in C(G)$ for every $\psi \in V^*$ and $v \in V$.

Lemma 1.8. *Let $h : G \to \mathrm{Is}(V)$ be a weakly continuous co-representation of G on V. For every $\psi \in V^*$ and $v \in V$ define*

$$L_\psi : V \to C(G), \quad R_v : V^* \to C(G), \quad \textit{where} \quad L_\psi(v) = R_v(\psi) = m_{v,\psi}.$$

Then

1. *L_ψ and R_v are bounded linear operators.*
2. *If ψ (resp., $v \in V$) is norm G-continuous, then $m_{v,\psi}$ is left (resp., right) uniformly continuous on G.*
3. (Eberlein) *If V is reflexive, then $m_{v,\psi} \in \mathrm{WAP}(G)$.*

Proof. See [27, Fact 3.5] and also Example 2.8 below. ∎

Assertion (3) of this lemma comes from Eberlein. The converse is also true: every wap function is a matrix coefficient of some (co-)representation on a reflexive space (see Theorem 2.10(2) below).

1.5. *Eberlein groups*

Definition 1.9. Following Eymard [12] denote by $B(G) = B(G, \mathbb{C})$ the set of all matrix coefficients of Hilbert representations for the group G. This is the collection of functions of the form

$$m_{u,v} : G \to \mathbb{C}, \quad g \mapsto \langle gu, v \rangle,$$

where we consider all possible continuous unitary representations $h : G \to U(H)$ into complex Hilbert spaces H. Then $B(G)$ is a subalgebra of $C(G)$ closed under pointwise multiplication and complex conjugation. This algebra is called the *Fourier–Stieltjes algebra* of G (see, for example, [12, 22]). Elements of this algebra will be called *Fourier–Stieltjes functions* on G.

Analogously can be defined the real version $B(G) = B(G, \mathbb{R}) \subset C(G, \mathbb{R})$ regarding real Hilbert space representations of G.

The algebra $B(G)$ is rarely closed in $C(G)$. Precisely, if G is locally compact then $B(G)$ is closed in $C(G)$ iff G is finite. Clearly, the set $P(G)$ of positive definite functions on G is a subset of $B(G)$ and every $m \in B(G)$ is a linear combination of some elements from $P(G)$. Every positive definite function is wap (see for example [3]). Hence always $B(G) \subset \mathrm{WAP}(G)$. The question whether $B(G)$ is dense in $\mathrm{WAP}(G)$ was raised by Eberlein (see [31]) and leads to the following definition of *Eberlein groups* [22] (originally defined for locally compact groups).

DEFINITION 1.10 (Chou [5], Mayer [22]). A topological group G is called an *Eberlein group* if the uniform closure $\mathrm{cl}(B(G))$ (denoted by $E(G) = E(G, \mathbb{C})$) of $B(G) = B(G, \mathbb{C})$ in $C(G) = C(G, \mathbb{C})$ is $\mathrm{WAP}(G) = \mathrm{WAP}(G, \mathbb{C})$ (or, equivalently, if every wap function on G can be uniformly approximated by Fourier–Stieltjes functions).

Replacing $\mathbb{C}$ by $\mathbb{R}$ in this definition we get the real-valued version. In this case one may say that G is $\mathbb{R}$-*Eberlein*. However, the following lemma shows that being $\mathbb{R}$-Eberlein and Eberlein is the same.

LEMMA 1.11. *Let G be a topological group and $f \in C(G, \mathbb{C})$. Consider the canonical representation $f(g) = f_1(g) + i f_2(g)$ by two real-valued bounded functions $f_1, f_2 \in C(G, \mathbb{R})$.*

(1) $f_1, f_2 \in \mathrm{WAP}(G, \mathbb{R})$ *if and only if* $f \in \mathrm{WAP}(G, \mathbb{C})$.
(2) $f_1, f_2 \in B(G, \mathbb{R})$ *if and only if* $f \in B(G, \mathbb{C})$.
(3) *G is $\mathbb{R}$-Eberlein if and only if G is Eberlein. That is*, $\mathrm{cl}(B(G, \mathbb{R})) = \mathrm{WAP}(G, \mathbb{R})$ *if and only if* $\mathrm{cl}(B(G, \mathbb{C})) = \mathrm{WAP}(G, \mathbb{C})$.

Proof. (1) Use for example DLP and Remark 1.2.

(2) If $f_1, f_2 \in B(G, \mathbb{R})$ then there exist:

(a) two real Hilbert spaces H_1 and H_2;
(b) continuous unitary representations $h_1 : G \to U(H_1)$ and $h_2 : G \to U(H_2)$;
(c) vectors $u_1, v_1 \in H_1$ and $u_2, v_2 \in H_2$

such that $f_1(g) = \langle g u_1, v_1 \rangle$ and $f_2(g) = \langle g u_2, v_2 \rangle$. Consider the orthogonal sum $H := H_1 \oplus H_2$ of real Hilbert spaces and the complexification $H_0 := \{a + ib \mid a, b \in H\}$ of H. We have an induced unitary representation $G \to U(H_0)$ of G on the complex Hilbert space H_0. Then $f(g) = \langle gu, v \rangle$, where $u := u_1 + i u_2 \in H_0$ and $v := v_1 + v_2 \in H \subset H_0$.

Conversely, if $f \in B(G, \mathbb{C})$ then there exist: a complex Hilbert space H with its complex (sesquilinear) inner product $\langle \, , \rangle$, a continuous unitary representation $h : G \to U(H)$ and two vectors $u, v \in H$ such that $f(g) =$

$\langle gu, v\rangle$ for every $g \in G$. Denote by $H_{\mathbb{R}}$ the corresponding real Hilbert space with the real inner product $H \times H \to \mathbb{R}$, $(x, y) \mapsto \mathrm{Re}(\langle x, y\rangle)$. Now observe that $f_1(g) = \mathrm{Re}(\langle gu, v\rangle)$ and $f_2(g) = \mathrm{Re}(-i\langle gu, v\rangle) = \mathrm{Re}(\langle gu, iv\rangle)$. This proves that $f_1, f_2 \in B(G, \mathbb{R})$.

(3) follows easily using (1) and (2). ∎

REMARK 1.12.

(a) By a result of Rudin [31] the group $\mathbb{Z}$ of all integers and the group $\mathbb{R}$ of all reals are not Eberlein.
(b) More generally, Chou [5] proved that no locally compact noncompact nilpotent group is Eberlein.
(c) By a result of Veech [37] every semisimple Lie group G with a finite center (e.g., $G := \mathrm{SL}_n(\mathbb{R})$) is Eberlein. In fact, $\mathrm{WAP}(G, \mathbb{C}) = C_0(G, \mathbb{C}) \oplus \mathbb{C}$ holds for $G := \mathrm{SL}_n(\mathbb{R})$. It follows that $\mathrm{WAP}(G, \mathbb{R}) = C_0(G, \mathbb{R}) \oplus \mathbb{R}$ for $G := \mathrm{SL}_n(\mathbb{R})$.

From now on, if not otherwise stated, we assume that $K = \mathbb{R}$; in particular, all Banach spaces and algebras are assumed to be real.

2. Actions on reflexive Banach spaces. We recall some old and new results about actions on reflexive spaces. Many of them can be found in [2], [3], [9], [13], [25], [8], [36].

2.1. *Dual actions.* Let V be a Banach space. Denote by B_V its closed unit ball. The dual Banach space of V will be denoted by V^*. For every strongly continuous (co-)representation $h : G \to \mathrm{Is}(V)$ the corresponding dual action of G on the weak star compact unit ball B_{V^*} of the dual space V^* is jointly continuous. Hence, B_{V^*} becomes a G-system. Sometimes (but not in general) this action is also jointly norm continuous, as follows from the result below.

THEOREM 2.1 ([23, Corollary 6.9]). *Let V be an Asplund (e.g., reflexive) Banach space. If a, not necessarily isometric, linear action $\pi : G \times V \to V$ is continuous then so is the dual action $\pi^* : V^* \times G \to V^*$, $(fg)(v) = f(gv)$.*

2.2. *Representations of flows on Banach spaces.* Denote by $\Theta(V)$ the semigroup of all contractive linear operators $\sigma : V \to V$, $\|\sigma\| \le 1$. It is a semitopological semigroup with respect to the weak operator topology. Moreover, $\Theta(V)$ is compact iff V is reflexive iff B_V is weakly compact. The compact semitopological semigroups $\Theta(V)$ and $\Theta(V^*)$ are topologically co-isomorphic for every reflexive V. Indeed, the desired co-isomorphism is the usual adjoint map.

The following definition of flow representations has already had some significant applications [27, 14–16].

Definition 2.2 (see [27]). Let $S \times X \to X$ be a separately continuous action on X of a semitopological semigroup S.

(1) A continuous *representation* of (S, X) on a Banach space V is a pair
$$(h, \alpha) : S \times X \rightrightarrows \Theta(V) \times V^*$$
where $h : S \to \Theta(V)$ is a weakly continuous co-homomorphism (equivalently, $h : S \to \Theta(V)^{\mathrm{op}}$ is a homomorphism) of semigroups and $\alpha : X \to V^*$ is a weak star continuous bounded S-mapping with respect to the dual action
$$S \times V^* \to V^*, \qquad (s\varphi)(v) := \varphi(h(s)(v)).$$
A continuous representation (h, α) is *proper* if α is a topological embedding.

(2) (S, X) is *reflexively representable* if there exists a proper representation of (S, X) on a reflexive space V. A reflexively representable compact flow is a dynamical version of Eberlein compacta in the sense of Amir & Lindenstrauss. If we can choose V to be Hilbert, then (S, X) is called *Hilbert representable*. The classes of Hilbert representable and reflexively representable compact systems will be denoted by Hilb and Ref respectively.

(3) (S, X) is called a *Hilbert* (resp., *reflexively*) *approximable* system if it can be represented as a subdirect product of Hilbert (resp., reflexively) representable systems. Denote by $\mathrm{Hilb}_{\mathrm{app}}$ and $\mathrm{Ref}_{\mathrm{app}}$ the classes of all Hilbert (resp., reflexively) approximable compact systems.

Remark 2.3.

(1) Let V be a reflexive (resp., Hilbert) space. As usual, denote by B_V and B_{V^*} the weakly compact unit balls of V and V^* respectively. Then the canonical separately continuous left actions $(\Theta(V)^{\mathrm{op}}, B_{V^*})$ and $(\Theta(V), B_V)$ are reflexively (resp., Hilbert) representable. The first case is immediate by our definitions. For the second case observe that $(\Theta(V), B_V)$ can be identified with $(\Theta(W)^{\mathrm{op}}, B_{W^*})$ for $W := V^*$.

(2) It follows that the definition of reflexive (resp., Hilbert) representability can be simplified. Precisely, (S, X) is reflexively (resp., Hilbert) representable iff there exists a continuous *homomorphism* $h : S \to \Theta(W)$ and a weakly continuous embedding $\alpha : X \hookrightarrow B_W$ such that $\alpha(sx) = h(s)\alpha(x)$, where W is reflexive (resp., Hilbert).

(3) Reflexive representability becomes especially simple for cascades. Let $(\mathbb{Z}, X)$ be an invertible cascade induced by a self-homeomorphism $\sigma : X \to X$. Then $(\mathbb{Z}, X)$ is reflexively (resp., Hilbert) representable if and only if there exist: a reflexive (resp., Hilbert) space V, a linear isometric operator $T \in \mathrm{Is}(V)$ and a topological embedding $\alpha : X \hookrightarrow$

B_V into the weakly compact unit ball B_V such that $\alpha(\sigma x) = T\alpha(x)$ for every $x \in X$.

(3) Let S be a subsemigroup of the compact semitopological semigroup $\Theta(V)$ (with V reflexive). Consider the natural left action of S on $\Theta(V)$. Then this action is reflexively approximable. Indeed, it can be approximated by actions of the form (S, B_V) (see proof of Lemma 4.5).

Recall [34, 30] that every Hausdorff topological group G admits a proper representation into a Banach space V. Namely, we can consider the algebra $V := \mathrm{RUC}(G)$ of all right uniformly continuous functions on G. Indeed, the mapping

$$h : G \to \mathrm{Is}(\mathrm{RUC}(G)), \quad h(g)(f)(x) = f(g^{-1}x),$$

is a topological group embedding ("Teleman's representation"). Defining $h(g)(f)(x) = f(gx)$ we get a co-embedding $h : G \to \mathrm{Is}(\mathrm{RUC}(G))$.

Remark 2.4.

(1) It is easy to see that every right uniformly continuous function $f \in \mathrm{RUC}(G)$ on G is a matrix coefficient of some continuous co-representation $h : G \to \mathrm{Is}(V)$. In fact, we can always take $V := \mathrm{RUC}(G)$ (see [27]). The use of "co-representations" of G seems to be unavoidable, in general. Indeed, if $h : G \to \mathrm{Is}(V)$ is a homomorphism then the matrix coefficient $m_{v,\psi}$ is defined by

$$m_{v,\psi} : G \to \mathbb{R}, \quad g \mapsto \langle gv, \psi\rangle = \langle v, \psi g\rangle,$$

where $gv := h(g)(v)$. If h is strongly continuous then as in Lemma 1.8(2) we see that $f = m_{v,\psi}$ is necessarily left uniformly continuous. It follows that if $f \in \mathrm{RUC}(G)$ but $f \notin \mathrm{LUC}(G)$ then f cannot be represented as a matrix coefficient of a *strongly continuous* representation on a Banach space.

(2) If V is reflexive then the situation is symmetric. Turning to the dual representation we can rewrite every matrix coefficient of a co-representation as a matrix coefficient defined by means of the dual representation. The reflexivity of V implies by Theorem 2.1 that the dual representation (resp., co-representation) of a given continuous co-representation (resp., representation) is also continuous. Similarly, matrix coefficients of a representation on a reflexive space can be treated as matrix coefficients of a co-representation.

(3) If V is reflexive and $S = G$ is a group then in Definition 2.2 the weakly continuous homomorphism $h : G \to \Theta(V)$ is necessarily strongly continuous (Theorem 1.6).

(4) Every compact G-flow X also admits a proper strongly continuous Banach representation in the sense of Definition 2.2(1). Indeed, de-

fine the strongly continuous co-homomorphism

$$h : G \to \mathrm{Is}(C(X)), \quad h(g)(f)(x) := f(gx),$$

and the weak star embedding $\alpha : X \to C(X)^*$, $\alpha(x) := \delta_x$, where δ_x is the point measure $C(X) \to \mathbb{R}$, $f \mapsto f(x)$.

2.3. *Compactifications.* A *G-compactification* of (G, X) is a continuous G-map $\nu : X \to Y$ with a dense range into a compact G-flow Y. The compactification is *proper* if ν is a topological embedding.

DEFINITION 2.5 (see for example [14, 15]).

(1) We say that a function $f \in C(X)$ on a G-space X *comes from* a compact G-system Y if there exist a G-compactification $\nu : X \to Y$ (so ν is onto if X itself is compact) and a function $F \in C(Y)$ such that $f = F \circ \nu$.
(2) A function $f \in C(G)$ *comes from* a pointed system (Y, y_0) if for some continuous function $F \in C(Y)$ we have $f(g) = F(gy_0)$ for all $g \in G$. Defining $\nu : X = G \to Y$ by $\nu(g) = gy_0$ observe that this is indeed a particular case of (1).

DEFINITION 2.6.

(1) Denote by $\mathrm{Hilb}(X)$ the set of all continuous functions on a G-space X which come from Hilbert representable G-compactifications $\nu : X \to Y$. In particular, for the canonical left G-space $X := G$ we get the definition of $\mathrm{Hilb}(G)$. Similarly one can define the sets $\mathrm{Hilb}_{\mathrm{app}}(X)$ and $\mathrm{Hilb}_{\mathrm{app}}(G)$.
(2) Replacing in (1) “Hilbert” by “reflexive” we get the definitions of the sets $\mathrm{Ref}(X)$ and $\mathrm{Ref}_{\mathrm{app}}(X)$.
(3) More generally, let Γ be a class of G-flows. Denote by $\Gamma(X)$ the set of all continuous functions on a G-space X which come from a G-compactification $\nu : X \to Y$ such that Y is in Γ.

In fact, always $\mathrm{Hilb}_{\mathrm{app}}(X) = \mathrm{Hilb}(X)$ and $\mathrm{Ref}_{\mathrm{app}}(X) = \mathrm{Ref}(X)$ (see Proposition 3.5). At the same time $\mathrm{Hilb}_{\mathrm{app}} \neq \mathrm{Hilb}$ and $\mathrm{Ref}_{\mathrm{app}} \neq \mathrm{Ref}$ even for the trivial group (just take a compact space which is not an Eberlein compactum).

Let $\mathcal{A}$ be a uniformly closed subalgebra of $C(X)$ for some topological space X. The corresponding Gelfand space (that is, the maximal ideal space of $\mathcal{A}$) will be denoted by $X^{\mathcal{A}}$. Let $\nu_{\mathcal{A}} : X \to X^{\mathcal{A}}$ be the associated compactification map. For instance, the *greatest ambit* of G is the compact G-space $G^{\mathrm{RUC}} := G^{\mathrm{RUC}(G)}$. It defines the *universal* (*right topological*) *semigroup compactification* of G. For $\mathcal{A} = \mathrm{WAP}(G)$ we get the *universal semitopological compactification* $u_{\mathrm{wap}} : G \to G^{\mathrm{WAP}}$ of G, which is also the universal wap compactification (see [21]) of G.

Remark 2.7. By [29] the natural projection $q : G^{\mathrm{RUC}} \to G^{\mathrm{WAP}}$ is a homeomorphism iff G is precompact. In the converse direction, by [24] there exists a Polish nontrivial group G, namely $G := H_+[0,1]$, such that the universal semitopological compactification G^{WAP} reduces to a singleton (equivalently, every wap function is constant).

2.4. *DFJP factorization theorem for actions*

Example 2.8.

(1) The next example goes back to Eberlein [9] (see also [2, Examples 1.2.f]). If V is reflexive, then every continuous representation (h, α) of a G-flow X on V and every pair (v, ψ) lead to a weakly almost periodic function $m_{v,\psi}$ on G. This follows easily from the (weak) continuity of the natural bounded operator $L_\psi : V \to C(G)$, where $L_\psi(v) = m_{v,\psi}$. Indeed, if the orbit vG is relatively weakly compact in V (e.g., if V is reflexive), then the same is true for the orbit $L_\psi(vG) = m_{v,\psi}G$ of $m_{v,\psi}$ in $C(G)$. Thus $m_{v,\psi}$ is wap. In fact, this argument proves Lemma 1.8(3).

(2) Analogously, every $v \in V$ (with V reflexive) defines a wap function $T_v : X \to \mathbb{R}$ on our G-flow X which naturally comes from the given flow representation (h, α). Precisely, define

$$T_v : X \to \mathbb{R}, \qquad x \mapsto \langle v, \alpha(x) \rangle.$$

Then $\{T_v\}_{v \in V} \subset \mathrm{WAP}(X)$. If in our example α is an embedding (which implies that X is reflexively representable) then $\{T_v\}_{v \in V}$ (and hence also $\mathrm{WAP}(X)$) separates the points of X. If, in addition, X is compact it follows that in fact $\mathrm{WAP}(X) = C(X)$ (because $\mathrm{WAP}(X)$ is always a closed subalgebra of $C(X)$). That is, in this case (G, X) is wap in the sense of Ellis & Nerurkar.

We proved in [27] that the converses to these facts are also true. We provide here a slightly improved result by direct arguments.

Theorem 2.9. *Let $S \times X \to X$ be a separately continuous action of a semitopological semigroup S on a compact space X. For every $f \in \mathrm{WAP}(X)$ there exist: a reflexive space V and an equivariant pair*

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V), B_V)$$

such that:

(i) *$h : S \to \Theta(V)$ is a weakly continuous homomorphism and $\alpha : X \to B_V$ is a weakly continuous S-map.*
(ii) *There exists $\phi \in V^*$ such that $f(x) = \langle \phi, \alpha(x) \rangle = \phi(\alpha(x))$ for every $x \in X$.*
(iii) *If S is separable we can assume that V is also separable.*

(iv) *If $S = G$ is a group then one can assume in addition that $h(G) \subset \mathrm{Is}(V)$ and $h : G \to \mathrm{Is}(V)$ is strongly continuous.*

Proof. Adjoining the isolated identity e (if necessary), one can assume that S is a monoid and $ex = x$.

(i) Since $f \in \mathrm{WAP}(X)$ the weak closure $Y := \mathrm{cl}_w(fS)$ of the S-orbit fS in $C(X)$ (with respect to the natural right action $C(X) \times S \to C(X)$) is weakly compact. Then the evaluation map $w : Y \times X \to \mathbb{R}$, $(y, x) \mapsto w(y, x) = \langle y, x \rangle$, is bounded, separately continuous and $\langle ys, x \rangle = \langle y, sx \rangle$ for every $s \in S$. Consider the induced linear left action $S \times C(Y) \to C(Y)$. Then the natural map $\alpha : X \to C(Y)$, $\alpha(x)(y) = \langle y, x \rangle$, is an S-map. Moreover, this map is weakly continuous by (Grothendieck's) Lemma 1.1 (because α is pointwise continuous and $\alpha(X)$ is pointwise compact and bounded, hence weakly compact). Denote by E the Banach subspace of $C(Y)$ topologically spanned by $\alpha(X)$. That is, $E = \mathrm{cl}(\mathrm{sp}(\alpha(X)))$.

Clearly every s-translation $\widetilde{s} : C(Y) \to C(Y)$ is a contractive linear operator. The orbit map $\widetilde{z} : S \to C(Y)$ is pointwise continuous for every $z \in \alpha(X) \subset C(Y)$. Again by Grothendieck's lemma, $\widetilde{z}$ is even weakly continuous. Then the same is true for every $u \in E = \mathrm{cl}(\mathrm{sp}(\alpha(X)))$ (as follows, for example, from [2, Proposition 6.1.2]). By the Hahn–Banach theorem, the weak topology of E is the same as its relative weak topology as a subset of $C(Y)$. We conclude that the action $S \times (E, w) \to (E, w)$ is separately continuous on (E, w).

Denote by W the convex, circled hull $\mathit{\Gamma}(\alpha(X))$ of $\alpha(X)$. By the Krein–Shmul'yan theorem, W is relatively weakly compact in E. We can apply an interpolation technique of [7]. For each natural n, set $K_n = 2^n W + 2^{-n} B_E$. Then K_n is convex, circled, bounded and radial (we use the terminology of [32]). Therefore the Minkowski functional $\|v\|_n$ of the set K_n is a well defined seminorm. Recall that $\|v\|_n = \inf \{\lambda > 0 \mid v \in \lambda K_n\}$. Then $\| \ \|_n$ is a norm on E equivalent to the given norm of E. For $v \in E$, let

$$N(v) := \Big(\sum_{n=1}^{\infty} \|v\|_n^2 \Big)^{1/2} \quad \text{and} \quad V := \{ v \in E \mid N(v) < \infty \}.$$

Denote by $j : V \to E$ the inclusion map (of sets).

(1) *$j : V \to E$ is a continuous linear injection and $\alpha(X) \subset W \subset B_V$.*

Indeed, if $w \in W$ then $2^n w \in K_n$. So, $\|w\|_n \leq 2^{-n}$. Thus, $N(w)^2 \leq \sum_{n \in \mathbb{N}} 2^{-2n} < 1$.

(2) *(V, N) is a reflexive Banach space* (see [7, Lemma 1]). *Hence the restriction of $j : V \to E$ to each bounded subset A of V induces a homeomorphism of A and $j(A)$ in the weak topologies.*

By our construction, W and B_E are S-invariant. Thus we get

(3) *V is an S-subset of E and $N(sv) \leq N(v)$ for every $v \in V$ and every $s \in S$.*
(4) *For every $v \in V$, the orbit map $\widetilde{v} : S \to V$, $\widetilde{v}(s) = sv$, is weakly continuous.*

Indeed, by (3), the orbit $\widetilde{v}(S) = Sv$ is N-bounded in V. Our assertion now follows from (2) (for $A = Sv$), taking into account that $\widetilde{v} : S \to E$ is weakly continuous.

By (3), for every $s \in S$, the translation map $\widetilde{s} : V \to V$, $v \mapsto sv$, is a linear contraction of (V, N). Therefore, we get the homomorphism $h : S \to \Theta(V)$, $h(s) = \widetilde{s}$, with $h(e) = 1_V$.

Now, directly from (4), we obtain the following assertion.

(5) *$h : S \to \Theta(V)$ is a w-continuous monoid homomorphism.*

By our construction the natural map $\alpha : X \to B_V$ is well defined by (1). It is clearly an S-map because V is an S-subset of $E \subset C(Y)$. Since $\alpha : X \to E$ is weakly continuous, (2) implies that $\alpha : X \to B_V$ is weakly continuous. This proves (i).

(ii) Denote by γ the continuous linear operator $V \to C(Y)$ defined as the composition $i \circ j$. Consider its adjoint $\gamma^* : C(Y)^* \to V^*$. Since S is a monoid, in our construction we can suppose that our original wap function f belongs to Y. Then the functional $f = ef \in Y \subset C(Y)^*$ defines a functional $\phi := \gamma^*(f) \in V^*$ such that $f(x) = \langle \phi, \alpha(x) \rangle = \phi(\alpha(x))$ for every $x \in X$.

(iii) If S is separable then so are fS and its weak closure Y. Therefore the compact space $\alpha(X) \subset C(Y)$ is metrizable in $(C(Y), w)$. Hence $\alpha(X)$ is separable in its weak topology. Then it is also norm separable. Indeed, if C is a countable weakly dense subset of $\alpha(X)$ then the norm (weak) closure of the convex hull $\mathrm{co}(C)$ of C is norm separable and contains $\alpha(X)$. This implies that E is also separable.

Now it suffices to show that $\alpha(V) \leq d(E)$. That is, we have to show that the canonical construction of [7] does not increase the density. Indeed, by construction, V is a (diagonal) subspace of the l_2-sum $Z := \sum_{n=1}^{\infty} (E, \|\cdot\|_n)_{l_2}$. So, $d(V) \leq d(Z)$. On the other hand, we know that every norm $\|\cdot\|_n$ is equivalent to the original norm on E. Hence, $d(E, \|\cdot\|_n) = d(E)$. Therefore, Z is an l_2-sum of countably many Banach spaces, each of density $d(E)$. It follows that

$$d(V) \leq d(Z) = d(E).$$

(iv) If $S = G$ is a group then $h(G) \subset \mathrm{Is}(V)$ because by our construction we can suppose that h is a monoid homomorphism and $h(e)$ represents the identity operator on V. Since $h : G \to \mathrm{Is}(V)$ is weakly continuous, Theorem 1.6 implies that it is strongly continuous. ∎

It is easy to derive from Theorem 2.9 the following results from [27] (taking into account Example 2.8 and Corollary 3.4).

THEOREM 2.10 ([27]). *Let G be a topological group.*

(1) *Every wap function on a G-space X comes from a reflexive representation. That is*, $\mathrm{WAP}(X) = \mathrm{Ref}(X)$. *Moreover*, *for every G the classes* $\mathrm{Ref}_{\mathrm{app}}$ *and WAP coincide.*
(2) *Every* $f \in \mathrm{WAP}(G)$ *is a matrix coefficient of a reflexive representation.*
(3) *If X is a compact metric G-space then it is wap if and only if X is reflexively representable.*

Note that these results remain true for semitopological semigroup actions as well. Theorems 2.10(2), 2.1 and Lemma 1.8 imply that every wap function on a topological group is left and right uniformly continuous (Helmer [17]). That is, $\mathrm{WAP}(G) \subset \mathrm{UC}(G)$, where $\mathrm{UC}(G) := \mathrm{LUC}(G) \cap \mathrm{RUC}(G)$.

2.5. *Ellis-Lawson's theorem.* Theorems 2.9 and 1.6 lead to a soft geometrical proof of the following version of Ellis–Lawson's theorem (see Lawson [20]).

FACT 2.11 (Ellis–Lawson's joint continuity theorem). *Let G be a subgroup of a compact semitopological monoid S. Suppose that $S \times X \to X$ is a separately continuous action with compact X. Then the action $G \times X \to X$ is jointly continuous and G is a topological group.*

Sketch of proof (see [27] for more details). It is easy to see that $C(X) = \mathrm{WAP}(S)$. Hence (S, X) is wap. By Theorem 2.9 and Remark 2.3(1) the proof can be reduced to the particular case of $(S, X) = (\Theta(V)^{\mathrm{op}}, B_{V^*})$ for some reflexive V with $G := \mathrm{Is}(V)$. Now by Theorem 1.6 the weak and strong operator topologies coincide on $G = \mathrm{Is}(V)$. In particular, G is a topological group and acts continuously on B_{V^*}.

3. Hilbert representability of flows

3.1. *General properties*

LEMMA 3.1. *Let Γ be a class of compact G-systems closed under subdirect products and G-isomorphisms.*

(1) *For every G-space X the set $\Gamma(X)$ of functions coming from a system Y with $Y \in \Gamma$ forms a uniformly closed G-subalgebra of $C(X)$. The corresponding Gelfand space $X^{\Gamma(X)}$ is the maximal* (*universal*) *G-compactification of X which belongs to Γ.*
(2) *The set $\Gamma(G)$ is a uniformly closed G-subalgebra of* $\mathrm{RUC}(G)$ *and the corresponding Gelfand space $G^{\Gamma(G)}$ is the universal G-factor of the greatest ambit G^{RUC} which belongs to Γ.*

Proof. The result and its proof are standard. See for example [14, Proposition 2.9]. ∎

LEMMA 3.2. *Let Γ be a class of compact G-spaces which is closed under G-isomorphisms. Assume that*:

(a) *Γ_0 is a subclass of Γ and Γ_0 generates Γ by subdirect products* (*that is*, *every $Y \in \Gamma$ is a subsystem of some product of members of Γ_0*).
(b) *Γ_0 is closed under subsystems and products with countably many members.*

Then for every (*not necessarily compact*) *G-space X we have $\Gamma(X) = \Gamma_0(X)$.*

Proof. Let $f \in \Gamma(X)$. Then there exist: a compact G-system $Y \in \Gamma$, a compactification map $\nu : X \to Y$ and a continuous function $F : Y \to \mathbb{R}$ such that $F \circ \nu = f$. By assumption (a), Y is a subsystem in a G-product $\prod_{i\in I} Y_i$ for some $Y_i \in \Gamma_0$. By the Stone–Weierstrass theorem it is easy to see that the map $F : Y \to \mathbb{R}$ "depends only on countably many coordinates". This fact is well known (see, for example, [6, Lemma 1] and [11, Exercise 3.2H]) for functions defined on products of compact spaces. By the normality of the compact product space $\prod_{i\in I} Y_i$ the function F is the restriction of some continuous function Φ defined on that product. There exists a countable subset $J \subset I$ of indices such that $\mathrm{pr}_J(y) = \mathrm{pr}_J(z)$ iff $F(y) = F(z)$ (for $y, z \in Y$) where $\mathrm{pr}_J : \prod_{i\in I} Y_i \to \prod_{j\in J} Y_j$ is the canonical projection. By construction the map $\Phi' : \prod_{j\in J} Y_j \to \mathbb{R}$, $\mathrm{pr}_J(y) \mapsto F(y)$, is well defined and $\Phi' \circ \mathrm{pr}_J = \Phi$. In particular, Φ' is continuous because pr_J is a quotient map.

Denote by Y_J the compact G-space $\mathrm{pr}_J(Y) \subset \prod_{j\in J} Y_j$ and by $F' : Y_J \to \mathbb{R}$ the restriction of Φ' to Y_J. Then $f = F' \circ \mathrm{pr}_J \circ \nu$. We see that $f : X \to \mathbb{R}$ comes from the compact G-space Y_J and the compactification $\mathrm{pr}_J \circ \nu : X \to Y_J$. Clearly, $Y_J \in \Gamma_0$ by assumption (b). We conclude that $f \in \Gamma_0(X)$. ∎

LEMMA 3.3. *For every semitopological semigroup S the classes* Ref *and* Hilb *are closed under countable products.*

Proof. Let X_n be a sequence of reflexively (resp., Hilbert) representable compact S-spaces. By definition there exists a sequence of proper reflexive (resp., Hilbert) representations

$$(h_n, \alpha_n) : (S, X_n) \rightrightarrows (\Theta(V_n), B_{V_n^*}).$$

We can suppose that $\|h_n(x)\| \le 2^{-n}$ for every $x \in X_n$. Consider the l_2-sum of these representations, that is,

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V), V^*)$$

where

$$V := \Big(\sum_{n\in\mathbb{N}} V_n\Big)_{l_2}, \qquad h(s)(v) = \sum_n h_n(s)(v_n),$$

for every $v = \sum_n v_n$ and $s \in S$. Define $\alpha(x) = \sum_n \alpha_n(x_n)$ for every $x = (x_1, x_2, \ldots) \in \prod_{n\in\mathbb{N}} X_n = X$. It is easy to show that $\alpha(x) \in B_{V^*}$ and $\alpha : X \to B_{V^*}$ is weak* (equivalently, weakly) continuous and injective. Hence, α is a topological embedding because X is compact. Now use the fact that the l_2-sum of reflexive (resp., Hilbert) spaces is again reflexive (resp., Hilbert). ■

Corollary 3.4. *If X is a separable metrizable S-space then X is Hilbert* (*resp.*, *reflexively*) *approximable iff X is Hilbert* (*resp.*, *reflexively*) *representable.*

Proof. Since X is second countable, there exists a countable family of Hilbert (resp., reflexive) representations of our S-space X which determines the topology of X. Hence we can apply Lemma 3.3. ■

Proposition 3.5.

(1) $\mathrm{Hilb}(X) = \mathrm{Hilb}_{\mathrm{app}}(X)$ *for every* (*not necessarily compact*) *G-space X. In particular*, $\mathrm{Hilb}(G) = \mathrm{Hilb}_{\mathrm{app}}(G)$ *for every topological group G.*
(2) $\mathrm{Hilb}(X)$ *is a closed G-subalgebra of* $C(X)$.
(3) $\mathrm{Hilb}(G)$ *is a closed G-subalgebra of* $\mathrm{RUC}(G)$.
(4) *If X is compact then it is Hilbert approximable iff* $\mathrm{Hilb}(X) = C(X)$.

Proof. (1) We can apply Lemmas 3.3 and 3.2.

(2) $\mathrm{Hilb}(X) = \mathrm{Hilb}_{\mathrm{app}}(X)$ by the first assertion. Now observe that $\mathrm{Hilb}_{\mathrm{app}}(X)$ is a closed subalgebra of $C(X)$ by Lemma 3.1(1).

(3) Use the first assertion and Lemma 3.1(2).

(4) Use (1) and Lemma 3.1. ■

Analogous facts are of course also true for the class of reflexively representable systems. This follows also from Theorem 2.10, which implies that always $\mathrm{Ref}(X) = \mathrm{WAP}(X)$ and $\mathrm{Ref}(G) = \mathrm{WAP}(G)$.

Let X be a G-space. From Lemma 3.1 it follows that the *universal Hilbert approximable G-compactification* (G-factor, if X is compact) X^{Hilb} of X is in fact the Gelfand space of the algebra $\mathrm{Hilb}(X)$. The Gelfand space G^{Hilb} of the algebra $\mathrm{Hilb}(G)$ is the universal Hilbert approximable (point transitive, of course) G-compactification of G. Analogous objects in the reflexive representability context, that is, $X^{\mathrm{Ref}(X)}$ and $G^{\mathrm{Ref}(G)}$ coincide in fact with the known objects X^{WAP} (see [14]) and G^{WAP}, respectively. The latter leads to the above mentioned (see [21]) universal semitopological semigroup compactification $u_{\mathrm{wap}} : G \to G^{\mathrm{WAP}}$.

Remark 3.6.

(1) Note that for non-locally compact groups the compactification $u_{\mathrm{wap}} : G \to G^{\mathrm{WAP}}$ is not in general an embedding and might even be trivial for nontrivial Polish groups. More precisely, let $G := H_+[0,1]$ be the Polish group of all orientation preserving homeomorphisms of the closed interval. In [24] we show that $\mathrm{WAP}(G) = \{\text{constants}\}$. Hence $\mathrm{Hilb}(G) = \{\text{constants}\}$. In this case for every reflexively representable compact G-space X the action is trivial. In particular, if X is a transitive G-space then X must be a singleton.
(2) On the other hand, $\mathrm{Hilb}(\mathrm{Is}(H)) = \mathrm{WAP}(\mathrm{Is}(H)) = \mathrm{UC}(\mathrm{Is}(H))$ for any Hilbert space H and the unitary group $\mathrm{Is}(H) = U(H)$. Indeed, Uspenskij proves in [35] that the completion of this group with respect to the Roelcke uniformity (= infimum of the left and right uniformities) is naturally equivalent to the embedding $\mathrm{Is}(H) \to \Theta(H)$ into the compact semitopological semigroup $\Theta(H)$. The action $(\mathrm{Is}(H), \Theta(H))$ is Hilbert approximable (Remark 2.3(4)). It follows that a function $f : \mathrm{Is}(H) \to \mathbb{R}$ can be approximated uniformly by matrix coefficients of Hilbert representations if and only if f is left and right uniformly continuous (i.e., $f \in \mathrm{UC}(\mathrm{Is}(H))$).

3.2. *Almost periodic functions and Hilbert representations.* A function $f \in C(X)$ on a G-space X is *almost periodic* if the orbit $fG := \{fg\}_{g \in G}$ forms a precompact subset of the Banach space $C(X)$. The collection $\mathrm{AP}(X)$ of AP functions is a G-subalgebra in $\mathrm{WAP}(X)$. The universal almost periodic compactification of X is the Gelfand space X^{AP} of the algebra $\mathrm{AP}(X)$. When X is compact this is the classical *maximal equicontinuous* factor of the system X. A compact G-space X is equicontinuous iff X is almost periodic (AP), that is, iff $C(X) = \mathrm{AP}(X)$. For a G-space X the collection $\mathrm{AP}(X)$ is the set of all functions which come from equicontinuous (AP) G-compactifications.

For every topological group G, treated as a G-space, the corresponding universal AP compactification is the well known *Bohr compactification* $b : G \to bG$, where bG is a compact topological group.

Proposition 3.7.

(1) *Let G be a compact group. Then every separable metrizable G-space X is Hilbert representable. Every Tikhonov G-space is Hilbert approximable.*
(2) *For every topological group G and a not necessarily compact G-space X we have*
$$\mathrm{AP}(X) \subset \mathrm{Hilb}(X) \subset \mathrm{WAP}(X)$$
and
$$\mathrm{AP}(G) \subset \mathrm{Hilb}(G) \subset \mathrm{WAP}(G) \subset \mathrm{UC}(G).$$

Proof. (1) It is well known [1] that if G is compact then there exists a proper G-compactification $\nu : X \to Y$. Moreover, we can suppose that Y is also separable and metrizable. By another well known fact there exists a unitary linearization of Y (see, for example, [38, Corollary 3.17]). Precisely, there exist: a Hilbert space H, a continuous homomorphism $h : G \to \mathrm{Is}(H)$ and a norm embedding $\alpha : Y \to B_H$ which is equivariant. Since Y is compact, α is also an embedding into the weakly compact unit ball B_V. Therefore X is Hilbert representable by Remark 2.3(2).

If X is a Tikhonov G-space then it is a G-subspace of a compact G-space Y. It can be approximated by a system $\{X_i\}_{i\in I}$ of compact metrizable G-spaces (see [1] or [14, Proposition 4.2]). As we already know, every X_i is Hilbert representable. Hence we conclude that X is Hilbert approximable.

(2) Let $f \in \mathrm{AP}(X)$. Then f comes from a G-compactification $\nu : X \to Y$ such that Y is a compact AP system. Then the enveloping semigroup $E(Y)$ is a compact topological group and the action $E(Y) \times Y \to Y$ is continuous. Therefore the proof can be reduced to the particular case when G and X are compact. So we can apply assertion (1). It follows that $\mathrm{AP}(X) \subset \mathrm{Hilb}(X)$.

The inclusion $\mathrm{Hilb}(X) \subset \mathrm{Ref}(X)$ is trivial. On the other hand, $\mathrm{Ref}(X) = \mathrm{WAP}(X)$ by Theorem 2.10(1). Finally, the inclusion $\mathrm{WAP}(G) \subset \mathrm{UC}(G)$ is the above mentioned result of Helmer [17]. ∎

3.3. *Eberlein property and Hilbert representability*

Definition 3.8 ([2, Definition VI.2.11]). Let S be a semitopological semigroup and $h : S \to \Theta(V)$ be a weakly continuous representation on a Banach space V. The *coefficient algebra* $\mathfrak{F}_h$ of this representation is the smallest, norm closed unital subalgebra of $C(S)$ containing all coefficients $m_{v,\psi} : S \to \mathbb{R}$ where $(v, \psi) \in V \times V^*$.

Example 3.9. For every locally compact group G consider the one-point compactification $\nu : G \to G_\infty := G \cup \{\infty\}$. Then G_∞ is a semitopological compactification. It can be embedded into the compact semigroup $\Theta(H)$ for some Hilbert space H. Indeed, one can consider the regular representation $\lambda_G : G \to \mathrm{Is}(H)$ of G on the complex Hilbert space $H := L_2(G, \mu_G)$ where μ_G is a Haar measure on G. Then by [12, 3.7] the corresponding coefficient algebra $\mathfrak{F}_h$ coincides with $C_0(G) \oplus \mathbb{C}$ and the weak closure of $\lambda_G(G)$ in $\Theta(H)$ is the semigroup $\lambda_G(G) \cup \{0\}$ (which can be identified with G_∞). In fact the similar result remains true for $K := \mathbb{R}$, real-valued functions and real Hilbert spaces (see also Lemma 4.5).

This observation implies the following result.

Lemma 3.10. *For every locally compact group G we have*:

(1) *G_∞ is Hilbert representable.*
(2) $C_0(G) \subset \mathrm{Hilb}(G)$.

For every reflexive representation $h : G \to \mathrm{Is}(V)$ the weak closure of $h(G)$ in $\Theta(V)$ is a semitopological compact semigroup which in fact is the enveloping semigroup of the action (G, B_V) of G on the weakly compact unit ball (B_V, w).

LEMMA 3.11. *Let S be a semitopological semigroup and $h : S \to \Theta(V)$ be a weakly continuous representation on a reflexive Banach space V. Denote by S_h the compact semitopological semigroup defined as the weak closure of $h(S)$ in $\Theta(V)$. Then the natural embedding*

$$\phi : C(S_h) \to C(S), \qquad \phi(f)(s) := f(h(s)),$$

induces an isomorphism $C(S_h) \cong \phi(C(S_h)) = \mathfrak{F}_h$.

Proof. Easily follows from the Stone–Weierstrass theorem. See for example de Leeuw–Glicksberg [21, Lemma 4.8] or [2, VI.2.12]. ∎

THEOREM 3.12. *For every topological group G the algebra* $\mathrm{Hilb}(G)$ *coincides with the Eberlein algebra $E(G) := \mathrm{cl}(B(G))$ (= the uniform closure of $B(G)$ in $C(G) = C(G, \mathbb{R})$).*

Proof. First observe that $B(G) \subset \mathrm{Hilb}(G)$. Indeed, let $f \in B(G)$. Then f is a matrix coefficient of some continuous representation of G on a Hilbert space H. By Remark 2.4(2) we can assume that h is a co-representation. This means that

$$f(g) = m_{v,\psi}(g) = \langle v, g\psi \rangle$$

for some continuous co-homomorphism $h : G \to U(H) = \mathrm{Is}(H)$ and some vectors $v, \psi \in H$. Consider the orbit closure $Y := \mathrm{cl}(G\psi)$ in (H, w). Then Y is a compact transitive G-flow and $\nu : G \to Y$, $g \mapsto g\psi$, is a G-compactification. The continuity of the action of G on Y can be derived for instance from the Ellis–Lawson theorem (see Fact 2.11). Indeed, observe that the action of the compact semitopological semigroup G_h (defined as the weak closure of $h(G)$ in $\Theta(H)$) on Y is well defined and separately continuous. Clearly, Y is Hilbert representable. The function $v_Y : Y \to \mathbb{R}$, $y \to \langle v, y \rangle$, (restriction of the functional $v : H \to \mathbb{R}$ to Y) is in $\mathrm{Hilb}(Y)$. Since $f(g) = v_Y(\nu(g))$ we see that f comes from Y. Thus, $f \in \mathrm{Hilb}(G)$.

Clearly, $B(G) \subset \mathrm{Hilb}(G) \subset C(G)$. This induces the inclusion $\mathrm{cl}(B(G)) \subset \mathrm{cl}(\mathrm{Hilb}(G))$ of the closures in $C(G)$. By Proposition 3.5(3) we know that $\mathrm{Hilb}(G)$ is closed in $C(G)$. Now it suffices to show that f is a uniform limit of Fourier–Stieltjes functions in $C(G)$ for any choice of $f \in \mathrm{Hilb}(G)$. By the definition of $\mathrm{Hilb}(G)$ the function f comes from a Hilbert representable transitive G-flow X (and the G-compactification $\nu : G \to X$, $g \mapsto gx_0$, where the orbit of x_0 is dense in X). By Remark 2.3(2) there exists a continuous homomorphism $h : G \to \mathrm{Is}(H)$ and a weakly continuous embedding $\alpha : X \to B_H$ such that $\alpha(gx) = h(g)\alpha(x)$. Also $F(\nu(g)) = f(g)$ for some continuous function $F : X \to \mathbb{R}$. Denote by ν_h the natural continuous onto map $G_h \to X$, $p \mapsto px_0$, defined on the compact semitopological semigroup G_h. Then $f = F \circ \nu_h \circ h$. By Lemma 3.11 we know that $F \circ \nu_h \circ h$ belongs to the coefficient algebra $\mathfrak{F}_h$. Therefore, $f \in \mathfrak{F}_h$. Clearly, $\mathfrak{F}_h \subset \mathrm{cl}(B(G))$. Thus $f \in \mathrm{cl}(B(G))$, as required. ∎

By Theorem 3.12 and Lemma 1.11 we find that Definition 1.10 of Eberlein groups can be reformulated as follows.

Corollary 3.13. *Let G be a topological group. Then G is Eberlein if and only if* $\mathrm{Hilb}(G) = \mathrm{WAP}(G)$.

Now we can prove the following result which distinguishes the reflexive and Hilbert representability of G-flows for many natural groups.

Theorem 3.14. *Let G be a separable topological group such that every reflexively representable metric compact transitive G-flow is Hilbert representable. Then G is an Eberlein group.*

Proof. By Corollary 3.13, G is Eberlein in the sense of Definition 1.10 if and only if $\mathrm{Hilb}(G) = \mathrm{WAP}(G)$. Always, $\mathrm{Hilb}(G) \subset \mathrm{WAP}(G)$ by Proposition 3.7(2). Let $f \in \mathrm{WAP}(G)$. We have to show that $f \in \mathrm{Hilb}(G)$.

Since G is separable, so is the closed G-invariant subalgebra $\mathcal{A}$ generated by the orbit fG in $\mathrm{RUC}(G)$. Consider the corresponding Gelfand space $G^{\mathcal{A}}$ and the canonical compactification map $\nu : G \to G^{\mathcal{A}}$. Since $\mathcal{A}$ is G-invariant and every wap function on G is right uniformly continuous (see Proposition 3.7(2)) it follows that $X := G^{\mathcal{A}}$ is a compact point transitive G-space and ν is a compactification of G-spaces. We know that X is a metrizable compact space (because $\mathcal{A}$ is separable). Moreover, (G, X) is wap. Indeed, every continuous function $\phi : X \to \mathbb{R}$ is wap because $j_\nu(C(X)) = \mathcal{A} \subset \mathrm{WAP}(G)$, where $j_\nu : C(X) \to C(G)$ is the operator induced by $\nu : G \to X$. Since X is metric and wap, by Theorem 2.10(3) we know that X is a reflexively representable G-system. Since $f \in j_\nu(C(X)) = \mathcal{A}$, there exists a continuous function $F : X \to \mathbb{R}$ such that $F(g\nu(e)) = f(g)$. That is, $f \in C(G)$ comes from the compact G-space X (Definition 2.5(1)). By our assumption X is a Hilbert representable G-system. So we conclude that $f \in \mathrm{Hilb}(G)$. ∎

The following result answers a question of T. Downarowicz (Dynamical Systems Conference, Prague, 2005).

Theorem 3.15.

(1) *There exists a transitive self-homeomorphism $\sigma : X \to X$ of a compact metric space X such that the corresponding invertible cascade $(\mathbb{Z}, X)$ is reflexively but not Hilbert representable.*

(2) *There exists a compact metric transitive* $\mathbb{R}$*-flow* X *which is reflexively but not Hilbert representable.*

Proof. By a result of Rudin [31] the groups $\mathbb{Z}$ and $\mathbb{R}$ are not Eberlein groups. Therefore, we can apply Theorem 3.14. ∎

REMARK 3.16.

(1) The G-spaces in Theorem 3.15 are not even Hilbert approximable, as follows from Corollary 3.4.
(2) The desired counterexamples in Theorem 3.15 come in a quite constructive way but up to the choice of an appropriate function. More precisely, according to the proof of Theorem 3.14 we start with a function $f : G \to \mathbb{R}$ which is wap but not uniformly approximated by Fourier–Stieltjes functions. We define X as the G-compactification $G^{\mathcal{A}}$ of G induced by the subalgebra $\mathcal{A} \subset \mathrm{WAP}(G)$ which is generated by the right orbit fG. In fact, it is the pointwise closure of the left orbit Gf in $\mathrm{WAP}(G)$, by general properties of such G-compactifications (see [14, Proposition 2.4]).
(3) The referee asks if one may produce a more explicit example of a wap cascade (say a subshift) which is not Hilbert representable.

DEFINITION 3.17. We say that a separable topological group G is *strongly Eberlein* if every reflexively representable metric compact transitive G-flow is Hilbert representable.

Theorem 3.14 justifies this definition because every strongly Eberlein group is Eberlein. If G is an abelian locally compact noncompact group, then G is not Eberlein and hence not strongly Eberlein.

EXAMPLE 3.18.

(1) The groups $G := \mathrm{SL}_n(\mathbb{R})$ are strongly Eberlein. Indeed, by Veech's result [37] (see also Chou [4] for the case of $\mathrm{SL}_2(\mathbb{R})$) we know (by Remark 1.12(4)) that $\mathrm{WAP}(G, \mathbb{R}) = C_0(G) \oplus \mathbb{R}$ for such groups. This means that every wap compactification of G is a one-point compactification which is Hilbert representable (see Example 3.9 and Lemma 3.10).
(2) The Polish group $G := H_+[0, 1]$ is strongly Eberlein by Remark 3.6(1).
(3) By Remark 3.6(2) we know that $\mathrm{Hilb}(G) = \mathrm{WAP}(G) = \mathrm{UC}(G)$ for the unitary group $G := U(H)$ for every Hilbert space H. In particular, $U(H)$ is Eberlein by Corollary 3.13. It is not clear, however, if this group is strongly Eberlein.

QUESTION 3.19. *Is every Polish Eberlein group strongly Eberlein? What if* $G := U(l_2)$*?*

Reflexively representable compact metric flows are closed under G-factors [27]. For Hilbert representability, this is unclear. That is, the following question is open.

QUESTION 3.20 (see also [28, Question 7.6]). *Are Hilbert representable compact metric G-spaces closed under G-factors?*

4. Semitopological semigroups and their representations. Recall that for every reflexive V the semigroup $\Theta(V)$ of all contractive linear operators is a compact semitopological semigroup with respect to the weak operator topology.

DEFINITION 4.1. A (weakly continuous) *representation* of a semitopological semigroup S on a Banach space V is a (weakly continuous) homomorphism $h : S \to \Theta(V)$. If h is a topological embedding we say that h is a *proper representation. S* is *reflexively* (resp., *Hilbert*) *representable* if it admits a proper representation on a reflexive (resp., Hilbert) space V.

For every locally compact topological group G consider the corresponding one-point compactification $\lambda : G \to G_\infty$. Then G_∞, as a compact semitopological semigroup, is Hilbert representable by Lemma 3.10.

FACT 4.2 ([33] and [25]). *Every compact semitopological semigroup S is topologically isomorphic to a subsemigroup of $\Theta(V)$ for a certain reflexive V. That is*, *every compact semitopological semigroup is reflexively representable.*

REMARK 4.3. If in addition S is metrizable then we may assume in Fact 4.2 that V is separable. See [25, Remark 3.2]. Alternatively, we can use Theorem 2.9 to produce a sequence $\{V_n\}_{n\in\mathbb{N}}$ of separable reflexive spaces such that S is embedded into $\Theta(W)$ where $W := (\sum_{n\in\mathbb{N}} V_n)_{l_2}$.

QUESTION 4.4. *Under which conditions a given compact semitopological semigroup* (*which is always reflexively representable*) *is Hilbert representable?*

For every isometry $u \in \mathrm{Is}(V)$ we have the associated compact monothetic semigroup $S_u := \mathrm{cl}_w(\{u^k\}_{k\in\mathbb{Z}})$, the weak closure of the cyclic subgroup $\{u^k\}_{k\in\mathbb{Z}}$ in $\Theta(V)$. Below we prove (Corollary 4.8) that there exists a separable reflexive Banach space V and a linear isometry $u \in \mathrm{Is}(V) \subset \Theta(V)$ such that the corresponding monothetic metrizable semitopological semigroup S_u is not Hilbert representable.

LEMMA 4.5. *Let S be a compact semitopological semigroup. The following are equivalent*:

(1) *S*, *as a semigroup*, *is Hilbert representable.*
(2) *The action (S, S) is Hilbert approximable.*

(*Furthermore*, *by Corollary* 3.4, *if S is in addition metrizable then "approximable" in the second assertion can be changed to "representable".*)

Proof. (1)⇒(2): Let S be a compact subsemigroup of the semigroup $\Theta(H)$ for some Hilbert H. Consider the inclusion $h : S \hookrightarrow \Theta(H)$ and the natural action of S on B_H. By Remark 2.3(2) the action (S, B_H) is Hilbert representable. Hence it suffices to show that there exists a family of weakly continuous S-maps from $X := S$ to B_H which separates the points of X. Now observe that for every vector $v_0 \in B_H$ the map $\alpha_{v_0} : S \to B_H$ defined by $\alpha(t) := tv_0$ is weakly continuous and S-equivariant.

(2)⇒(1): We can assume that S is a monoid. Denote by X the left regular action of S on itself. By our assumption and Remark 2.3(2) there exist: a family $\{H_i\}_{i\in I}$ of Hilbert spaces, a family $\{h_i : S \to \Theta(H_i)\}_{i\in I}$ of weakly continuous homomorphisms, and a family $\{\alpha_i : X \to B_{H_i}\}_{i\in I}$ of weakly continuous S-equivariant maps such that the latter family separates points of $X = S$. Since S is a monoid, the family $\{h_i : S \to \Theta(H_i)\}_{i\in I}$ separates the points of S. Then the induced homomorphism $h : S \to \Theta(H)$, where $H := (\bigoplus_{i\in I} H_i)_{l_2}$ is an orthogonal l_2-sum, is weakly continuous and injective. Since S is compact, h is the desired embedding. ∎

We need the following very useful fact.

FACT 4.6 (Downarowicz [8, Fact 2]; see also [13, Theorem 1.48]). *Let X be a compact transitive wap G-flow. If G is commutative then the enveloping semigroup $E(X)$ is commutative and the flows $(G, E(X))$ and (G, X) are topologically isomorphic.*

THEOREM 4.7. *There exists a compact metrizable monothetic* (*hence commutative*) *semitopological semigroup S such that S is not Hilbert representable* (*being reflexively representable*).

Proof. By Theorem 3.15(1) there exists a transitive pointed compact metrizable cascade X such that $(\mathbb{Z}, X)$ is reflexively but not Hilbert representable. Then $(\mathbb{Z}, X)$ is wap (Theorem 2.10) and the enveloping semigroup $E(X) = E(\mathbb{Z}, X)$ is a compact semitopological semigroup (Fact 1.5). Take the corresponding natural $\mathbb{Z}$-compactification $\gamma : \mathbb{Z} \to S := E(X)$ of the group $G := \mathbb{Z}$. Then S is reflexively representable by Fact 4.2. Clearly, S is a monothetic semigroup by our construction. It is also easy to see that S is metrizable because X is wap and hence by Lemma 1.4 all elements of $E(X)$ are continuous self-maps $X \to X$ of a compact metric space X. Indeed, $E(X)$, as a topological space, is embedded into the product space X^D where D is a countable dense subset of X.

We claim that $S := E(X)$ is the desired semigroup. We only have to show that S is not Hilbert representable. Assuming the contrary let $j : E(X) \hookrightarrow \Theta(H)$ be an embedding of compact semitopological semigroups where H

is a Hilbert space. Then Lemma 4.5 implies that the natural action of the cyclic group $G := \mathbb{Z}$ on $E(X)$ is Hilbert representable. This is a contradiction because the flows $(\mathbb{Z}, E(X))$ and $(\mathbb{Z}, X)$ are topologically isomorphic by Fact 4.6 and the assumption that $(\mathbb{Z}, X)$ is not Hilbert representable. ∎

COROLLARY 4.8. *There exists a separable reflexive Banach space V and a linear isometry $u \in \mathrm{Is}(V) \subset \Theta(V)$ such that the corresponding monothetic metrizable semitopological semigroup $S_u \subset \Theta(V)$ is not Hilbert representable. That is, S_u is not topologically isomorphic to a subsemigroup of $\Theta(H)$ for any Hilbert space H.*

Proof. Apply Theorem 4.7 using Remark 4.3. ∎

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