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MINIMAL MODELS FOR $\mathbb{Z}^d$-ACTIONS

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Abstract. We prove that on a metrizable, compact, zero-dimensional space every $\mathbb{Z}^d$-action with no periodic points is measurably isomorphic to a minimal $\mathbb{Z}^d$-action with the same, i.e. affinely homeomorphic, simplex of measures.

1. Basics. In 1970 Robert Jewett proved that for any weakly mixing dynamical system there exists an isomorphic strictly ergodic (i.e. uniquely ergodic and minimal) topological dynamical system. Extended by Wolfgang Krieger to the class of all ergodic transformations, it was one of the first major results concerning modelling measure-theoretical dynamical systems by topological systems with preassigned topological conditions, like minimality. One of the recent theorems of this kind was proved by Tomasz Downarowicz in [1]: an aperiodic, continuous map of a compact, metric, zero-dimensional space is Borel* isomorphic to a minimal one. Borel* isomorphism is a relation which involves not only a measurable isomorphism between dynamical systems, but also an affine homeomorphism between simplices of invariant measures. Our present paper is a sequel of [1]—we adapt the methods used there to obtain such an isomorphism theorem for continuous $\mathbb{Z}^d$-actions.

We consider a compact zero-dimensional metrizable space X and a collection $T = \{T_1, \ldots, T_d\}$ of commuting homeomorphisms of X. We call a pair (X, T) a *d-dimensional dynamical system.* For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ we write $T^{\mathbf{n}}$ for the superposition $T_1^{n_1} \ldots T_d^{n_d}$. We say that a system (X, T) is *aperiodic* if $T^{\mathbf{n}}(x) \neq x$ for all $x \in X$ and all $\mathbf{n} \neq (0, \ldots, 0)$. It is *minimal* if X contains no proper nonempty closed subset which is invariant (a set F is *invariant* if $T_i F = F$ for $i = 1, \ldots, d$). Equivalently, (X, T) is minimal if and only if the orbit $\{T^{\mathbf{n}}x : \mathbf{n} \in \mathbb{Z}^d\}$ of every $x \in X$ is dense.

We denote by $\mathcal{P}_T(X)$ the set of all Borel probability measures on X invariant under T (i.e. under all T_i, $i = 1, \ldots, d$). It is well known that in our case $\mathcal{P}_T(X)$ endowed with the weak* topology is a compact, metrizable

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and convex subset of the space of all Borel probability measures on X. Every point of $\mathcal{P}_T(X)$ has a unique representation as a barycenter of a certain Borel measure concentrated on the Borel set of all ergodic measures. These properties are usually abbreviated by saying that $\mathcal{P}_T(X)$ is a *Choquet simplex* (see [4] for details). A set $E \subset X$ is called *full* if $\mu(E) = 1$ for every $\mu \in \mathcal{P}_T(X)$.

DEFINITION. We say that two d-dimensional dynamical systems (X, T) and (Y, S) are *Borel* isomorphic* if there exists an equivariant Borel-measurable bijection $\Phi : X_0 \to Y_0$ between full invariant subsets $X_0 \subset X$ and $Y_0 \subset Y$ such that the conjugate map $\Phi^* : \mathcal{P}_T(X) \to \mathcal{P}_S(Y)$ given by the formula $\Phi^*(\mu) = \mu \circ \Phi^{-1}$ is an (affine) homeomorphism with respect to weak* topologies.

We will extensively use a special type of dynamical systems, namely d-dimensional symbolic systems over a compact alphabet Λ. These are defined in the following way: on a compact space $\Lambda^{\mathbb{Z}^d}$ we define shift maps σ_i setting $(\sigma_i(y))_{\mathbf{n}} = y_{\mathbf{n}+\mathbf{e}_i}$ for all $y \in \Lambda^{\mathbb{Z}^d}$, $\mathbf{n} \in \mathbb{Z}^d$ and $i = 1, \ldots, d$, where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ with the only 1 occurring at the ith place. A *d-dimensional symbolic system* is a nonempty closed subset Y of $\Lambda^{\mathbb{Z}^d}$ which is invariant under all σ_i.

We use the following conventions. For a set Λ a function $M : \mathbb{Z}^d \to \Lambda$, i.e. an element of $\Lambda^{\mathbb{Z}^d}$, is called an *array*. For a finite set $A \subset \mathbb{Z}^d$ and an array M we define the *configuration* M_A to be M restricted to A. In particular, for $\mathbf{n} \in \mathbb{Z}^d$ we denote by $M_{\mathbf{n}}$ the single symbol $M_{\{\mathbf{n}\}}$. If $\widetilde{A} = A + \mathbf{m}$ for some $\mathbf{m} \in \mathbb{Z}^d$, and $(\widetilde{M}_{\widetilde{A}})_{\mathbf{n}} = (M_A)_{\mathbf{n}+\mathbf{m}}$ for every $\mathbf{n} \in \widetilde{A}$, then we say that M_A and $\widetilde{M}_{\widetilde{A}}$ have the same *pattern*. In this case both A and $\widetilde{A}$ are called the *shape* of the pattern. More formally, shapes and patterns are cosets of the equivalence relation based on the translation of the domain. Thus one can define inclusion for shapes S, S' as follows: $S' \subset S$ if $A' \subset A$ for some $A' \subset \mathbb{Z}^d$ representing S' and $A \subset \mathbb{Z}^d$ representing S. A shape S is *bounded* if sets representing S are bounded. A *cube* with *maximal vertex* $\mathbf{v} = (v_1, \ldots, v_d)$ and edge length b is the set

$$\mathcal{K}_b^{\mathbf{v}} = \{\mathbf{n} \in \mathbb{Z}^d : v_i - b < n_i \le v_i\};$$

then $(v_1 - b + 1, \ldots, v_d - b + 1)$ will be called the *minimal vertex* of $\mathcal{K}_b^{\mathbf{v}}$. For $b \in \mathbb{N}_0$, $\mathbf{v} = (b, \ldots, b)$ we also write $\mathcal{K}_b = \mathcal{K}_{b+1}^{\mathbf{v}}$ for the cube fixed at the origin. We will also use the name "cube" for shapes based on cubes in $\mathbb{Z}^d$. It will be convenient to denote $(0, \ldots, 0) \in \mathbb{Z}^d$ by $\mathbf{0}$, $(1, \ldots, 1) \in \mathbb{Z}^d$ by $\mathbf{1}$, and $(k, \ldots, k) \in \mathbb{Z}^d$ by $k \cdot \mathbf{1}$.

In a symbolic system (Y, σ), by *blocks* we will mean patterns having bounded shapes. A restriction of a block of the shape S to some shape $S' \subset S$ is called a *subblock*. A block B *occurs* in $y \in \Lambda^{\mathbb{Z}^d}$ if it is a pattern of some configuration y_A; B occurs in a system (Y,σ) if it occurs in some $y \in Y$. Let d_Λ be a metric on the alphabet Λ. On the set of all blocks of the same shape we define a distance D to be the supremum of distances d_Λ between symbols occupying identical positions. Note that if B_1',B_2' are identically shaped subblocks of B_1 and B_2, respectively, and $D(B_1,B_2)<\varepsilon$, then $D(B_1',B_2')<\varepsilon$.

The following theorem is the main result of this work.

Theorem 1. *If X is a metrizable, compact, zero-dimensional space then every d-dimensional aperiodic dynamical system (X,T) is Borel* isomorphic to a minimal dynamical system $(\widetilde{X},\tau)$ (with $\widetilde{X}$ being also metrizable, compact and zero-dimensional).*

The first step of the construction of $(\widetilde{X},\tau)$ will be to replace (X,T) by a conjugate, thus having "the same" simplex of measures, d-dimensional symbolic system (X^*,σ) over the infinite alphabet $\Lambda=(X\cup\overline{\mathbb{N}}_0)^{\mathbb{N}_0}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers and $\overline{\mathbb{N}}_0$ is the set $\mathbb{N}_0\cup\{\infty\}$. Elements of Λ and $X\cup\overline{\mathbb{N}}_0$ will be referred to as *symbols* and *characters*, respectively. Then we will construct a Borel* isomorphism Φ between (X^*,σ) and a minimal symbolic system $(\widetilde{X},\tau)$ with the same alphabet. The map Φ will be defined as the pointwise limit of a sequence of topological conjugacies given by block codes.

We will now mention two of the difficulties typical for the multidimensional case. Similarly to [1], the construction relies on a choice of a decreasing sequence of clopen sets, called markers. For every x in the underlying space, each of these markers induces a division of the trajectory of x into nonoverlapping blocks in such a way that every block created for the $(n+1)$st marker is a concatenation of blocks specified by the nth marker. In several dimensions, the operation of dividing trajectories into blocks requires much more effort. Rectangular blocks are not possible and even Voronoi regions seem to be unsuitable for our purposes, so we develop a new algorithm. The second problem, which was not present in dimension one, concerns boundaries of blocks induced by markers. The elements with badly behaving boundaries have to be ruled out, which forces another calculation to ensure that we get rid only of a set of measure zero.

2. Markers.

For $p\in\mathbb{N}_0$ we denote the central cube with edge length $2p+1$ by

$$\overline{\mathcal{K}}_p=\{n=(n_1,\ldots,n_d)\in\mathbb{Z}^d:\max\{|n_1|,\ldots,|n_d|\}\le p\}.$$

Definition. A set $F\subset X$ is a *marker of order* $p\in\mathbb{N}_0$ or simply a *p-marker* if:

(i) elements of $\{T^{\mathbf{n}}F\colon \mathbf{n}\in\overline{\mathcal{K}}_p\}$ are pairwise disjoint,

(ii) $\{T^{\mathbf{n}}F\colon \mathbf{n} \in \overline{\mathcal{K}}_N\}$ is a cover of X for some $N \in \mathbb{N}_0$. The number $2N+1$ with minimal such N will be called the *covering constant* of the marker F.

We say that (X,T) has the *marker property* if X contains a clopen p-marker for every $p \in \mathbb{N}_0$.

LEMMA 2 (Marker lemma). *Any aperiodic $\mathbb{Z}^d$-action (X,T) on a compact zero-dimensional Hausdorff space has the marker property. Moreover, for any increasing sequence (p_t) of positive integers there is a descending sequence of p_t-markers, with the covering constant q_t of the p_t-marker equal to $4p_t + q_{t-1}$.*

Proof. It is clear that the whole space X is a 0-marker with covering constant 1. We will show that given a clopen k-marker F^k and an integer $p > k$ we can find a clopen p-marker $F^p \subset F^k$. The covering constant of F^k will be denoted by $2K+1$.

For every $x \in F^k$ we choose a clopen neighbourhood E_x of x, contained in F^k, such that $\{T^{\mathbf{n}}E_x\colon \mathbf{n} \in \overline{\mathcal{K}}_{2p}\}$ consists of pairwise disjoint sets. From the cover $\{E_x\colon x \in F^k\}$ of the clopen set F^k we choose a finite subcover $\mathcal{V} = \{V_l : l = 1, \ldots, L\}$. Now we set

$$F_1 = V_1, \qquad F_{l+1} = F_l \cup \Big(V_{l+1} \setminus \bigcup_{\mathbf{m}\in\overline{\mathcal{K}}_{2p}} T^{\mathbf{m}}F_l\Big).$$

Finally, $F^p = F_L$. Obviously, F^p is clopen.

We skip the induction that proves disjointness of $T^{\mathbf{n}}F^p$ for $\mathbf{n} \in \overline{\mathcal{K}}_p$, but we show that $\{T^{\mathbf{n}}F^p\colon \mathbf{n} \in \overline{\mathcal{K}}_{2p+K}\}$ is a cover. Every $x \in F^k$ belongs to one of V_l's. Either it was appended to $F_l \subset F^p$ at the lth step of the construction or it had already been contained in $T^{\mathbf{m}}F_{l-1} \subset T^{\mathbf{m}}F^p$ for some $\mathbf{m} \in \overline{\mathcal{K}}_{2p}$. Thus $F^k \subset \bigcup_{\mathbf{n}\in\overline{\mathcal{K}}_{2p}} T^{\mathbf{n}}F^p$, and $X \subset \bigcup_{\mathbf{n}\in\overline{\mathcal{K}}_{2p+K}} T^{\mathbf{n}}F^p$. ∎

3. The space X^*. Fix the summable sequence $\varepsilon_t = 1/2^{t+3}$, $t \in \mathbb{N}_0$. Let $d_{\overline{\mathbb{N}}_0}$ be the metric on $\overline{\mathbb{N}}_0$ given by $d_{\overline{\mathbb{N}}_0}(k,l) = \sum_{t=k+1}^{l} \varepsilon_t$ for $k \le l$. Let d_X denote a metric on X. We define a compact metric d on $X \cup \overline{\mathbb{N}}_0$ by

$$d(x,y) = \begin{cases} d_X(x,y) & \text{for } x,y \in X,\\ \operatorname{diam}(X) & \text{for } x \in X,\ y \in \overline{\mathbb{N}}_0 \text{ or } x \in \overline{\mathbb{N}}_0,\ y \in X,\\ d_{\overline{\mathbb{N}}_0}(x,y) & \text{for } x,y \in \overline{\mathbb{N}}_0,\end{cases}$$

and the distance d_Λ between $\mathbf{x} = (x^0, x^1, \ldots)$ and $\mathbf{y} = (y^0, y^1, \ldots)$ in Λ by

$$d_\Lambda(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{\infty} 2^{-i} d(x^i, y^i).$$

Note that (Λ, d_Λ) is a compact metric space.

For an array $M \in \Lambda^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$ let $M_{\mathbf{n}}^k$ denote the kth character of the symbol $M_{\mathbf{n}}$. The function mapping $\mathbf{n} \in \mathbb{Z}^d$ to $M_{\mathbf{n}}^k$ is called the *kth level* of M.

Fix an increasing sequence (p_t) and let (F_t) be a descending sequence of p_t-markers with covering constants q_t (see Lemma 2). Let $Q_t = \sum_{i=0}^{t} q_i$. In Section 5 we will give more information about the choice of the sequence (p_t). In particular, the inequality $Q_t < p_{t+1}$ will be satisfied.

In the first step of the construction of (X^*, σ) we replace each $x \in X$ by an array $[x] : \mathbb{Z}^d \to \Lambda$ such that $[x]_{\mathbf{n}}^k = 0$ for $k > 1$, $[x]_{\mathbf{n}}^1 = T^{\mathbf{n}}x$ and $[x]_{\mathbf{n}}^0 = t$ if $T^{\mathbf{n}}x \in F_{p_t}$, $T^{\mathbf{n}}x \notin F_{p_{t+1}}$ or $[x]_{\mathbf{n}}^0 = \infty$ if x belongs to all markers. We say that $[x]$ has the marker t at position $\mathbf{n} \in \mathbb{Z}^d$ if $[x]_{\mathbf{n}}^0 = t$. The space $X^* = \{[x] : x \in X\}$ is homeomorphic to X and the collection σ of shifts σ_i is topologically conjugate to T. According to the definition of a marker, every $[x] \in X^*$ has the following properties:

(i) every configuration in $[x]$ based on a cube with edge length p_t has (at some position) at most one marker $\geq t$,

(ii) every configuration in $[x]$ based on a cube with edge length q_t has at least one marker $\geq t$.

4. t-blocks. In the current section we describe an inductive algorithm of partitioning every $[x] \in X^*$ into disjoint configurations. The sequence of partitions, induced by a fixed sequence of markers, thus depending only on the zero level of $[x]$, will be the base of our construction of a topological conjugacy between X^* and a minimal system.

On every cone $\mathbf{n} + \mathbb{N}_0^d = \{\mathbf{m} \in \mathbb{Z}^d : \mathbf{m} \geq \mathbf{n}\}$, where $\mathbf{n} \in \mathbb{Z}^d$, we define a maximolexicographic order "$<^*$" as follows. For $\mathbf{m} \in \mathbb{N}^d$ let $\mathrm{sort}(\mathbf{m})$ denote the element of $\mathbb{Z}^d$ whose coordinates are equal to those of $\mathbf{m}$, but arranged in nonincreasing order, and let "$\prec$" be the usual lexicographic order. We write $\mathbf{m} <^* \mathbf{m}'$ if

- $\mathrm{sort}(\mathbf{m}) \prec \mathrm{sort}(\mathbf{m}')$ or
- $\mathrm{sort}(\mathbf{m}) = \mathrm{sort}(\mathbf{m}')$ and $\mathbf{m} \prec \mathbf{m}'$.

Figure 1 presents the scheme of the order for $d = 2$. The relation "$<^*$" is a linear order. The operation of taking minimum with respect to this order will be denoted by "$\min^*$".

9	11	13	15
4	6	8	14
1	3	7	12
0	2	5	10

Fig. 1. The scheme of the maximolexicographic order for $d = 2$. Number 0 is the vertex of a cone; consecutive integers are placed according to the maxlex order on this cone.

Let $Q_{-1} = 0$ and $p_0 = q_0 = 1$. First we define 0-configurations as single symbols $[x]_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$. To proceed with the induction, we assume that we have defined t-configurations in such a way that every t-configuration contains on the zero level exactly one marker $u \geq t$. Let us denote the position of this marker in a t-configuration $[x]_A$ by $\mathbf{n}(t, A)$. We define a $(t+1)$-configuration as a concatenation of t-configurations as follows. Every $(t+1)$-configuration $[x]_C$ consists of exactly one t-configuration $[x]_A$ with a marker $u \geq t+1$ and some other t-configurations $[x]_{A'}$ such that $\mathbf{n}(t, A) = \min^*\{\mathbf{m} \geq^* \mathbf{n}(t, A') : [x]^0_{\mathbf{m}} \geq t+1\}$, where the ordering "$\geq^*$" is inverse to "$<^*$" defined for the cone $\mathbf{n}(t, A') + \mathbb{N}_0^d$. We obtain $\mathbf{n}(t+1, C) = \mathbf{n}(t, A)$. Roughly speaking, the t-marker of A' searches for the nearest (in "$<^*$") $(t+1)$-marker of some A, and then the t-configuration A' is glued to A. Figure 2 pictures the distribution of 1-blocks and 2-blocks in two dimensions.

Patterns of t-configurations will be called *t-blocks*. The collection of all t-blocks which occur in the system X^* will be denoted by $\mathcal{B}_t$. Below we summarize the main properties of t-blocks.

LEMMA 3. *Let B be a t-block.*

(1) *B is a finite concatenation of $(t-1)$-blocks $(t > 0)$.*
(2) *B has exactly one marker $u \geq t$.*
(3) *The marker $u \geq t$ is situated at the maximal vertex of B, i.e. at the maximal vertex of the smallest cube containing the domain of B.*
(4) *The shape of B contains a cube with edge length $p_t - Q_{t-1}$.*
(5) *The shape of B is contained in a cube with edge length Q_t.*

Proof. Properties (1) and (2) follow immediately from the construction. Properties (3)–(5) will be proved by induction.

Let $[x] \in X^*$. Observe that 0-blocks obey these rules. Assume that conditions (3)–(5) hold for every t-block in $\mathcal{B}_t$. Consider a $(t+1)$-configuration $[x]_C$ with marker $\geq t+1$ at $\mathbf{n}(t+1, C)$. We will show that the cube $\mathcal{K}^{\mathbf{v}}_{p_{t+1}-Q_t}$ with $\mathbf{v} = \mathbf{n}(t+1, C) - (Q_t \cdot \mathbf{1})$ is a subset of C.

Let $\mathbf{n} \in \mathcal{K}^{\mathbf{v}}_{p_{t+1}-Q_t}$. The point $\mathbf{n}$ is in the domain of $[x]_A$ for some t-configuration $[x]_A$ with marker t at $\mathbf{n}(t, A)$. By the induction hypothesis, the domain A of $[x]_A$ is a subset of a cube of edge length Q_t and the marker t is situated at the maximal vertex of A. Hence, having in mind that $Q_t < p_{t+1}$, we get $\mathbf{n}(t, A) \in \mathcal{K}^{\mathbf{n}(t+1,C)}_{p_{t+1}}$. Therefore $\mathbf{n}(t+1, C)$ lies in the cube $\mathcal{L}$ with minimal vertex $\mathbf{n}(t, A)$ and edge length p_{t+1}. At $\mathbf{n}(t+1, C)$ there is a marker $\geq t+1$ and in $\mathcal{L}$ there are no other markers $\geq t+1$. So A must be a subset of C and $\mathbf{n}$ lies in C.

To prove (5), we will show that the domain of the $(t+1)$-configuration $[x]_C$ is a subset of the cube $\mathcal{K}^{\mathbf{n}(t+1,C)}_{Q_{t+1}}$. Let $\mathbf{n}$ be situated outside this cube. The position $\mathbf{n}$ lies in the domain of some t-configuration $[x]_A$ with marker

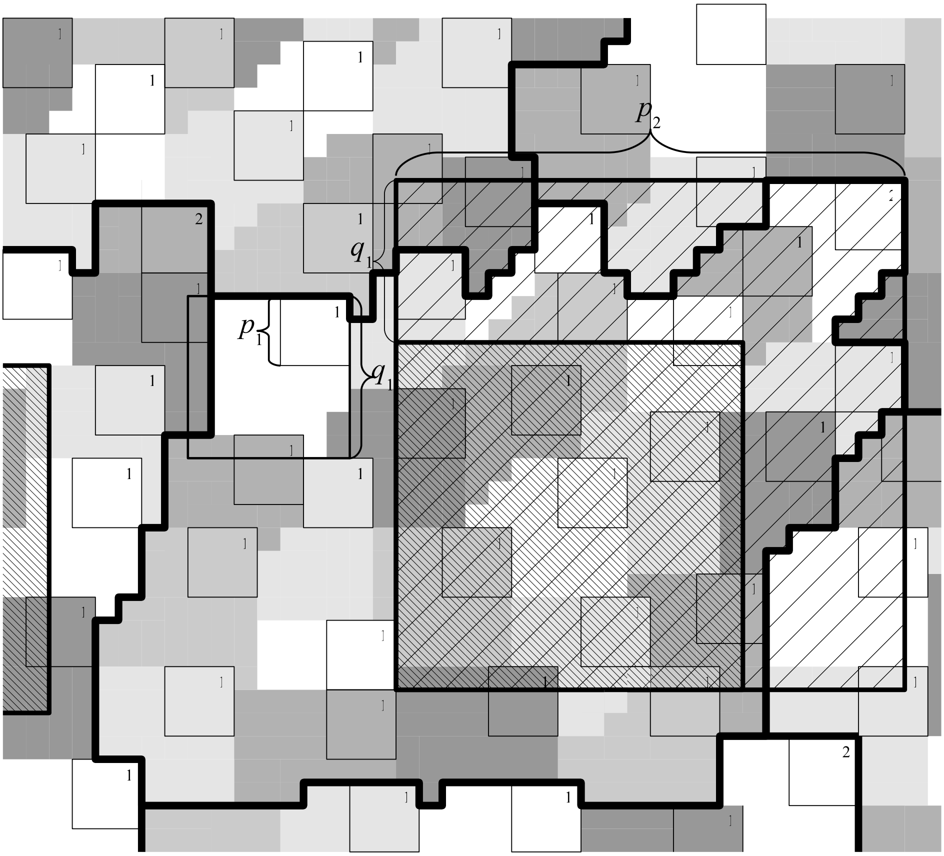

Fig. 2. The construction of 1- and 2-blocks in two dimensions for $p_1 = 3$, $q_1 = 7$, $p_2 = 22$. 1-blocks are distinguished by shades of grey. The bold line separates 2-blocks. Each of marked squares with edge length p_1 has a unique 1-marker in the right upper corner. The big hatched square is an area with a unique 2-marker.

$\geq t$ at $\mathbf{n}(t, A)$. The cube with minimal vertex $\mathbf{n}(t, A)$ and edge length q_{t+1} contains at least one marker $\geq t+1$ and it does not contain the position $\mathbf{n}(t+1, C)$ (because the cube $\mathcal{K}^{\mathbf{n}(t+1,C)}_{Q_{t+1}-Q_t}$ does not contain $\mathbf{n}(t, A)$). Therefore the t-configuration $[x]_A$ is part of a $(t+1)$-configuration with marker $\geq t+1$ outside $[x]_C$. Hence $\mathbf{n} \notin C$.

Observe that the marker $\geq t+1$ of $[x]_C$ lies at the maximal vertex of the cube $\mathcal{K}^{\mathbf{n}(t+1,C)}_{Q_{t+1}}$ which contains C. This proves (3). ∎

Recall that on t-blocks of the same shape we have a metric D, defined as the supremum of the distances d_Λ between symbols occurring at identical positions. Since there are only finitely many shapes available for t-blocks, the metric D is compact on every $\mathcal{B}_t$.

5. Block codes ϕ_t. Now, we will simultaneously define a sequence (p_t) of marker constants and a sequence (ϕ_t) of codes, with each ϕ_t acting on t-blocks. To start the induction we set $p_0 = q_0 = 1$ (as in the previous section) and let ϕ_0 be the identity. We also choose a finite ε_0-dense collection $\mathcal{B}_0^{\varepsilon_0}$ from the set $\mathcal{B}_0$ (with metric D) and put $r_0 = \#\mathcal{B}_0^{\varepsilon_0}$.

In step $t+1$ we assume that we have already defined p_s, q_s and ϕ_s for $s \le t$ and that each ϕ_s maps s-blocks into patterns of the same shape (images of s-blocks under ϕ_s will be called *s-images*). Every $[x] \in X^*$ is a uniquely determined concatenation of s-blocks so we can define a mapping Φ_s on X^*, which applies ϕ_s to every s-block of $[x]$. Moreover, we assume that the orbit of x was moved by ϕ_s from the first level of $[x]$ to a level not farther than $(s+1)$st. Let ψ_s be an auxiliary mapping on s-blocks that only changes every marker $u \ge s$ into marker s, and let $\bar{\phi}_s$ denote $\psi_s \circ \phi_s$. Let $\mathcal{B}_t^{\varepsilon_t}$ be an ε_t-dense subset of $\mathcal{B}_t$ and $r_t = \#(\mathcal{B}_t^{\varepsilon_t})$. We put

$$p_{t+1} = \frac{Q_t(\lceil \sqrt[d]{r_t}\rceil + 2)}{\varepsilon_{t+1}}.$$

Let $\widetilde{B} \in \mathcal{B}_t$ occur in $[x] \in X^*$ on a domain A. We will define $\phi_{t+1}(\widetilde{B})$ as a pattern of the same shape, by describing a configuration M on A.

By Lemma 3 the domain A contains a cube $\mathcal{K} = \mathcal{K}^{\mathbf{v}}_{p_{t+1}-Q_t}$ for some $\mathbf{v} \in A$. Let $\mathcal{K}' = \mathcal{K}^{\mathbf{w}}_{b_{t+1}}$ be a smaller cube with edge length $b_{t+1} = Q_t\lceil \sqrt[d]{r_t}\rceil$ and maximal vertex $\mathbf{w} = \mathbf{v} - \lceil (p_{t+1} - Q_t - b_{t+1})/2\rceil \cdot \mathbf{1}$. Let $[x]_W$ denote the concatenation of all t-configurations whose domains have nonempty intersections with $\mathcal{K}'$. The configuration $[x]_W$ will be called a *buffer*. Observe that $W \subset A$. The code ϕ_t preserves shapes of t-blocks and it will follow from this construction that it differs from ϕ_{t-1} only inside buffers. The buffer has to be large enough to enclose the whole ε_t-dense collection $\mathcal{B}_t^{\varepsilon_t}$ of cardinality r_t. Syndetic appearance of buffers will then imply minimality of the final model. On the other hand, buffers must be relatively small compared to whole blocks in order to preserve the set of invariant measures.

We start the construction of M by inserting in M_W all images $\bar{\phi}_t(B)$ of t-blocks B from the ε_t-dense collection $\mathcal{B}_t^{\varepsilon_t}$, so that their markers t lie at positions $\mathbf{w} - Q_t \cdot \mathbf{m}$, where $\mathbf{m} \in \mathcal{K}_{\lceil \sqrt[d]{r_t}\rceil}$. The rest of M_W (let U denote its domain) will be filled with $\bar{\phi}_s(B_s)$ for $B_s \in \mathcal{B}_s$, $s \le t$, in the following way. Put $U_t = \{\mathbf{n} \in U : \mathcal{K}^{\mathbf{n}}_{Q_t} \subset U\}$, the set of possible maximal vertices for cubes with edge length Q_t, totally contained in U. Consider the order $<^*$ on the cone $(\min\{n_i : \mathbf{n} \in U_t\})_{i=1,\dots,d} + \mathbb{N}_0^d$. If U_t is nonempty, choose $B_t \in \mathcal{B}_t$ and place $\bar{\phi}_t(B_t)$ in M_W so that its marker t lies at $\min^* U_t$. Reduce the set U by subtracting the area where $\bar{\phi}_t(B_t)$ was placed and create new U_t for the reduced U. Until U_t is empty repeat this procedure choosing blocks from $\mathcal{B}_t$ and pasting their images in M_W so that markers lie at minimal points of U_t.

Then repeat this procedure for what has remained of U, replacing t by $t-1$, then by $t-2$ and so on. In the last step for $t=0$ we fill up the whole M_W with 0-blocks.

Now we complete M outside the buffer W. The configuration $[x]_{A\setminus W}$ is a concatenation of t-configurations. For every C being the domain of such a t-configuration with pattern B_t we place in M_C a t-image $\phi_t(B_t)$. By the induction hypothesis, the $(t+2)$nd level of M_W consists of zeros. So for $\mathbf{n}\in W$ we set $M^{t+2}_{\mathbf{n}}=[x]^1_{\mathbf{n}}$.

Having defined M_W made of s-images for $s\le t$ and $M_{A\setminus W}$ made of t-images, we have determined the whole configuration M, whose pattern is $\phi_{t+1}(\widetilde{B})$.

It has to be stressed that the construction of levels 0 to $t+1$ of M_W may be performed in such a way that it depends only on the shape of the buffer W. We do so to ensure that if two $(t+1)$-blocks have buffers of the same shape then their images coincide in buffers on every level except $t+2$.

Properties of the codes are summarized in the following lemma.

LEMMA 4.

(1) *The orbit of x can be read in $\Phi_t([x])$ on the level not farther than $(t+1)$st guaranteeing that Φ_t is one-to-one.*
(2) *ϕ_t and Φ_t differ from ϕ_{t-1} and Φ_{t-1} only in buffers of t-blocks.*
(3) *ϕ_t and Φ_t do not change markers $\ge t$.*
(4) *ϕ_t is continuous on $\mathcal{B}_t$.*
(5) *If $D(B,B')<\varepsilon_t$ for t-blocks B and B', then $D(\phi_t(B),\phi_t(B'))<\varepsilon_t$.*
(6) *Let $B\in\mathcal{B}_t$. Inside the buffer, the image $\phi_t(B)$ is a concatenation of s-images for $s\le t-1$ (with markers $u>s$ changed to s). Outside the buffer it is a concatenation of $(t-1)$-images.*

Proof. Properties (1), (2) and (6) follow directly from the construction of ϕ_t. All others are clearly satisfied for ϕ_0. To prove (3) note that by (2) it suffices to check the markers in the buffer; but every marker in the buffer of a $(t+1)$-block is less than or equal to t, while ϕ_{t+1} replaces s-blocks $(s\le t)$ from the buffer only by s-images with markers changed to s. We leave the straightforward verification of (4) and (5) to the reader. ∎

Let $\widetilde{\mathcal{B}}_t$ denote the collection of all t-images and let

$$\widetilde{\mathcal{B}}^{\varepsilon_t}_t=\{\overline{\phi}_t(B):B\in\mathcal{B}^{\varepsilon_t}_t\}.$$

Note that by (5) of the above lemma, $\widetilde{\mathcal{B}}^{\varepsilon_t}_t$ is ε_t-dense in $\widetilde{\mathcal{B}}_t$.

6. Frequency of buffers and borders. Let $\mathcal{A}=\{A_x\colon x\in X\}$ be a collection of subsets of $\mathbb{Z}^d$. We will say that $\mathcal{A}$ occurs in a system X with *frequency* $\le\alpha$ if there exist a and b such that $a/b^d\le\alpha$ and for every $x\in X$

in any cube $\mathcal{K}_b^{\mathbf{v}}$ lying in the domain of $[x]$ the cardinality of $A_x \cap \mathcal{K}_b^{\mathbf{v}}$ is less than or equal to a.

A position $\mathbf{n}$ in the domain of a t-configuration M belongs to the *border of the t-configuration* (or, simply, to the *t-border*) if at least one of the $2d$ positions $\mathbf{n} \pm \mathbf{e}_i$ belongs to the domain of another t-configuration.

LEMMA 5. *For the sequences (p_t) and (q_t) defined above we have*:

(1) *if $\mathcal{A}$ denotes a set of t-buffers, i.e. A_x is the union of all buffers of t-configurations in $[x]$, then $\mathcal{A}$ occurs in X with frequency $\leq (4\varepsilon_t)^d$*,
(2) *the set of t-borders occurs in X with frequency $\leq (1-10^{-2d})^t$.*

Proof. Throughout the proof $c_t = \lceil \sqrt[d]{r_t}\rceil + 2$.

(1) Consider an arbitrary array $[x] \in X^*$ and a cube $\mathcal{K}$ with edge length $p_t/2$, lying in the domain of $[x]$. The domain of every t-configuration $[x]_A$ contains a cube with edge length $p_t - Q_{t-1} > p_t/2$, carrying a buffer of a t-block. The buffer is situated at positions lying at least $3p_t/16$ from the closest face of this cube. It follows that $\mathcal{K}$ may intersect the domains of buffers of at most 2^d t-configurations. A buffer of a t-configuration $[x]_A$ is contained in a cube with edge length $Q_{t-1}c_{t-1}$. Thus, among all positions in $\mathcal{K}$, at most $2^d(Q_{t-1}c_{t-1})^d$ positions lie in t-buffers. By the recursive definition of p_t we have $Q_{t-1}c_{t-1} = p_t\varepsilon_t$. So the frequency of the set of symbols lying in t-buffers is less than or equal to

$$\frac{2^d(Q_{t-1}c_{t-1})^d}{(p_t/2)^d} = \frac{2^d(p_t\varepsilon_t)^d}{(p_t/2)^d} = (4\varepsilon_t)^d.$$

(2) Since $q_{t-1} \leq Q_{t-1}$ and $\varepsilon_t < 1 < c_t$ for every $t > 0$,

$$\frac{p_t}{q_t} = \frac{p_t}{4p_t + q_{t-1}} = \frac{Q_{t-1}c_{t-1}}{\varepsilon_t\left(4\frac{Q_{t-1}c_{t-1}}{\varepsilon_t} + q_{t-1}\right)} \geq \frac{c_{t-1}}{4c_{t-1} + \varepsilon_t} > \frac{1}{5}.$$

Fix $[x] \in X^*$. In every cube with edge length q_t one can find a marker $u \geq t$. Every domain of a t-configuration contains a cube with edge length $p_t - Q_{t-1}$. Hence, in every cube with edge length $q_t + p_t$ there is a cube with edge length $p_t - Q_{t-1}$, totally contained in the domain of one t-configuration. Cutting off a border of thickness one we obtain a cube with edge length $p_t - Q_{t-1} - 2$, no position of which belongs to a t-border.

Set $\widetilde{p}_0 = p_0$ and $\widetilde{q}_0 = q_0$. Inductively, let $\widetilde{p}_t$ be the largest integer multiple of $\widetilde{q}_{t-1}$ less than or equal to $p_t/4$ and let $\widetilde{q}_t$ be the smallest integer multiple of $\widetilde{p}_t$ greater than or equal to $2q_t$. For $t \geq 1$ we have

$$\widetilde{p}_t \geq \frac{p_t}{4} - \widetilde{q}_{t-1} \geq \frac{p_t}{4} - 2q_{t-1} - \widetilde{p}_{t-1} \geq \frac{p_t}{4} - 2\,\frac{p_{t-1}}{16} - \frac{p_{t-1}}{4} \geq \frac{7p_t}{64},$$

so

$$\frac{\widetilde{p}_t}{\widetilde{q}_t} \geq \frac{\widetilde{p}_t}{2q_t + \widetilde{p}_t} \geq \frac{7p_t}{64(2q_t + p_t/4)} \geq 10^{-2}.$$

In the domain of $[x]$ consider a cube $\mathcal{K}$ with edge length being an integer multiple of $\widetilde{q}_t$. The cube $\mathcal{K}$ consists of disjoint cubes $\mathcal{L}_1, \ldots, \mathcal{L}_n$, with edges of common length $\widetilde{q}_t$. Every cube $\mathcal{L}_i$ can be divided into cubes $\mathcal{L}_{i,j}$ with edges $\widetilde{p}_t$. Since $2\widetilde{p}_t < p_t - Q_{t-1} - 2$ and $\widetilde{q}_t > q_t + p_t$, for every i at least one of $\mathcal{L}_{i,j}$'s does not contain any position from the t-border. Suppose that $\mathcal{L}_{i,1}$ is such. Put $\mathcal{L}^t = \bigcup_i \mathcal{L}_{i,1}$. The set $\mathcal{L}^t$ covers at least a 10^{-2d} fraction of the cube $\mathcal{L}$. In the next step we perform an analogous reasoning, replacing $\mathcal{L}$ by $\mathcal{L} \setminus \mathcal{L}_t$, and t by $t-1$ (note that $\mathcal{L} \setminus \mathcal{L}_t$ is a concatenation of cubes with edge length $\widetilde{q}_{t-1}$). We define $\mathcal{L}^{t-1}$, which again covers 10^{-2d} of a new $\mathcal{L}$, and remove it from $\mathcal{L}$. In the kth step we divide the current set $\mathcal{L}$ into cubes $\mathcal{L}_i$ with edge length $\widetilde{q}_{t-k+1}$, and $\mathcal{L}_i$'s into $\mathcal{L}_{i,j}$'s with edge length $\widetilde{p}_{t-k+1}$, and define $\mathcal{L}^{t-k+1} = \bigcup_i \mathcal{L}_{i,1}$ that occupies 10^{-2d} of $\mathcal{L}$. The algorithm is repeated until $k = t$. In every step we diminish $\mathcal{L}$ at least by 10^{-2d} of it, so after t steps we obtain an $\mathcal{L}$ which is at least $(1 - 10^{-2d})^t$ times smaller, but it contains the whole t-border. ■

7. The space $(\widetilde{X}, \tau)$. We will distinguish a full subset of X^* which will become the support of a Borel* isomorphism Φ defined as the pointwise limit of maps Φ_t.

Let X_t^{buf} denote the set of all $[x] \in X^*$ whose position $\mathbf{0}$ lies in a t-buffer. Using the first part of Lemma 5 and Tempel'man's ergodic theorem for d commuting endomorphisms of a probability space (see [2]), for any ergodic measure μ_E on X^* we obtain

$$\mu_E(X_t^{\mathrm{buf}}) = \lim_{r\to\infty} \frac{1}{r^d} \sum_{\mathbf{n}\in\mathcal{K}_r} \mathbf{1}_{X_t^{\mathrm{buf}}} \circ \sigma^{\mathbf{n}} \le (4\varepsilon_t)^d.$$

Let X^{buf} be the set of those $[x] \in X^*$ whose zero position lies in a t-buffer for at most finitely many t, and let X' be the set of $[x]$ each of whose positions lies in a t-buffer for at most finitely many t. Then

$$X^{\mathrm{buf}} = X \setminus \bigcap_{s=1}^{\infty} \bigcup_{t=s}^{\infty} X_t^{\mathrm{buf}} \quad \text{and} \quad X' = \bigcap_{\mathbf{n}\in\mathbb{Z}^d} \sigma^{\mathbf{n}}(X^{\mathrm{buf}}).$$

The sequence $((4\varepsilon_t)^d)_t$ is summable, so $\mu_E(X^{\mathrm{buf}}) = 1$. Since μ_E is invariant, we also obtain $\mu_E(X') = 1$. This holds for any ergodic $\mu_E \in \mathcal{P}(X^*)$, thus $\mu(X') = 1$ for any measure $\mu \in \mathcal{P}(X^*)$ and X' is a full subset of X^*.

Similarly we define X_t^{border} to be the subset of X' consisting of the points whose zero position belongs to a t-border,

$$X^{\mathrm{border}} = X' \setminus \bigcap_{s=1}^{\infty} \bigcup_{t=s}^{\infty} X_t^{\mathrm{border}} \quad \text{and} \quad X'' = \bigcap_{\mathbf{n}\in\mathbb{Z}^d} \sigma^{\mathbf{n}}(X^{\mathrm{border}}).$$

Analogously to the above considerations, the second part of Lemma 5 and the ergodic theorem yield $\mu_E(X_t^{\text{border}}) \leq (1-10^{-2d})^t$ for any ergodic measure μ_E, hence the sets X^{border} and X'' are full subsets of X^*. It follows that for all $y \in X''$ any cube appearing in the domain of y is covered by one t-configuration for sufficiently large t.

Recall that according to Lemma 4 the map Φ_{t+1} differs from Φ_t only in buffers of $(t+1)$-blocks. Consequently, for any y from X'' each position is changed by Φ_t only for a finite number of t. Thus we can define a map Φ on X'' as the pointwise limit of the maps Φ_t as $t \to \infty$. Let $\widetilde{X}$ be the closure of $\overline{\Phi(X'')}$ in $\Lambda^{\mathbb{Z}^d}$, where $\Lambda = (X \cup \overline{\mathbb{N}}_0)^{\mathbb{N}_0}$, and let τ be the set of shift maps on $\widetilde{X}$.

For each t every element y of $\Phi(X'')$ is a concatenation of s-images for $s \leq t$. Shapes of s-images are the same as shapes of s-blocks, so they satisfy (4) and (5) of Lemma 3. As in the proof of Lemma 5 every cube with edge length $p_t/2$ in the domain of y intersects at most 2^d buffers of t-configurations. Hence for elements of $\Phi(X'')$ we obtain the same upper bound on the frequency of t-buffers as in Lemma 5. We now show that this bound is also valid for elements of $\widetilde{X}$. Pick $y = \lim_k y_k \in \widetilde{X}$, where $(y_k)_{k\in\mathbb{N}} \subset \Phi(X'')$. For a given t consider a cube $\mathcal{K} = \mathcal{K}^{\mathbf{v}}_{p_t/2}$ in the domain of y. If for any y the cube $\mathcal{K}$ intersects the domain of a buffer of some s-configuration for $s \leq t$, then its marker belongs to $\mathcal{L} = \mathcal{K}^{\mathbf{v}+p_t}_{3p_t/2}$, a larger cube sharing the minimal vertex with $\mathcal{K}$. Hence, if y_k converges to y, location of s-buffers for $s \leq t$ on $(y_k)_{\mathcal{K}}$ is for sufficiently large k the same as on $y_{\mathcal{K}}$. In particular, we get an upper bound on the frequency of t-buffers for the whole $\widetilde{X}$ as in Lemma 5. In the same way as in X^* we define a full subset $\widetilde{X}'$ of $\widetilde{X}$, consisting of the points each of whose positions lies in a t-buffer for finitely many t only.

On $\widetilde{X}' \cap \Phi(X'')$ a position belongs to a t-border either if it belonged to a t-border in X'' and it does not lie in any of u-buffers for $u > t$, or if it lies in u-buffers for $u > t$ and it has fallen into a t-border by filling the buffer for the largest such u. The first case happens with frequency $\leq (1-10^{-2d})^t$, according to Lemma 5. The frequency of the second case is bounded from above by the sum of the frequencies of the u-buffers for $u > t$, which is equal to $\sum_{i=t+1}^{\infty}(4\varepsilon_i)^d$. Hence the frequency of observing a t-border is bounded on $\widetilde{X}' \cap \Phi(X'')$ by the terms of the summable sequence $(1-10^{-2d})^t + 4^d\varepsilon_t^d$. To prove that this bound is also valid for $\widetilde{X}'$ consider $y = \lim_k y_k \in \widetilde{X}'$, where $(y_k)_{k\in\mathbb{N}} \subset \Phi(X'')$. In the domain of y select a cube $\mathcal{K} = \mathcal{K}^{\mathbf{v}}_b$. Let $\mathcal{L} = \mathcal{K}^{\mathbf{v}+Q_t\cdot\mathbf{1}}_{b+2Q_t}$. Since y is the limit of y_k's, positions of markers t in $(y_k)_{\mathcal{L}}$ are for sufficiently large k the same as in $y_{\mathcal{L}}$, except for the markers t which will be replaced by higher markers during the construction. Therefore t-configurations in $(y_k)_{\mathcal{K}}$ for large k have the same shapes as t-configurations in $y_{\mathcal{K}}$, and they have the same t-borders. It follows that on $\widetilde{X}'$ the frequency of t-borders has the same upper bound as on $\widetilde{X}' \cap \Phi(X'')$. Setting $\widetilde{X}''$ to be the set of all elements of $\widetilde{X}'$ whose positions belong to t-borders for at most finitely many t, we obtain again a full subset of $\widetilde{X}'$, hence also of $\widetilde{X}$.

REMARK 6. Note also that for $\widetilde{y} \in \widetilde{X}''$ every cube $\mathcal{K}_b^{\mathbf{v}}$ in $\widetilde{y}$ has the same distribution of markers as an identical cube in some $y \in \Phi(X'')$. Thus the structure of t-configurations on $\mathcal{K}_{b-2Q_t}^{\mathbf{v}-Q_t}$ in $\widetilde{y}$ is the same as in y.

8. $(\widetilde{X}, \tau)$ **is minimal.** A set $A \subset \mathbb{Z}^d$ is *syndetic* with constant $L \geq 0$ if $A \cap \mathcal{K}_L^{\mathbf{v}} \neq \varnothing$ for every $\mathbf{v} \in \mathbb{Z}^d$. We say that a block B with the shape contained in some cube with edge length k *appears* in an array y *syndetically with a constant* L if it appears as a subblock of every cube with edge length $L + k$. We skip the standard proof of the following lemma.

LEMMA 7. *Let Y be a d-dimensional symbolic system over a compact alphabet Λ. Let $\mathcal{B}'_Y$ be a countable collection of blocks satisfying the following condition: for every $\varepsilon > 0$ and every block B occurring in Y one can find $B' \in \mathcal{B}'_Y$ such that $D(B, B'') < \varepsilon$ for a certain subblock B'' of B'.*

If there exists a dense set $Y' \subset Y$ consisting of elements y in which every $B \in \mathcal{B}'_Y$ occurs syndetically with constant depending only on B, then the symbolic system (Y, σ) is minimal.

We will use the above lemma for $Y = \widetilde{X}$, taking as $\mathcal{B}'_{\widetilde{X}}$ the collection $\bigcup_t \widetilde{\mathcal{B}}_t^{\varepsilon_t}$, and as a dense subset of $\widetilde{X}$ the set $\Phi(X'')$.

Consider a block $\widetilde{B}$ with the shape of cube, occurring in $\widetilde{X}$. For arbitrarily small ε the block is ε-close to a configuration in some element of $\Phi(X'')$. For large t this configuration is contained in a t-image $B_t \in \widetilde{\mathcal{B}}_t$, whose ε_t-approximation will be denoted by $B'_t \in \widetilde{\mathcal{B}}_t^{\varepsilon_t}$. Pick any $y \in \Phi(X'')$. It suffices to show that B'_t (with entries at positions on levels farther than $t+1$ possibly changed from zeros to other characters) occurs syndetically in y. The distance D between $\widetilde{B}$ and an appropriate subblock of B'_t will be bounded by $\varepsilon + (1 + \operatorname{diam}(X))\varepsilon_t$.

The block B'_t appears in the buffer of every $(t+1)$-image. Fix $\mathbf{n} \in \mathbb{Z}^d$. We will show that for some subset E of $\mathcal{K}_{3Q_{t+1}}^{\mathbf{n}+Q_{t+1}}$ the configuration y_E corresponds to a $(t+1)$-image. It will prove that B'_t appears syndetically with constant $3Q_{t+1} - Q_t$.

Recall that the array y is a concatenation of $(t+1)$-images apart from u-buffers for $u > t+1$, while the buffer of a u-block is a concatenation of s-images for $s < u$. Consider a block consisting of the cube $\mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$ in y together with all u-blocks, $u > t+1$, whose buffers intersect this cube and will not be changed by higher codes (note that any two buffers, possibly of different order, are either disjoint or ordered by inclusion). If no such blocks exist, then $\mathbf{n}$ belongs to the domain of some $(t+1)$-image, which will be the final outcome of the action of Φ. This block is contained in a cube with edge length Q_{t+1}, so its domain is contained in $\mathcal{K}_{2Q_{t+1}}^{\mathbf{n}+Q_{t+1}}$. On the other hand, if there are some $u > t+1$ such that u-buffers intersect $\mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$, then pick one of those u-buffers and study its structure. It was concatenated of u'-images for $u' < u$, whose buffers may intersect $\mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$. Each of these u'-images again is a certain concatenation and so on. Let u_0 denote the least $u' > t+1$ such that a u'-buffer intersects our cube. If $\mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$ is completely covered by the buffer of a u_0-block, then it intersects the domain of some s-block, where $t+1 \leq s < u_0$ (because of the algorithm of filling the buffer). Hence, there is a position $\mathbf{n}' \in \mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$, which belongs to the domain of an s-image (take the smallest such $s \geq t+1$), but not to its buffer. Then it lies in the domain of some $(t+1)$-image, which is contained in $\mathcal{K}_{2Q_{t+1}}^{\mathbf{n}'+Q_{t+1}} \subset \mathcal{K}_{3Q_{t+1}}^{\mathbf{n}+Q_{t+1}}$.

But if the cube $\mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$ is not completely covered by the buffer of a u_0-block, then a certain position $\mathbf{n}'' \in \mathcal{K}_{Q_{t+1}}^{\mathbf{n}}$ lies outside this buffer and inside the same u_0-image. Thus $\mathbf{n}''$ belongs to the domain of a $(t+1)$-block, which is contained in the cube $\mathcal{K}_{2Q_{t+1}}^{\mathbf{n}''+Q_{t+1}} \subset \mathcal{K}_{3Q_{t+1}}^{\mathbf{n}+Q_{t+1}}$.

9. Φ is a Borel* isomorphism. It remains to prove that (X^*, σ) and $(\widetilde{X}, \tau)$ are Borel* isomorphic. The sets X'' and $\widetilde{X}''$ are full subsets of X^* and $\widetilde{X}$, respectively. We will show that Φ is a bijection between them.

Since for every $[x] \in X''$ we have $\Phi([x])_{\mathbf{0}}^k \neq 0$ for at most finitely many k, and the last nonzero level contains x, Φ is injective. To prove that it is also surjective, choose $\widetilde{y} \in \widetilde{X}''$. By Remark 6, at every position only a finite number of nonzero levels is allowed. Let x be the character that appears at the last level of position $\mathbf{0}$ in $\widetilde{y}$. The character is a member of the original space X, and its array representation in X^* is denoted by $[x]$. Consider a central cube $\overline{\mathcal{K}}_b$ in the domain of $\widetilde{y}$. It is contained in the domain of some t-configuration, representing a block $\widetilde{B}$. Since $\widetilde{y} \in \widetilde{X}$, there exists a sequence (B_k) of t-blocks such that $\widetilde{B} = \lim_k \phi_t(B_k)$. Note that since $\widetilde{B}$ has x at the last nonzero level of the position corresponding to position $\mathbf{0}$ of $\widetilde{y}$, B_k's must approach x at the first level of the same position. The metric D on the set of t-blocks is compact, thus we can choose a subsequence (B_k') of (B_k) convergent to some t-block B, which surrounds position $\mathbf{0}$ in $[x]$. Recall that ϕ_t is continuous, hence

$$\widetilde{B} = \lim_k \phi_t(B_k) = \lim_k \phi_t(B_k') = \phi_t(\lim_k B_k') = \phi_t(B).$$

Thus the equality $\widetilde{y} = \Phi_t([x])$ holds on the whole $\overline{\mathcal{K}}_b$. Since $\overline{\mathcal{K}}_b$ can be taken arbitrarily large and the calculation above is correct for any sufficiently large t, every position of $[x]$ lies in buffers of at most finitely many t-blocks and $\widetilde{y} = \Phi([x])$. To show that a position $\mathbf{n}$ of $[x]$ visits borders finitely many times, fix t and note that the set of t-border positions in $[x]$ coincides with the t-border of $\Phi_u([x])$, $u \geq t$, apart from s-buffers for $t < s \leq u$. Take $\overline{\mathcal{K}}_b \ni \mathbf{n}$ and t so large that for $u \geq t$ we have $\widetilde{y} = \Phi_u([x])$ on $\overline{\mathcal{K}}_b$ and $[x]_{\mathbf{n}}$ lies outside u-buffers. From the fact that $\widetilde{y}_{\mathbf{n}}$ belongs only to a finite number of borders, we get the same property for $[x]_{\mathbf{n}}$.

Measurability of Φ follows from the fact that it is the pointwise limit of a sequence of continuous maps Φ_t, and measurability of its inverse is thus granted by the Kuratowski theorem (see [3]).

Since $\widetilde{X}$ is metric, the space $C(\widetilde{X})$ is separable. Choosing a dense countable set $\{\widetilde{f}_k\} \subset C(\widetilde{X})$ and setting $f_k = \widetilde{f}_k/\|\widetilde{f}_k\|_\infty$ we can define a metric on $\mathcal{P}(\widetilde{X})$, compatible with the weak* topology, by the formula

$$\varrho(\widetilde{\mu}, \widetilde{\nu}) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \left| \int f_k \, d\widetilde{\mu} - \int f_k \, d\widetilde{\nu} \right|.$$

We can also demand that the set $\{\widetilde{f}_k\}$ consists of simple functions combined from characteristic functions of clopen cylinders.

We will show that the sequence of maps $\Phi_t^* \colon \mathcal{P}(X^*) \to \mathcal{P}(\widetilde{X})$, $t \in \mathbb{N}$, converges uniformly, by verifying the Cauchy criterion. Fix $\varepsilon > 0$. We need to find T such that $\varrho(\Phi_t^*(\mu), \Phi_T^*(\mu)) < \varepsilon$ for all $t > T$ and $\mu \in \mathcal{P}(X^*)$. Since all Φ_t^*'s are affine, $\mu \mapsto \varrho(\Phi_t^*(\mu), \Phi_T^*(\mu))$ is convex (and continuous), hence attains its maximum on the set of extremal points of $\mathcal{P}(X^*)$. Thus it is enough to consider ergodic measures.

Find $K \in \mathbb{N}$ such that $\sum_{k=K+1}^{\infty} 1/2^k < \varepsilon/2$. For every pair of measures $\widetilde{\mu}, \widetilde{\nu}$, the kth element of the series $\varrho(\widetilde{\mu}, \widetilde{\nu})$ is bounded by $1/2^k$, so the task boils down to finding T such that for every $t > T$ and every ergodic $\mu \in \mathcal{P}(X^*)$,

$$\sum_{k=1}^{K} \frac{1}{2^{k+1}} \left| \int f_k \, d\Phi_t^*(\mu) - \int f_k \, d\Phi_T^*(\mu) \right| < \frac{\varepsilon}{2}.$$

Since the above sum is finite and each f_k is a linear combination of characteristic functions of cylinders, it is enough to prove that for every $\delta > 0$ and every cylinder A there exists T such that for every $t > T$ and every ergodic measure $\mu \in \mathcal{P}(X^*)$,

$$\left| \int \mathbf{1}_A \, d\Phi_t^*(\mu) - \int \mathbf{1}_A \, d\Phi_T^*(\mu) \right| = |\mu(\Phi_t^{-1} A) - \mu(\Phi_T^{-1} A)| < \delta.$$

Note that Φ_T and Φ_t differ only in buffers of s-markers for $T < s \leq t$. Thus the above inequality follows from the $\mathbb{Z}^d$-ergodic theorem by estimating the frequency of visits of $\Phi_T(x)$ and $\Phi_t(x)$ in A.

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