COLLOQUIUM MATHEMATICUM

VOL. 110 2008 NO. 2

MARKOV PARTITIONS FOR FIBRE EXPANDING SYSTEMS

BY

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Abstract. Fibre expanding systems have been introduced by Denker and Gordin. Here we show the existence of a finite partition for such systems which is fibrewise a Markov partition. Such partitions have direct applications to the Abramov–Rokhlin formula for relative entropy and certain polynomial endomorphisms of $\mathbb{C}^2$.

1. Fibre expanding systems. Let Y be a compact metric space and $T : Y \to Y$ be a continuous surjective map. Consider a fibred situation of a dynamical system (Y, T), where the map T is foliated over a continuous map $S : X \to X$ on some compact metric space X, with a continuous surjective factor map $\pi : Y \to X$ which semiconjugates T to S:

$$\begin{array}{ccc} Y & \xrightarrow{T} & Y \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X & \xrightarrow{S} & X \end{array}$$

X is called the *base space* and S the *base transformation*. According to [6] T is said to be *fibre expanding* if there exist $a > 0$ and $\lambda \in (0, 1)$ such that the following holds: If $u, v' \in Y$, $\pi(T(u)) = \pi(v')$ and $d(T(u), v') < 2a$, there exists a unique $v \in Y$ such that $\pi(v) = \pi(u)$, $T(v) = v'$ and $d(u, v) < 2a$. Furthermore,

$$d(u, v) \le \lambda d(T(u), T(v)).$$

The situation was examined from a purely topological point of view by Roy [17], who discussed the relations to expansiveness and openness of fibre maps.

When extending the thermodynamical formalism to the fibred situation, the first question is to generalize the notion of Gibbs measures to the relative case. A *Gibbs measure* is defined by the usual property that the Jacobian (of the disintegration measures) under fibre maps has a prescribed fibrewise

2000 *Mathematics Subject Classification*: 37D05, 37A05, 37A35, 37F10.

Key words and phrases: Markov partition, fibred system, entropy, polynomial map.

This research is partially supported by the Graduiertenkolleg *Gruppen und Geometrie* at Göttingen University. M. Denker would like to thank the University of Kyoto, Graduate School of Environmental Science, for their hospitality and support.

(Hölder) continuous version. Problems of this type in the relativized context of fibred systems have been considered in the literature. In the work of Ferrero and Schmitt ([8]) and later of Bogenschütz and Gundlach ([2], [4]), this problem was considered when the base transformation S is an invertible measure-preserving map of some probability space. The case of non-invertible transformations and fibrewise expanding systems was considered in [6], while a Gibbs family for certain fibrewise expansive systems appears in [18].

In the present situation the classical Frobenius–Perron theory or spectral theory is not applicable, since the transfer operators act between different function spaces (see [7]). However, the construction of equilibrium measures has been accomplished in some cases (e.g. [12], [13], [11], [7]). The associated pressure (in the relative setting) is defined in [16] and a variational formula is proved. Bogenschütz in [2] and [4] and Kifer in [14] studied pressure functions for random bundle maps and their relative variational formulas. A new relative variational formula using the Abramov–Rohklin relative entropy formula has been derived in [7], where the maximum of the sum of conditional entropy and expectation over potentials is described by the integral of some generalized eigenvalue function over the module of functions constant on fibres.

The existence of relative generators has been studied in the invertible case by Kifer and Weiss (cf. [15]), and independently by Danilenko and Park ([5]). In the non-invertible case the problem is more delicate (as in the non-relativized case). Some results in this direction are contained in [20].

The theory has been developed without the use of Markov partitions. This is because the existence of such a partition does not seem to be known. In this note we prove that there is a partition which, restricted to each fibre, is a generator and has the Markov property with respect to fibre maps. When the base space is reduced to a one-point space, this partition will be a Markov partition in the usual sense (cf. [19]). It should be noted that our proof (when reduced to this particular case) gives a new and direct proof for the existence of a Markov partition for expanding and open maps.

2. Markov partitions. Let (Y,T) denote a fibred system which is fibrewise expanding over (X,S) as defined in Section 1. The fibres over X will be denoted by $Y_x=\pi^{-1}\{x\}$. Furthermore, $T_x : Y_x \to Y_{S(x)}$ denotes the map T restricted to the fibre over $x\in X$. We shall prove the following theorem:

THEOREM. *There exists a finite partition γ of Y such that*:

(A) $T_x(G\cap Y_x)=\bigcup_{g\in\gamma\,:\,g\cap T(G)\neq\emptyset} g\cap Y_{S(x)}$ *for all $G\in\gamma$ and $x\in X$.*

(B) *There is a constant C such that*

$$\sup_{x\in X}\sup_{G\in\gamma_0^n}\operatorname{diam}(G\cap Y_x)\le C\lambda^n,$$

where $\gamma_0^n=\gamma\vee T^{-1}\gamma\vee\cdots\vee T^{-n+1}\gamma$.

Proof. Let a and λ be as in the definition of the fibrewise expanding property. Choose δ so small that $\delta/(1-\lambda) < a$. Let $\mathcal{U}_0$ be a finite open cover of Y by sets of diameter $\leq \delta$.

For $U \in \mathcal{U}_0$ define

$$\mathcal{U}(U) = \{V \in \mathcal{U}_0 : V \cap T(U) \neq \emptyset\}.$$

Recursively, we let

$$\Psi_0(U) = U,$$

$$\Psi_n(U) = \left\{ y \in \pi^{-1}(\pi(U)) : \begin{array}{l} T(y) \in \Psi_{n-1}(V) \text{ for some } V \in \mathcal{U}(U) \\ \text{and } d(y,U) < a \end{array} \right\}.$$

We first claim that

(a) $U \subset \Psi_{n-1}(U) \subset \Psi_n(U) \subset B(U, (\lambda + \ldots + \lambda^n)\delta)$,
(b) $\pi(\Psi_n(U)) = \pi(U)$,
(c) $T_x(\Psi_n(U) \cap Y_x) = \bigcup_{V \in \mathcal{U}(U)} \Psi_{n-1}(V) \cap Y_{S(x)}$ for all $x \in \pi(U)$.

Statement (a) is proved by induction over n; (b) and (c) follow from this.

For $n = 1$ we obtain:

(a) $y \in U$, $T(y) \in V = \Psi_0(V) \in \mathcal{U}(U)$ imply $y \in \Psi_1(U)$, hence $U \subset \Psi_1(U)$. If $y \in \Psi_1(U) \cap Y_x$, there exist $V \in \mathcal{U}(U)$ and $z \in U \cap Y_x$ such that $T(y), T(z) \in V \cap Y_{S(x)}$. Therefore $d(T(z), T(y)) < \delta$, and by the expanding property, $d(z, y) < \lambda\delta$, i.e. $y \in B(U, \lambda\delta)$.

(b) Let $y \in \Psi_1(U)$. Then $\pi(y) \in \pi(U)$ by definition, so $\pi(\Psi_1(U)) \subset \pi(U)$. The converse follows from $U \subset \Psi_1(U)$.

(c) Let $x \in \pi(U)$ and $y \in \Psi_1(U) \cap Y_x$. Then there exists $V \in \mathcal{U}(U)$ such that $T_x(y) = T(y) \in V$. Therefore

$$T_x(\Psi_1(U) \cap Y_x) \subset \bigcup_{V \in \mathcal{U}(U)} V \cap Y_{S(x)}.$$

Conversely, if $z \in V \cap Y_{S(x)}$, where $V \in \mathcal{U}(U)$, then there exists $y \in T(U)$ with $d(z, T(y)) < \delta < 2a$. By the expanding property there exists $z' \in B(y, 2a) \cap Y_x$ such that $T(z') = z$ and $d(z', y) < \lambda\delta < a$, whence $z' \in \Psi_1(U)$.

Assume that (a) holds for $n - 1$.

(a) From the induction hypothesis we have

$$U \subset \Psi_{n-2}(U) \subset \Psi_{n-1}(U) \subset B(U, (\lambda + \cdots + \lambda^{n-1})\delta).$$

Let $y \in \Psi_{n-1}(U)$. Then

(i) $T(y) \in \Psi_{n-2}(V) \subset \Psi_{n-1}(V)$ for some $V \in \mathcal{U}(U)$,
(ii) $y \in \pi^{-1}(\pi(U))$,
(iii) $d(y, U) < a$,

hence $y \in \Psi_n(U)$ and $U \subset \Psi_{n-1}(U) \subset \Psi_n(U)$.

Let now $y \in \Psi_n(U)$. Then there exists $V \in \mathcal{U}(U)$ such that $T(y) \in \Psi_{n-1}(V)$. Choose $z \in \overline{V}$ such that $d(z, T(y)) = d(V, T(y))$ and $z' \in U$ such that $T(z') \in V$. Then

$$d(T(y), T(z')) \leq d(T(y), z) + d(z, T(z')) \leq (\lambda + \cdots + \lambda^{n-1})\delta + \delta < 2a,$$

hence by the expanding property,

$$d(y, z') \leq \lambda(1 + \lambda + \cdots + \lambda^{n-1})\delta = (\lambda + \cdots + \lambda^n)\delta.$$

(b) Let $y \in \Psi_n(U)$. Then $\pi(y) \in \pi(U)$ by definition, so $\pi(\Psi_n(U)) \subset \pi(U)$. The converse follows from $U \subset \Psi_n(U)$ (using (a) as proved above).

(c) Let $x \in \pi(U)$ and $y \in \Psi_n(U) \cap Y_x$. Then there exists $V \in \mathcal{U}(U)$ such that $T_x(y) = T(y) \in \Psi_{n-1}(V)$. Therefore

$$T_x(\Psi_n(U) \cap Y_x) \subset \bigcup_{V \in \mathcal{U}(U)} \Psi_{n-1}(V) \cap Y_{S(x)}.$$

Conversely, if $z \in \Psi_{n-1}(V) \cap Y_{S(x)}$, where $V \in \mathcal{U}(U)$, then there exist $z_1 \in V$ and $T(y) = z_2 \in T(U)$ with $d(z, z_1) \leq (\lambda + \cdots + \lambda^{n-1})\delta$ and $d(z_1, z_2) < \delta < 2a$. Therefore $d(z, z_2) < 2a$ and by the expanding property there exists $z' \in B(y, 2a) \cap Y_x$ such that $T(z') = z$ and $d(z', y) < \lambda\delta < a$, whence $z' \in \Psi_n(U) \cap Y_x$.

The theorem follows from (a)–(c) in a canonical way. Define

$$\Psi(U) = \lim_{n \to \infty} \Psi_n(U).$$

Then, with $\Lambda = \lambda/(1 - \lambda)$, we have

(a) $U \subset \Psi(U) \subset B(U, \Lambda\delta)$,
(b) $\pi(\Psi(U)) = \pi(U)$,
(c) for $x \in \pi(U)$,

$$\begin{aligned} T_x(\Psi(U) \cap Y_x) &= \lim_{n \to \infty} T_x(\Psi_n(U) \cap Y_x) \\ &= \lim_{n \to \infty} \bigcup_{V \in \mathcal{U}(U)} \Psi_{n-1}(V) \cap Y_{S(x)} \\ &= \bigcup_{V \in \mathcal{U}(U)} \Psi(V) \cap Y_{S(x)} \end{aligned}$$

Now we construct the partition γ. Write $\mathcal{U}_0 = \{U_1, \ldots, U_s\}$ for some $s \geq 1$ and define the atoms of γ by

$$G = \bigcap_{j \in I(G)} \Psi(U_j) \cap \bigcap_{j \notin I(G)} \Psi(U_j)^{\mathrm{c}},$$

where $I(G)$ is any (non-empty) subset of $\{1, \ldots, s\}$. If

$$H = \bigcap_{j \in I} \Psi(U_j)$$

for some $I \subset \{1, \ldots, s\}$, then by invertibility of T_x on sets of diameter $< 2a$,

$$T_x(H \cap Y_x) = \bigcap_{j \in I} T_x(\Psi(U_j)) \cap Y_{S(x)} = \bigcap_{j \in I} \bigcup_{V \in \mathcal{U}(U_j)} \Psi(V) \cap Y_{S(x)}$$
$$= \bigcup_{V_j \in \mathcal{U}(U_j);\, j \in I} \bigcap_{j \in I} \Psi(V_j) \cap Y_{S(x)}.$$

Hence $T_x(H \cap Y_x)$ is a union of elements in $\gamma \cap Y_{S(x)}$. This proves (A) by taking differences of appropriate sets.

It is left to show (B). Clearly, since $\operatorname{diam}(G) < \delta$ for every $G \in \gamma$ and since γ has the Markov property, we have

$$\operatorname{diam}\Big(\bigcap_{j=0}^{n-1} T^{-j}(G_{i_j}) \cap Y_x\Big) = \operatorname{diam}(T^{-n+1}(G_{i_{n-1}}) \cap Y_x) \le \lambda^{n-1}\delta. \ \blacksquare$$

3. Conditional entropy. Let (Y, T) be a dynamical system which is fibred over the base (X, S). We fix a T-invariant measure μ on Y and denote the conditional entropy of measurable partitions ξ and η by $H(\xi \mid \eta)$. We denote by ϵ_Y (resp. ϵ_X) the partition of Y (resp. X) into points. Let $\{\mu_x : x \in X\}$ denote the disintegration of μ with respect to $\pi^{-1}\epsilon_X$ and let $\nu = \mu \circ \pi^{-1}$.

The *relative entropy* $h(T \mid S)$ of the endomorphism T with respect to its factor S is defined by the expression

$$h(T \mid S) = \sup\{h(T \mid S, \xi) : \xi \text{ a measurable partition of } Y \text{ such that } H(\xi \mid \pi^{-1}\epsilon_X) < \infty\},$$

where

$$h(T \mid S, \xi) = \lim_{n\to\infty} H(T^{-n}\xi \mid T^{-(n+1)}\xi^- \vee \pi^{-1}\epsilon_X)$$

is called the *entropy of* ξ *relative to* $T \mid S$, and where $\xi^- = \bigvee_{n=0}^{\infty} T^{-n}\xi$. It is known ([7]) that

$$h(T \mid S, \xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi_0^{(n)} \mid \pi^{-1}\epsilon_X \vee T^{-n}\xi^-)$$

and

$$h(T \mid S, \xi) = \lim_{n\to\infty} \frac{1}{n} H(\xi_0^{(n)} \mid \pi^{-1}\epsilon_X). \tag{1}$$

COROLLARY. *Let (Y, T) be fibre expanding. For every T-invariant measure μ on Y with disintegration μ_x on Y_x we have*

$$h_\nu(T \mid S) = \lim_{n\to\infty} \frac{1}{n} \int_X H_{\mu_x}(\gamma_0^{n-1} \cap Y_x)\,\nu(dx),$$

where γ denotes the Markov partition of Section 2.

Proof. Since
$$\pi^{-1}(\epsilon_X) \vee \bigvee_{n\ge 0} T^{-n}\gamma = \epsilon_Y,$$
γ is a unilateral relative generator for T and $\pi^{-1}\epsilon_X$. By Proposition 3.9 in [7],
$$h(T\,|\,S,\gamma) = h(T\,|\,S) = \lim_{n\to\infty} H(T^{-n}\epsilon_Y\,|\,T^{-(n+1)}\epsilon_Y \vee \pi^{-1}\epsilon_X).$$
The corollary follows from (1). ∎

From the corollary we immediately obtain the following version of the Abramov–Rokhlin formula for the entropies $h(T)$ of the transformation T with respect to the invariant measure μ and the entropy $h(S)$ of S with respect to the image measure ν:
$$h(T) - h(S) = \lim_{n\to\infty} \frac{1}{n} \int_X H_{\mu_x}(\gamma_0^{n-1} \cap Y_x)\,\nu(dx).$$

4. Polynomial endomorphisms of $\mathbb{C}^2$. Let $\widehat{T}$ denote a polynomial mapping of $\mathbb{C}^2$. Such a mapping can be written in the form
$$\widehat{T}(x,y) = (p(x,y), q(x,y)),$$
where p and q are polynomials in $x, y \in \mathbb{C}$. It is called (d,d')-*regular*, where $d \in \mathbb{Q}$ and $d' \in \mathbb{N}$, if there are constants $k_1, k_2 > 0$ and $r \ge 0$ such that for every $z \in \mathbb{C}^2$, $\|z\| \ge r$,
$$k_1\|z\|^d \le \|\widehat{T}(z)\| \le k_2\|z\|^{d'}.$$
In the case $d = d'$, $\widehat{T}$ is called *strict*. A special case are *skew products* when p does not depend on y. Then $\pi \circ \widehat{T} = p \circ \pi$ where $\pi : \mathbb{C}^2 \to \mathbb{C}$ denotes the projection map onto the first factor.

A point $z \in \mathbb{C}^2$ is called *weakly normal* if there exists an open neighbourhood V of z and a family $\{\mathcal{K}_x : x \in V\}$ of at least one-dimensional complex analytic sets $\mathcal{K}_x$ such that $x \in \mathcal{K}_x$ and the family $\{\widehat{T}^n|_{\mathcal{K}_x} :\ n \ge 0\}$ is normal in x. The complement of the set of normal points is called the *Julia set* of $\widehat{T}$ and is denoted by $J(\widehat{T})$. It is shown in [9] that, for regular polynomial mappings, $J(\widehat{T})$ is compact and fully invariant. In particular, it follows that a regular skew product restricted to $J(\widehat{T})$ is a fibred system, but not a skew product in general. It is worth mentioning that $J_x = \pi^{-1}(x) \cap J(\widehat{T})$ is the fibre over x, and (for certain maps $\widehat{T}$ at least) is the Julia set of $\widehat{T}_x^n$ in case x is periodic with period n.

Let $\widehat{T} : \mathbb{C}^2 \to \mathbb{C}^2$ be a skew product and $T = \widehat{T}|_{J(\widehat{T})}$ be its restriction to the Julia set $J = J(\widehat{T})$. Denote by J_x^* the Julia set for the family of maps
$$q_{p^n(x)} \circ q_{p^{n-1}(x)} \circ \cdots \circ q_x \qquad (n \ge 0).$$

Then $J = \bigcup_{x \in J(p)} \{x\} \times J_x^*$ if each q_x for $x \in J(p)$ is hyperbolic (see [10]). If, in addition, p is a hyperbolic polynomial, then T is hyperbolic as well (on J) (see [9] and Theorem 2.3.1 in [10]). In particular, these maps are fibrewise expanding. Hence we infer from our Theorem that T has a fibrewise Markov partition.

The same result can be proved for hyperbolic rational semigroups (see e.g. [21] for a definition).

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Received 3 November 2006;
revised 10 April 2007 (4815)