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MULTIPARAMETER ERGODIC CESÀRO-α AVERAGES

BY

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Abstract. Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space. Associated with k Lamperti operators on $L^p(\nu)$, $T_1, \ldots, T_k$, $\bar{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $0 < \alpha_j \le 1$, we define the ergodic Cesàro-$\bar{\alpha}$ averages

$$\mathcal{R}_{\bar{n},\bar{\alpha}} f = \frac{1}{\prod_{j=1}^k A_{n_j}^{\alpha_j}} \sum_{i_k=0}^{n_k} \cdots \sum_{i_1=0}^{n_1} \prod_{j=1}^{k} A_{n_j-i_j}^{\alpha_j-1} T_k^{i_k} \cdots T_1^{i_1} f.$$

For these averages we prove the almost everywhere convergence on X and the convergence in the $L^p(\nu)$ norm, when $n_1, \ldots, n_k \to \infty$ independently, for all $f \in L^p(d\nu)$ with $p > 1/\alpha_*$ where $\alpha_* = \min_{1\le j\le k} \alpha_j$. In the limit case $p = 1/\alpha_*$, we prove that the averages $\mathcal{R}_{\bar{n},\bar{\alpha}} f$ converge almost everywhere on X for all f in the Orlicz–Lorentz space $\Lambda(1/\alpha_*, \varphi_{m-1})$ with $\varphi_m(t) = t(1 + \log^+ t)^m$. To obtain the result in the limit case we need to study inequalities for the composition of operators T_i that are of restricted weak type (p_i, p_i). As another application of these inequalities we also study the strong Cesàro-$\bar{\alpha}$ continuity of functions.

1. Introduction. Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space and T a bounded linear operator on $L^p(\nu)$. The operator T is called a *Lamperti operator* on $L^p(\nu)$ if it preserves disjointness of supports. It is known that Lamperti operators include L^p isometries, $p \ne 2$, positive L^2 isometries and invertible linear operators T such that both T and T^{-1} are positive (see e.g. [11], [12] and [13]). It follows from the results in [11] that if T is a Lamperti operator on $L^p(\nu)$, $1 < p < \infty$, power bounded, i.e., $\|T^n\|_p \le K < \infty$ for $n = 0, 1, \ldots$ and such that the adjoint T^* of T separates supports, then the ergodic averages

$$R_n f = \frac{1}{n+1} \sum_{k=0}^{n} T^k f$$

converge almost everywhere and in the $L^p(\nu)$ norm for all $f \in L^p(\nu)$. Under different assumptions on T, the same result was obtained in [14] (see also [16]). In these articles the authors considered an invertible Lamperti

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operator T on $L^p(\nu)$, $1 < p < \infty$, such that its linear modulus $|T|$ is Cesàro bounded, that is,

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |T|^k \right\|_p < \infty.$$

The linear modulus $|T|$ of a Lamperti operator T on $L^p(\nu)$ is also a Lamperti operator and it satisfies $|Tf| = |T|\,|f|$ (see [11] for more details). For positive contractions (not necessarily invertible) this result was obtained by Akcoglu [1]. For contractions in $L^1(\nu)$ and in $L^\infty(\nu)$ the result is due to Dunford and Schwartz [7].

The convergence of $\{R_n f\}$ is the convergence in the Cesàro-1 sense of the sequence $\{T^n f\}$. In general, we say that the sequence $\{T^n f\}$ *converges in the Cesàro-α sense*, with $0 < \alpha \leq 1$, if the limits of the Cesàro-α averages

$$R_{n,\alpha} f = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k f$$

exist, where $A_n^\alpha = (\alpha+1)\cdots(\alpha+n)/n!$ if $n \neq 0$ and $A_0^\alpha = 1$. Convergence in the Cesàro-α sense with $0 < \alpha < 1$ is stronger than convergence in the Cesàro-1 sense. The following result for the averages $R_{n,\alpha} f$ was obtained in [4].

Theorem 1.1 ([4]). *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space, $0 < \alpha \leq 1$, $1/\alpha < p < \infty$, and let T be an invertible Lamperti operator on $L^p(\nu)$ such that*

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |T|_\alpha^k \right\|_{p\alpha} < \infty,$$

where $|T|_\alpha f = [|T|(f^\alpha)]^{1/\alpha}$ for $f \geq 0$. Then

(i) *the ergodic Cesàro-α maximal operator $M_\alpha f = \sup_{n\geq 0} |R_{n,\alpha} f|$ is bounded on $L^p(\nu)$,*
(ii) *the set $D = \{g + (h - Th) : g, h \in L^p(\nu),\ g = Tg$ and h simple$\}$ is a dense subset of $L^p(\nu)$,*
(iii) *the averages $R_{n,\alpha} f$ converge almost everywhere and in the $L^p(\nu)$ norm for all $f \in L^p(\nu)$.*

The above theorem was proved in [15] under the additional assumption that T and its inverse T^{-1} are positive operators. For positive contractions this result was obtained by Irmisch [9]. If T is not invertible but satisfies the hypothesis of Kan [11] then T is controlled by a positive Lamperti contraction on $L^p(\nu)$ (see [11, Corollary 4.1]) and, from Irmisch's result, we can obtain the following result whose proof will be outlined in Section 3.

THEOREM 1.2. *Let* $(X, \mathcal{F}, \nu)$ *be a* σ*-finite measure space,* $0 < \alpha \leq 1$, $1/\alpha < p < \infty$, *and let* T *be a power bounded Lamperti operator on* $L^p(\nu)$ *such that* T^* *separates supports. Then* (i)–(iii) *of Theorem* 1.1 *hold.*

For the limit case $p = 1/\alpha$ and if $Tf(x) = f(\tau x)$, where τ is a measure preserving transformation, Broise, Déniel and Derriennic [6] observed that it is possible to obtain a weak type inequality with the Lebesgue space $L^{1/\alpha}(\nu)$ replaced by the Lorentz space $L(1/\alpha, 1)(\nu)$ (see the definition below). In the setting of invertible Lamperti operators the corresponding result was established in [3]:

THEOREM 1.3 ([3]). *Let* $(X, \mathcal{F}, \nu)$ *be a* σ*-finite measure space,* $0 < \alpha \leq 1$, *and let* T *be an invertible Lamperti operator on* $L^{1/\alpha}(\nu)$ *such that*

$$\sup_{n \geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |T|_{\alpha}^{k} \right\|_1 < \infty \quad \textit{and} \quad \sup_{n \in \mathbb{Z}} \|T^n\|_\infty < \infty.$$

Then

(i) *the maximal operator* M_α *is of restricted weak type* $(1/\alpha, 1/\alpha)$, *that is,* M_α *maps the Lorentz space* $L(1/\alpha, 1)(\nu)$ *into weak-*$L^{1/\alpha}(\nu)$,
(ii) *for all* f *in* $L(1/\alpha, 1)(\nu)$, *the averages* $R_{n,\alpha} f$ *converge almost everywhere.*

Now, given k linear operators $T_1, \ldots, T_k$, we define the ergodic averages

$$\mathcal{R}_{\bar{n}} f(x) = \frac{1}{\prod_{j=1}^{k}(n_j+1)} \sum_{i_k=0}^{n_k} \cdots \sum_{i_1=0}^{n_1} T_k^{i_k} \cdots T_1^{i_1} f(x),$$

where $\bar{n} = (n_1, \ldots, n_k)$. If each T_i is a contraction of both $L^1(\nu)$ and $L^\infty(\nu)$ (such operators are called *Dunford–Schwartz operators*) then, for all f in $L^p(\nu)$ with $1 < p < \infty$, the averages $\mathcal{R}_{\bar{n}} f$ converge (when $n_1, \ldots, n_k \to \infty$ independently) almost everywhere and in the $L^p(\nu)$ norm (see [7]). The limit case $p = 1$ was studied in [8]. In fact, N. Fava [8] proved that if each T_i is positive and is a contraction of both $L^1(\nu)$ and $L^\infty(\nu)$, then the averages $\mathcal{R}_{\bar{n}} f$ converge almost everywhere, as $n_1, \ldots, n_k \to \infty$ independently, for every f in the Orlicz space $L^{\varphi_{k-1}}(\nu)$ associated to the Young function $\varphi_k(t) = t(1 + \log^+ t)^k$ (see the definition of Orlicz space below).

Given $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1, \ldots, \alpha_k \in (0, 1]$, we define the ergodic Cesàro-$\bar{\alpha}$ averages by

$$\begin{aligned}\mathcal{R}_{\bar{n},\bar{\alpha}} f(x) &= R_{n_k,\alpha_k} \circ \cdots \circ R_{n_1,\alpha_1} f(x)\\ &= \frac{1}{\prod_{j=1}^{k} A_{n_j}^{\alpha_j}} \sum_{i_k=0}^{n_k} \cdots \sum_{i_1=0}^{n_1} \prod_{j=1}^{k} A_{n_j - i_j}^{\alpha_j - 1} T_k^{i_k} \cdots T_1^{i_1} f(x).\end{aligned}$$

It is clear that $\mathcal{R}_{\bar{n}} = \mathcal{R}_{\bar{n},\bar{\alpha}}$ when $\bar{\alpha} = (1, \ldots, 1)$. The main purpose of this paper is to extend the results of [7] and [8] to the averages $\mathcal{R}_{\bar{n},\bar{\alpha}} f$ associated to Lamperti operators T_i that satisfy the conditions in Theorems 1.1–1.3. In order to state them, we introduce some function spaces.

Given a Young function $\varphi : [0, \infty) \to [0, \infty)$, that is, φ is a nondecreasing, continuous and convex function such that $\varphi(0) = 0$, $\varphi(t) > 0$ if $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$, and given $p \geq 1$, the *Orlicz–Lorentz space* $\Lambda(p, \varphi)$ is defined by

$$\Lambda(p,\varphi) := \Big\{ f \in \mathcal{F} : \Psi_{p,\varphi}(cf) = \int_0^\infty \varphi(cf^*(t)) t^{1/p-1}\, dt < \infty \text{ for some } c > 0 \Big\},$$

where $f^*(t) = \inf\{s : \lambda_f(s) \leq t\}$ is the nonincreasing rearrangement function of f and $\lambda_f(s) = \nu(\{x \in X : |f(x)| > s\})$ is its distribution function. This space $\Lambda(p, \varphi)$ is a Banach space with the Luxemburg norm defined by

$$\|f\|_{p,\varphi} = \inf\{c > 0 : \Psi_{p,\varphi}(f/c) \leq 1\}.$$

The *Orlicz space* L^φ is $\Lambda(1, \varphi)$, and if $\varphi(t) = t$ then the space $\Lambda(p, \varphi)$ is the *Lorentz space* $L(p, 1)$.

Now we state our results.

THEOREM 1.4. *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space and let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, k$, $\alpha_* = \min_{1\leq j\leq k} \alpha_j$ and $p > 1/\alpha_*$. For $1 \leq j \leq k$, let T_j be a Lamperti operator on $L^p(\nu)$ with one of the following properties:*

(a) *T_j is an invertible operator and*

$$\sup_{n\geq 0} \Big\| \frac{1}{n+1} \sum_{k=0}^{n} |T_j|^k_{\alpha_j} \Big\|_{p\alpha_j} < \infty,$$

(b) *T_j is power bounded and T_j^* separates supports.*

Then:

(i) *the ergodic Cesàro-$\bar{\alpha}$ maximal operator $\mathcal{M}_{\bar{\alpha}} f(x) = \sup_{\bar{n}>0} |\mathcal{R}_{\bar{n},\bar{\alpha}} f(x)|$ is bounded on $L^p(\nu)$, where $\bar{n} = (n_1, \ldots, n_k) > 0$ means $n_j > 0$ for all $1 \leq j \leq k$,*

(ii) *the ergodic Cesàro-$\bar{\alpha}$ averages $\mathcal{R}_{\bar{n},\bar{\alpha}} f$ converge almost everywhere on X when $\bar{n} \to \infty$ ($n_1, \ldots, n_k \to \infty$ independently) and in the $L^p(\nu)$ norm for all $f \in L^p(\nu)$.*

THEOREM 1.5. *Let $(X, \mathcal{F}, \nu)$ be a σ-finite measure space and let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $0 < \alpha_j \leq 1$, $j = 1, \ldots, k$. Let $\alpha_* = \min_{1\leq j\leq k} \alpha_j$ and suppose that the minimum α_* is reached at exactly m numbers α_j. For $1 \leq j \leq k$, let T_j be an invertible Lamperti operator on $L^{1/\alpha_j}(\nu)$ such* *that*

$$\sup_{n\geq 0}\left\|\frac{1}{n+1}\sum_{k=0}^{n}|T_j|_{\alpha_j}^k\right\|_1 < \infty \quad \textit{and} \quad \sup_{n\in\mathbb{Z}}\|T_j^n\|_\infty < \infty.$$

Assume also that the operators T_j all commute with one another. Then:

(i) *the operator $\mathcal{M}_{\bar\alpha}$ satisfies*

$$\nu(\{x\in X : \mathcal{M}_{\bar\alpha}f(x) > t\}) \leq \Big(C\varphi_{m-1}(1/t)\int_0^\infty \varphi_{m-1}(f^*)(s)s^{\alpha_*-1}\,ds\Big)^{1/\alpha_*},$$

where $\varphi_m(t) = t(1+\log^+ t)^m$,

(ii) *the averages $\mathcal{R}_{\bar n,\bar\alpha}f$ converge almost everywhere on X for all f in the Orlicz–Lorentz space $\Lambda(1/\alpha_*, \varphi_{m-1})$ when $\bar n\to\infty$.*

Note that

$$\mathcal{M}_{\bar\alpha}f(x) \leq M_{\alpha_k}\circ\cdots\circ M_{\alpha_1}f(x), \tag{1.1}$$

where the operators M_{α_i} are the ergodic Cesàro-α_i maximal operators associated with the linear operators T_i. Under the hypothesis on the operators T_i given in Theorem 1.5, we find that each maximal operator M_{α_i} is bounded on $L^\infty(\nu)$ and is of restricted weak type (p_i, p_i) with $p_i = 1/\alpha_i$. In order to prove Theorem 1.5 the main tool is the study of the boundedness of the composition of this type of operators, which will be accomplished in Section 2. Section 3 is devoted to proving Theorems 1.2, 1.4 and 1.5. Finally, in Section 4, we study the strong Cesàro-$\bar\alpha$ continuity as a consequence of the theorems in Section 2.

Throughout this paper we will denote by C a nonnegative constant that can be different at each occurrence.

2. Composition operators. Let $(X,\mathcal{F},\nu)$ be a σ-finite measure space. In this section we shall deal with sublinear operators T_i, $i=1,\dots,k$, defined on measurable functions so that all of them are of strong type (∞,∞) and each of the operators T_i is of restricted weak type (p_i,p_i) with $1\leq p_i<\infty$. We say that an operator T is of *strong type* (∞,∞) if there exists a constant $C>0$ such that for any measurable function f,

$$\|Tf\|_\infty \leq C\|f\|_\infty.$$

We say that T is of *restricted weak type* (p,p), $1\leq p<\infty$, if there exists a constant $C>0$ such that

$$\|Tf\|_{p,\infty} \leq C\|f\|_{p,1},$$

where $\|\cdot\|_{p,q}$, $1 \le p, q \le \infty$, is the quasi-norm in the Lorentz space $L(p,q)$ defined for $q = \infty$ by

$$\|f\|_{p,\infty} = \sup_{t>0} t(\lambda_f(t))^{1/p} = \sup_{t>0} t^{1/p} f^*(t)$$

and for $1 \le p, q < \infty$ by

$$\|f\|_{p,q} = \left[\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right]^{1/q} \approx \left[\int_0^\infty (\lambda_f(s)^{1/p} s)^q \frac{ds}{s}\right]^{1/q}. \tag{2.1}$$

It is well known that a sublinear operator T is of strong type (∞,∞) and of restricted weak type (p,p) if and only if

$$\lambda_{Tf}(t) \le \left(\frac{C}{t} \int_{t/C}^\infty \lambda_f(s)^{1/p}\, ds\right)^p \quad \text{for all } t > 0, \tag{2.2}$$

where C is a positive constant independent of f and t (see for example [5] or [19, p. 91] for $p = 1$).

Now we state the main result of this section.

THEOREM 2.1. *Let $1 \le p_1 \le \cdots \le p_k$ be real numbers such that either they are all equal or there exists an integer ℓ with $0 \le \ell < k-1$ such that $1 \le p_1 \le \cdots \le p_{k-\ell-1} < p_{k-\ell} = \cdots = p_k$. For each $i = 1, \ldots, k$, let T_i be a sublinear operator of strong type (∞,∞) and of restricted weak type (p_i, p_i). Then $T = T_1 \circ \cdots \circ T_k$ satisfies the inequality*

$$\nu(\{x \in X : |Tf(x)| > t\}) \le C\Big(\varphi_\ell(1/t) \int_0^\infty \varphi_\ell(f^*(s)) s^{1/p_k - 1}\, ds\Big)^{p_k}$$

for all $t > 0$, where $\varphi_\ell(t) = t(1 + \log^+ t)^\ell$ with $\log^+ u = \max\{0, \log u\}$.

The above result generalizes the following result obtained in [5], where all the operators T_i are of strong type (∞,∞), and for a given $p \ge 1$, of restricted weak type (p,p).

THEOREM 2.2 ([5]). *Let T_i, $i = 1, \ldots, j$, be sublinear operators such that all of them are of strong type (∞,∞), and for a given $p \ge 1$, of restricted weak type (p,p). Then $T = T_1 \circ \cdots \circ T_j$ satisfies*

$$\nu(\{x : |Tf(x)| > t\}) \le \left(\frac{C}{(j-1)!\, t} \int_t^\infty [\lambda_f(s/C^j)]^{1/p} [\log(s/t)]^{j-1}\, ds\right)^p \tag{2.3}$$

for all $t > 0$, where the nonnegative constant C depends only on the boundedness constants of the operators T_i.

First of all we study a particular case of Theorem 2.1, when the parameter p_k is greater than the other, i.e., $\ell = 0$.

THEOREM 2.3. *Let $1 \le p_1 \le \cdots \le p_{k-1} < p_k$ be real numbers. For each $i = 1, \ldots, k$, let T_i be a sublinear operator of strong type (∞, ∞) and of restricted weak type (p_i, p_i). Then $T = T_1 \circ \cdots \circ T_k$ is of strong type (∞, ∞) and of restricted weak type (p_k, p_k).*

Proof. It is easy to see that it is sufficient to prove the theorem for only two operators. So consider two operators T_1 and T_2 that satisfy inequality (2.2) with $p = p_1$ and $p = p_2$ respectively, and assume that $p_1 < p_2$. Moreover suppose that the constant in (2.2) is the same in both cases. Then $T = T_1 \circ T_2$ satisfies (2.2) with $p = p_2$. In fact, using Minkowski's integral inequality and by Theorem 2.2 we get

$$\begin{aligned}\lambda_{T_1 \circ T_2 f}(t) &\le \bigg[\frac{C}{t}\int_{t/C}^{\infty}\bigg(\frac{C}{s}\int_{s/C}^{\infty}\lambda_f(u)^{1/p_2}\,du\bigg)^{p_2/p_1} ds\bigg]^{p_1} \\ &\le C\Big(\frac{1}{t}\Big)^{p_1}\bigg[\int_{t/C^2}^{\infty}\lambda_f(u)^{1/p_2}\Big(\int_{t/C}^{\infty}(1/s)^{p_2/p_1}\,ds\Big)^{p_1/p_2} du\bigg]^{p_2} \\ &\le \bigg(\frac{\tilde{C}}{t}\int_{t/\tilde{C}}^{\infty}\lambda_f(s)^{1/p_2}\,ds\bigg)^{p_2}\end{aligned}$$

for some constant $\tilde{C}$. ■

Proof of Theorem 2.1. We begin the proof by showing that if S_1 is an operator that satisfies (2.2) with $p = p_1$, and S_2 is an operator that satisfies (2.3) for any $j \ge 2$ and $p = p_2$ with $1 \le p_1 < p_2$, then $S = S_1 \circ S_2$ satisfies (2.3) with the same parameters j and p_2. In fact,

$$\begin{aligned}\lambda_{S_1 \circ S_2 f}(t) &\le \bigg(\frac{C}{t}\int_{t/C}^{\infty}\lambda_{S_2 f}(s)^{1/p_1}\,ds\bigg)^{p_1} \\ &\le \bigg[\frac{C}{t}\int_{t/C}^{\infty}\bigg(\frac{1}{(j-1)!s}\int_{s}^{\infty}\lambda_f(u/C^j)^{1/p_2}[\log(u/s)]^{j-1}\,du\bigg)^{p_2/p_1} ds\bigg]^{p_1}\end{aligned}$$

and, by using Minkowski's integral inequality, we see that $\lambda_{S_1 \circ S_2 f}(t)$ is bounded by

$$\begin{aligned}&\Big(\frac{C}{t}\Big)^{p_1}\bigg[\frac{1}{(j-1)!}\int_{t/C}^{\infty}\lambda_f(u/C^j)^{1/p_2}[\log(uC/t)]^{j-1}\Big(\int_{t/C}^{\infty}s^{-p_2/p_1}\,ds\Big)^{p_1/p_2} du\bigg]^{p_2} \\ &\qquad\le \tilde{C}\bigg(\frac{1}{(j-1)!t}\int_{t}^{\infty}\lambda_f(u/\tilde{C}^j)^{1/p_2}[\log(u/t)]^{j-1}\,du\bigg)^{p_2}.\end{aligned}$$

Now, let $S = T_1 \circ \cdots \circ T_{k-\ell-1}$ and $\tilde{S} = T_{k-\ell} \circ \cdots \circ T_k$. By interpolation (see for example [17, Theorem 3.15]) we see that $T_{k-\ell-1}$ is of strong type (p, p), and consequently of restricted weak type (p,p), for all $p > p_{k-\ell-1}$. Let $\epsilon_{k-\ell-1} > 0$ be such that $p_{k-\ell-1} + \epsilon_{k-\ell-1} < p_{k-\ell}$. From Theorem 2.3 we see that S satisfies (2.2) with $p = p_{k-\ell-1} + \epsilon_{k-\ell-1}$. On the other hand, from Theorem 2.2 the operator $\tilde{S}$ satisfies (2.3) with $p = p_k$ and $j = \ell+1$. Applying the above results with $S_1 = S$ and $S_2 = \tilde{S}$ we get

$$\begin{aligned}\lambda_{Tf}(t) &\le \tilde{C}\biggl(\frac{1}{\ell! t}\int_t^\infty \lambda_f(u/\tilde{C}^{\ell+1})^{1/p_k}[\log(u/t)]^\ell\,du\biggr)^{p_k}\\ &\le \tilde{C}\biggl(\frac{1}{\ell! t}\int_0^\infty \lambda_f(u/\tilde{C}^{\ell+1})^{1/p_k}\phi_\ell(u/t)\,du\biggr)^{p_k}\\ &\le \tilde{C}\biggl(\frac{\phi_\ell(1/t)}{t}\int_0^\infty \lambda_f(u)^{1/p_k}\phi_\ell(u)\,du\biggr)^{p_k},\end{aligned}$$

where $\phi_\ell(t) = (1+\log^+ t)^\ell$. Since $\varphi_\ell(t) = t(1+\log^+ t)^\ell$, it is easy to see that $\phi_\ell(t) \le \varphi'_\ell(t) \le (\ell+1)\phi_\ell(t)$ for all $t > 0$. The theorem follows from the equivalence

$$\int_0^\infty s^{1/p-1}\varphi_\ell(f^*(s))\,ds \approx \int_0^\infty \phi_\ell(s)\lambda_f(s)^{1/p}\,ds,$$

whose proof follows the same ideas as the proof of the equivalence between the quasi-norms in (2.1). ∎

3. Proofs of Theorems 1.2, 1.4 and 1.5. We start by proving Theorem 1.2. In order to show the density result we shall need to use the following properties of the Cesàro numbers A_n^α, $\alpha > -1$ (see [20]):

(C1) $A_n^\alpha - A_{n-1}^\alpha = (\alpha/n)A_{n-1}^\alpha$ for all $n \ge 1$.
(C2) There exist positive constants C_1 and C_2 depending only on α such that, for all $n \ge 0$,

$$C_1(n+1)^\alpha \le A_n^\alpha \le C_2(n+1)^\alpha.$$

Proof of Theorem 1.2. (i) Given an operator T satisfying the hypothesis of the theorem, from [11, Corollary 4.1] we find that there exists a positive Lamperti contraction S on $L^p(\nu)$ such that

$$|T^n f| \le KS^n|f| \quad \text{for each } f \in L^p(\nu),\ n = 0,1,\ldots. \tag{3.1}$$

Hence $M_{\alpha,T}(f) \le KM_{\alpha,S}(|f|)$. Then, using Irmisch's result for positive contractions, we deduce that the operator $M_\alpha = M_{\alpha,T}$ is bounded on $L^p(\nu)$.

(ii) From [11] we get the norm convergence of the ergodic averages $R_n f = (n+1)^{-1}\sum_{k=0}^n T^k f$. Then, by [7, Corollary VIII.5.2], the set $D = \{g + (h - Th) : g,h \in L^p,\ g = Tg,\ h \text{ simple}\}$ is a dense subset of $L^p(\nu)$.

(iii) First, we note that $R_{n,\alpha}g = g$ for all g such that $g = Tg$. It remains to prove, for a simple function h, that $R_{n,\alpha}(h - Th)(x)$ converges for almost every $x \in X$. As in [9] (see also [15] or [4]), using the definition of the numbers A_n^α and property (C1), we get

$$\begin{aligned} R_{n,\alpha}(h - Th)(x) &= \sum_{i=0}^{n} \frac{A_{n-i}^{\alpha-1}}{A_n^\alpha}(T^i h(x) - T^{i+1}h(x)) \\ &= \frac{A_n^{\alpha-1}}{A_n^\alpha}h(x) - \frac{A_n^{\alpha-1}}{A_n^\alpha}Th(x) + \sum_{i=1}^{n} \frac{A_{n-i}^{\alpha-1}}{A_n^\alpha}T^i h(x) - \sum_{i=1}^{n} \frac{A_{n-i}^{\alpha-1}}{A_n^\alpha}T^{i+1}h(x) \\ &= \frac{A_n^{\alpha-1}}{A_n^\alpha}h(x) - \frac{1}{A_n^\alpha}T^{n+1}h(x) + \frac{1}{A_n^\alpha}\sum_{i=1}^{n}(A_{n-i}^{\alpha-1} - A_{n+1-i}^{\alpha-1})T^i h(x) \\ &= \frac{\alpha}{\alpha+n}h(x) - \frac{T^{n+1}h(x)}{A_n^\alpha} + \frac{1-\alpha}{A_n^\alpha}\sum_{i=1}^{n}\frac{A_{n-i}^{\alpha-1}}{n+1-i}T^i h(x) \\ &= A_n(x) + B_n(x) + C_n(x). \end{aligned}$$

Clearly, $\lim_{n\to\infty} A_n(x) = 0$ for a.e. x. Let

$$F_1(x) = \sum_{n=0}^{\infty} |B_n(x)|^p \quad \text{and} \quad F_2(x) = \sum_{n=0}^{\infty} |C_n(x)|^p.$$

Using property (C2) of the Cesàro numbers and (3.1), we get

$$\int_X |F_1(x)|\, d\nu \lesssim \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha p}} \big\|S^{n+1}|h|\big\|_p^p \lesssim \|h\|_p^p \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha p}} < \infty$$

and

$$\begin{aligned} \int_X |F_2(x)|\, d\nu &\lesssim \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha p}} \bigg\|\sum_{i=1}^{n} \frac{A_{n-i}^{\alpha-1}}{n+1-i}T^i h(x)\bigg\|_p^p \\ &\lesssim \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha p}} \bigg(\sum_{i=1}^{n} \frac{A_{n-i}^{\alpha-1}}{n+1-i}\big\|S^i|h|\big\|_p\bigg)^p \\ &\lesssim \|h\|_p^p \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha p}} \Big(\sum_{i=1}^{n}(n+1-i)^{\alpha-2}\Big)^p \\ &\lesssim \|h\|_p^p \bigg(\sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha p}}\bigg)\Big(\sum_{k=1}^{\infty} k^{\alpha-2}\Big)^p < \infty. \end{aligned}$$

Hence $\lim_{n\to\infty} B_n(x) = \lim_{n\to\infty} C_n(x) = 0$ for a.e. x. Finally, the Banach principle (see e.g. [2, p. 237] and [12, Th. 7.2, p. 64]) implies almost everywhere convergence on the whole space, and Lebesgue's dominated convergence theorem implies norm convergence. ∎

Proof of Theorem 1.4. (i) Let $p > 1/\alpha_*$. Since each Lamperti operator T_j satisfies the hypothesis of Theorem 1.1 or Theorem 1.2 with $\alpha = \alpha_j$ and

$p > 1/\alpha_* \geq 1/\alpha_j$, each ergodic maximal operator M_{α_j} is bounded on $L^p(\nu)$. Thus the boundedness on $L^p(\nu)$ of $\mathcal{M}_{\bar{\alpha}}$ follows trivially from (1.1).

(ii) First, we shall assume for each $j = 1, \ldots, k$, either

(a′) T_j is positive with positive inverse and satisfies hypothesis (a) of the theorem, or

(b′) T_j is a positive Lamperti contraction on $L^p(\nu)$.

As usual, we first study the pointwise convergence of the averages $\mathcal{R}_{\bar{n},\bar{\alpha}}$ in a dense subset of $L^p(\nu)$. As in [15, proof of Theorem 3.1] or by Irmisch's result we know that the sets

$$D_j = \{g + (h - T_j h) : g, h \in L^p,\ g = T_j g,\ h \text{ simple}\}$$

are dense subsets of $L^p(\nu)$ for all $j = 1, \ldots, k$. As in [7] we shall prove the convergence of the averages $\mathcal{R}_{\bar{n},\bar{\alpha}} f$ for all $f \in D_1$, using induction on the number of operators.

If $k = 1$, the result was proved in [15] if the operator satisfies (a′) or in [9] if it satisfies (b′). Now suppose that the result holds for $k - 1$ operators $T_2, \ldots, T_k$, where each T_j satisfies (a′) or (b′), i.e., for any $f \in D_1$, the limit $R_{n_k,\alpha_k} \circ \cdots \circ R_{n_2,\alpha_2} f(x)$ exists for almost every $x \in X$ when $n_k, \ldots, n_2 \to \infty$. For simplicity set $\tilde{n} = (n_2, \ldots, n_k)$, $\tilde{\alpha} = (\alpha_2, \ldots, \alpha_k)$ and $\mathcal{R}_{\tilde{n},\tilde{\alpha}} = R_{n_k,\alpha_k} \circ \cdots \circ R_{n_2,\alpha_2}$. Let $g \in L^p(\nu)$ be such that $T_1 g = g$. Then

$$\mathcal{R}_{\bar{n},\bar{\alpha}} g(x) = R_{n_k,\alpha_k} \circ \cdots \circ R_{n_2,\alpha_2} g(x) = \mathcal{R}_{\tilde{n},\tilde{\alpha}} g(x).$$

By the inductive hypothesis $\mathcal{R}_{\bar{n},\bar{\alpha}} g(x)$ converges for almost every $x \in X$ as $\bar{n} \to \infty$. It remains to prove, for a simple function h, that $\mathcal{R}_{\bar{n},\bar{\alpha}}(h - T_1 h)(x)$ converges for almost every $x \in X$. It is sufficient to study the convergence of $\mathcal{R}_{\bar{n},\bar{\alpha}}(\chi_A - T_1\chi_A)(x)$ with A a measurable subset of X with $0 < \nu(A) < \infty$. From [15, Proposition 3.2] or from Irmisch's result, we know that

$$\lim_{n_1 \to \infty} R_{n_1,\alpha_1}(\chi_A - T_1\chi_A)(x) = 0 \quad \text{a.e.}$$

Notice that the operators $\mathcal{R}_{\tilde{n},\tilde{\alpha}}$ are positive and $\sup_{\tilde{n}>0} |\mathcal{R}_{\tilde{n},\tilde{\alpha}} f| \in L^p(\nu)$ for $f \in L^p(\nu)$ with $p > 1/\alpha^*$, because

$$\sup_{\tilde{n}>0} |\mathcal{R}_{\tilde{n},\tilde{\alpha}}(f)| \leq M_{\alpha_k} \circ \cdots \circ M_{\alpha_2}(f).$$

Now, applying the inductive hypothesis and using a general reduction principle of Sucheston (see [18, Proposition 1.1]) we get

$$\mathcal{R}_{\bar{n},\bar{\alpha}}(\chi_A - T_1\chi_A)(x) = \mathcal{R}_{\tilde{n},\tilde{\alpha}}(R_{n_1,\alpha_1}(\chi_A - T_1\chi_A))(x) \to 0$$

as $\bar{n} \to \infty$. Then the Banach principle implies almost everywhere convergence on the whole space $L^p(\nu)$.

Now, let T_j, $j = 1, \ldots, k$, be as in the hypothesis of the theorem. By using Theorem 1.1(ii) or 1.2(ii) and repeating the induction argument, we only need to show that $\mathcal{R}_{\bar{n},\bar{\alpha}}(\chi_A - T_1\chi_A)(x)$ converges for a.e. x. Notice that the operator $\mathcal{R}_{\tilde{n},\tilde{\alpha}}$ can be dominated by the corresponding one associated to positive operators $T_2^+,\dots,T_k^+$, where T_j^+ can be the linear modulus of T_j or the associated positive contraction S_j (it is the linear modulus if T_j satisfies (a) and a positive contraction if T_j satisfies (b)); we denote this operator by $\mathcal{R}^+_{\tilde{n},\tilde{\alpha}}$. The operators T_j^+, $j=2,\dots,k$, satisfy hypothesis (a′) or (b′). Then, from the previous results we infer that $\mathcal{R}^+_{\tilde{n},\tilde{\alpha}}f$ converges a.e. for every $f\in L^p(\nu)$. Now, since

$$|\mathcal{R}_{\bar{n},\bar{\alpha}}(\chi_A-T_1\chi_A)|\le \mathcal{R}^+_{\tilde{n},\tilde{\alpha}}|R_{n_1,\alpha_1}(\chi_A-T_1\chi_A)|$$

and $\lim_{n_1\to\infty}|R_{n_1,\alpha_1}(\chi_A-T_1\chi_A)(x)|=0$ for a.e. x, we obtain the desired result by applying Sucheston's principle again. Finally, the Banach principle implies almost everywhere convergence on the whole space, and Lebesgue's dominated convergence theorem implies norm convergence. ■

Proof of Theorem 1.5. (i) As mentioned in the Introduction, the main tool to prove this theorem is Theorem 2.1. In fact, from the assumptions on T_j we see that each operator M_{α_j} is of restricted weak type $(1/\alpha_j,1/\alpha_j)$ and bounded in $L^\infty(\nu)$. Then, by Theorem 2.1 and inequality (1.1), we obtain the boundedness of $\mathcal{M}_{\bar{\alpha}}$.

(ii) In the proof of Theorem 1.3 (see [3, p. 235]) the authors showed that if a Lamperti operator satisfies the hypothesis of Theorem 1.1 then it also satisfies the hypothesis of Theorem 1.3. Applying this fact we deduce that by Theorem 1.4 the averages $\mathcal{R}_{\bar{n},\bar{\alpha}}f$ converge for all $f\in L^p$ with $p>1/\alpha_*$. Let $D=L^p(\nu)\cap\Lambda(1/\alpha_*,\varphi_{m-1})$ with $p>1/\alpha_*$. The set D is a dense subset of $\Lambda(1/\alpha_*,\varphi_{m-1})$ since the set of simple functions is. Thus we get the convergence of $\mathcal{R}_{\bar{n},\bar{\alpha}}f$ for almost every $x\in X$ and all $f\in D$, and (ii) follows. In fact, let $A_t(f)=\{x:\limsup_{\bar{n}\to\infty}|\mathcal{R}_{\bar{n},\bar{\alpha}}f(x)-f(x)|>t\}$ and let g be a simple function. Then

$$\begin{aligned}|A_t(f)|\le|A_{t/2}(f-g)|&\le 2|\{x:\mathcal{M}_{\bar{\alpha}}(f-g)(x)>t/4\}|\\&\le 2[\varphi_{m-1}(4/t)\Psi_{1/\alpha_*,\varphi_{m-1}}(f-g)]^{1/\alpha_*}.\end{aligned}$$

The desired result follows since given $f\in\Lambda(1/\alpha_*,\varphi_{m-1})$, for any $\epsilon,t>0$, we can choose a simple function g such that the last term above is less than ϵ. ■

4. Application to strong Cesàro-α continuity. For a function $f:\mathbb{R}^n\to\mathbb{R}$ we say that f is *Cesàro-α continuous* at x, for $\alpha>0$, if the Cesàro-α averages

$$P^\alpha_\epsilon f(x)=\frac{c(n,\alpha)}{|Q(x,\epsilon)|^{1+(\alpha-1)/n}}\int\limits_{Q(x,\epsilon)}f(y)d(y,\partial Q(x,\epsilon))^{\alpha-1}\,dy$$

converge to $f(x)$ as $\epsilon \to 0$, where $Q(x,\epsilon) = \prod_{i=1}^{n}[x_i - \epsilon, x_i + \epsilon]^n$, $\partial Q(x,\epsilon)$ is the border of $Q(x,\epsilon)$, $d(y, \partial Q(x,\epsilon)) = \min_{1\le i\le n}\{x_i+\epsilon-y_i, y_i-(x_i-\epsilon)\}$ is the distance in the infinity norm from y to the border of $Q(x,\epsilon)$, and the constant $c(n,\alpha)$ can be written in terms of the β function as $c(n,\alpha) = \frac{2^{\alpha-1}}{n\beta(\alpha,n)}$, where $\beta(m,n) = \int_0^1 (1-t)^{m-1} t^{n-1}\,dt$, $m,n \ge 0$. If $\alpha = 1$,

$$P_\epsilon^1 f(x) = \frac{1}{|Q(x,\epsilon)|} \int\limits_{Q(x,\epsilon)} f(y)\,dy, \tag{4.1}$$

and Lebesgue's differentiation theorem establishes that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then f is Cesàro-1 continuous at almost every x.

If in (4.1) we replace the cubes $Q(x,\epsilon)$ by rectangles with sides parallel to the axes, $R(x,\bar\epsilon) = [x_1-\epsilon_1, x_1+\epsilon_1] \times \cdots \times [x_n-\epsilon_n, x_n+\epsilon_n]$, the theorem of Jensen, Marcinkiewicz and Zygmund shows that if f belongs to the Orlicz space $L^\varphi(\mathbb{R}^n)$ with $\varphi(t) = t(1+\log^+ t)^{n-1}$, then the averages

$$\mathcal{P}_{\bar\epsilon}^1 f(x) = \frac{1}{|R(x,\bar\epsilon)|} \int\limits_{R(x,\bar\epsilon)} f(y)\,dy$$

converge to f at x for almost every x as $\bar\epsilon \to 0$, that is, $\epsilon_1 \to 0, \dots, \epsilon_n \to 0$ independently. In this case we say that f is strongly Cesàro-1 continuous at x. In general, for $\bar\alpha = (\alpha_1, \dots, \alpha_n)$, we say that f is *strongly Cesàro-$\bar\alpha$ continuous* at x if the averages

$$\mathcal{P}_{\bar\epsilon}^{\bar\alpha} f(x) = \frac{c(n,\bar\alpha)}{\prod_{i=1}^n |I(x_i,\epsilon_i)|^{\alpha_i}} \int\limits_{R(x,\bar\epsilon)} f(y) \prod_{i=1}^n d(y_i, \partial I(x_i,\epsilon_i))^{\alpha_i-1}\,dy$$

converge to f at almost every x as $\bar\epsilon \to 0$, where $I(x_i,\epsilon_i) = [x_i-\epsilon_i, x_i+\epsilon_i]$ and $c(n,\bar\alpha) = 2^{|\bar\alpha|-n}\prod_{j=1}^n \alpha_j$ with $|\bar\alpha| = \sum_{j=1}^n \alpha_j$.

We want to apply the results of Section 2 to the study of the convergence of the averages $\mathcal{P}_{\bar\epsilon}^{\bar\alpha} f$. We shall work with slightly more general averages. For $L \in \mathbb{N}$, we consider the space $\mathbb{R}^n$ factored into L blocks, $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_L}$. If $G_1 := \{j \in \mathbb{N} : 1 \le j \le n_1\}$ and $G_i := \{j \in \mathbb{N} : n_1 + \cdots + n_{i-1} + 1 \le j \le n_2 + \cdots + n_i\}$ for $i = 2, \dots, L$, then we denote by x^i the set of variables $(x_j : j \in G_i) \in \mathbb{R}^{n_i}$, $i = 1, \dots, L$. Given $\bar\alpha = (\alpha_1, \dots, \alpha_L)$ with $0 < \alpha_i \le 1$ and $\bar\epsilon = (\epsilon_1, \dots, \epsilon_L)$ with $\epsilon_i > 0$, we define the averages

$$\mathcal{P}_{\bar\epsilon}^{\bar\alpha} f(x) = \frac{c(\bar n,\bar\alpha)}{\prod_{i=1}^L |Q_i|^{1+(\alpha_i-1)/n_i}} \int\limits_{Q_1} \cdots \int\limits_{Q_L} f(y) \prod_{i=1}^L d(y^i, \partial Q_i)^{\alpha_i-1}\,dy^L \cdots dy^1,$$

where $Q_i = Q(x^i,\epsilon_i)$ are cubes in $\mathbb{R}^{n_i}$ and $c(\bar n,\bar\alpha) = \prod_{i=1}^L c(n_i,\alpha_i) = \prod_{i=1}^L \frac{2^{\alpha_i-1}}{n_i\beta(\alpha_i,n_i)}$. The purpose of this section is to prove the following result.

THEOREM 4.1. *Given $\bar{\alpha} = (\alpha_1, \dots, \alpha_L)$, let $\alpha_* = \min_{1\le i\le L} \alpha_i$ and assume that there are exactly m numbers α_i, with $1 \le m \le L$, such that $\alpha_* = \alpha_i$. Then*

$$\lim_{\bar{\epsilon}\to 0} \mathcal{P}_{\bar{\epsilon}}^{\bar{\alpha}} f(x) = f(x)$$

for almost every $x \in \mathbb{R}^n$, for all $f \in L^p(\mathbb{R}^n)$ with $p > 1/\alpha_*$ and for all $f \in \Lambda(1/\alpha_*, \varphi_{m-1})$ where $\varphi_k(t) = t(1+\log^+ t)^k$.

Proof. Given $x \in \mathbb{R}^n$, we denote $\tilde{x}_i = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^L)$. For $f : \mathbb{R}^n \to \mathbb{R}$, let $f_{\tilde{x}_i} : \mathbb{R}^{n_i} \to \mathbb{R}$ be given by $f_{\tilde{x}_i}(x^i) = f(x^1, \dots, x^i, \dots, x^L)$. For $0 < \gamma \le 1$ and $\delta > 0$, we define the averages

$$\begin{aligned} P_\delta^{i,\gamma} f(x) &= P_\delta^{\gamma}(f_{\tilde{x}_i})(x^i) \\ &= \frac{c(n_i,\gamma)}{|Q(x^i,\delta)|^{1+(\gamma-1)/n_i}} \int\limits_{Q(x^i,\delta)} |f_{\tilde{x}_i}(y^i)| d(y^i, \partial Q(x^i,\delta))^{\gamma-1}\, dy^i. \end{aligned}$$

Associated with these averages we define the maximal operators

$$M_\gamma^i f(x) = M_\gamma(f_{\tilde{x}_i})(x^i) = \sup_{\delta>0} P_\delta^{i,\gamma}|f|(x),$$

where $M_\gamma f(x) = \sup_{\delta>0} P_\delta^{\gamma}|f|(x)$.

From the results in [10], the operators M_γ^i are bounded on $L^p(\mathbb{R}^n)$ for all $p > 1/\gamma$ and are of restricted weak type $(1/\gamma, 1/\gamma)$. In fact, by Minkowski's integral inequality and the restricted weak type of M_γ we get

$$\begin{aligned} |\{x \in \mathbb{R}^n : M_\gamma^i f(x) > t\}| &= \int\limits_{\mathbb{R}^{n-n_i}} \int\limits_{\mathbb{R}^{n_i}} \chi_{\{y^i : M_\gamma f_{\tilde{x}_i}(y^i) > t\}}(x^i)\, dx^i\, d\tilde{x}_i \\ &\le \int\limits_{\mathbb{R}^{n-n_i}} \left(\frac{C}{t} \int\limits_0^\infty |\{x^i : |f_{\tilde{x}_i}(x^i)| > s\}|^\gamma\, ds \right)^{1/\gamma} d\tilde{x}_i \\ &\le \left(\frac{C}{t} \int\limits_0^\infty \Big(\int\limits_{\mathbb{R}^{n-n_i}} |\{x^i : |f(x)| > s\}|\, d\tilde{x}_i \Big)^\gamma ds \right)^{1/\gamma} \\ &= \left(\frac{C}{t} \int\limits_0^\infty |\{x \in \mathbb{R}^n : |f(x)| > s\}|^\gamma\, ds \right)^{1/\gamma}. \end{aligned}$$

Notice that $\mathbb{M}_{\bar{\alpha}}$, the maximal operator associated to the averages $\mathcal{P}_{\bar{\epsilon}}^{\bar{\alpha}} f$, satisfies the pointwise inequality

$$\mathbb{M}_{\bar{\alpha}} f(x) = \sup_{\bar{\epsilon}>0} \mathcal{P}_{\bar{\epsilon}}^{\bar{\alpha}}|f|(x) \le M_{\alpha_1}^1 \circ \cdots \circ M_{\alpha_L}^L f(x), \tag{4.2}$$

where $\bar{\epsilon} > 0$ means $\epsilon_i > 0$ for all $i = 1, \dots, L$. Then it is clear that the operator $M_{\bar{\alpha}}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p > 1/\alpha_*$. On the other hand, since it is possible to change the order of the operators $M_{\alpha_i}^i$ in (4.2), from the above results for each operators $M^i_{\alpha_i}$ and Theorem 2.1 we get

$$|\{x : \mathbb{M}_{\bar\alpha} f(x) > t\}| \le C\Big(\varphi_{m-1}(1/t)\int_0^\infty s^{\alpha_*-1}\varphi_{m-1}(f^*(s))\,ds\Big)^{1/\alpha_*}$$

for all $t > 0$.

Finally, following the same arguments as in the proofs of Theorems 1.4 and 1.5, using for example the set of continuous functions with compact support as a dense subset of $L^p(\mathbb{R}^n)$, we obtain the convergence of the averages $\mathcal{P}^{\bar\alpha}_{\bar\epsilon} f$ in the desired spaces. ■

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