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# NOTE ON BESSAGA–KLEE CLASSIFICATION

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**Abstract.** We collect several variants of the proof of the third case of the Bessaga–Klee relative classification of closed convex bodies in topological vector spaces. We were motivated by the fact that we have not found anywhere in the literature a complete correct proof. In particular, we point out an error in the proof given in the book of C. Bessaga and A. Pełczyński (1975). We further provide a simplified version of T. Dobrowolski's proof of the smooth classification of smooth convex bodies in Banach spaces which also works in the topological case.

**1. Introduction.** A well-known result due to Bessaga and Klee (see, for example, [2, Section III.6]) provides a classification of pairs $(X, U)$, where $X$ is a Hausdorff topological vector space and $U \subset X$ a closed convex body, up to homeomorphism. Let us recall this result.

Let $X$ be a Hausdorff topological vector space and $U \subset X$ a closed convex body (i.e., a closed convex set with nonempty interior). The *characteristic cone* of $U$ (denoted by $\operatorname{cc} U$) is the set of those $x \in X$ such that the half-line $a+[0,\infty)x$ is contained in $U$ for some $a \in U$. If $0 \in \operatorname{Int} U$, then $\operatorname{cc} U$ is exactly the zero set of the Minkowski functional of $U$ (see, e.g., [2, Section III.1]).

Then the classification is summed up in the following theorem:

THEOREM 1. *Let $X$ be a Hausdorff topological vector space and $U \subset X$ a closed convex body.*

(i) *If $\operatorname{cc} U$ is a linear subspace of finite codimension $m$, then the pair $(X, U)$ is homeomorphic to $(\operatorname{cc} U \times \mathbb{R}^m, \operatorname{cc} U \times [0,1]^m)$.*
(ii) *If $\operatorname{cc} U$ is a linear subspace of infinite codimension, then $(X, U)$ is homeomorphic to $(X, X^+)$, where $X^+$ is a closed half-space of $X$.*
(iii) *If $\operatorname{cc} U$ is not a linear subspace, then $(X, U)$ is also homeomorphic to $(X, X^+)$.*

We have studied this result at a student seminar using the book [2] and we encountered a difficulty in proving assertion (iii). On page 112 of that book a formula is given, illustrated by a picture and followed by the

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claim that 'it is not difficult to check' that this gives the required homeomorphism. After a certain effort we realized that this claim is not true—the formula need not provide a homeomorphism. This is explained in Section 3.2 below.

After finding the error we tried to correct it and to look for a correct proof in the literature. The original reference for the result is the paper [1]. However, this paper does not contain an explicit formulation of the theorem. The desired statement is a special case of the more general [1, Lemma 1.3]. And again, in its proof a formula is given and followed by the claim that 'it is tedious but not difficult to verify' that the formula gives the desired homeomorphism. In this case the claim is correct. In fact, the proof is not even too tedious. In Section 3.1 we describe this method, applied directly to the above theorem.

Before finding and analyzing the original paper we established a correction of the proof from [2]. This correction is described in Section 3.3. It is quite complicated, but we think it contains several interesting features. Later, after analyzing the original method we came to think that the error in [2] is probably due to a misprint. And this yields the second proof described in Section 3.4. The proof is a bit more complicated than the original one.

Finally, we found the paper [4] where an analogous classification of $C^p$-smooth convex bodies in Banach spaces up to $C^p$-diffeomorphism is given. As a special case $p = 0$ the homeomorphic classification is given. The proof of case (iii) takes only half a page. It refers to the implicit function theorem [3, Theorem 10.2.5]. However, the key parts of the proof are omitted (for example the proof that the relevant maps are bijections and that the Fréchet differential at each point is an onto isomorphism). Further, there is one small mistake in the definition of one of the important sets. In Section 4 below we give a proof using the method of [4] for the homeomorphism case. Under the additional smoothness assumptions the same proof provides the classification up to diffeomorphism. Further, our proof is more elementary, since it uses only a simple version of the implicit function theorem (see Theorem 5 below).

In view of the above, we decided to write down several variants of the proof because we think that such a result deserves it.

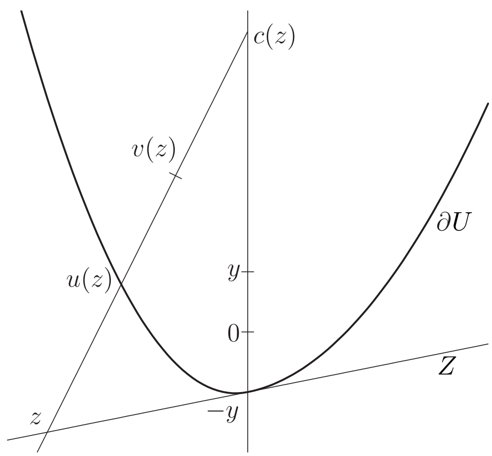
Let us fix some notation. We adopt the notation of [2]; the notation in the other two works is different.

If $U$ is a convex set containing 0 in its interior, we denote by $w_U$ the Minkowski functional of $U$. Further, $\operatorname{cs} U$ is the set of those $x \in U$ such that the line $a + \mathbb{R}x$ is contained in $U$ for some $a \in X$. In other words, $\operatorname{cs} U = \operatorname{cc} U \cap \operatorname{cc}(-U)$.

**2. The basic method of the proof.** We will review below several possibilities of proving assertion (iii) of Theorem 1. Not surprisingly, all the proofs follow the same pattern. Let us describe it.

Let $U \subset X$ be a closed convex body such that $\operatorname{cc} U$ is not a linear subspace. This means that there is $y \in \operatorname{cc} U$ such that $-y \notin \operatorname{cc} U$. Without loss of generality we may suppose that $0 \in \operatorname{Int} U$. Then $[0, \infty)y \subset U$, $(-\infty, 0]y \not\subset U$ and there is some $\varepsilon > 0$ such that $(-\varepsilon, 0]y \subset U$. Hence, without loss of generality we may suppose that $-y \in \partial U$. If we define a linear functional on $\mathbb{R}y$ by the formula $\psi_0(ty) = -t$, then $\psi_0(ty) \le w_U(ty)$ for each $t \in \mathbb{R}$. So, the Hahn–Banach theorem implies that there is a linear functional $\psi$ on $X$ extending $\psi_0$ such that $\psi(x) \le w_U(x)$ for each $x \in X$. Set $\varphi = -\psi$. Then $\varphi$ is a linear functional on $X$ such that $\varphi(-y) = -1$ and $\varphi(x) \ge -1$ for $x \in U$. In particular, $|\varphi(x)| \le 1$ on $U \cap (-U)$, so $\varphi$ is continuous. Set $Z = \{x \in X : \varphi(x) = -1\}$.

Now, a basic method of constructing a homeomorphism of $(X, U)$ onto $(X, \varphi^{-1}([-1, \infty))$ is the following: To any $z \in Z$ assign some $c(z) \in [-1, \infty)y$. Let $u(z)$ be the last point of the segment $[c(z), z]$ contained in $U$ and let $v(z)$ be a suitable point of the segment $(c(z), u(z))$. Next, we choose a self-homeomorphism $h_z$ of the half-line $c(z) + (0, \infty)(z - c(z))$ which is the identity on the segment $(c(z), v(z)]$, and the segment $[v(z), u(z)]$ is mapped onto $[v(z), z]$. Finally, we define the global homeomorphism $H$ to be $h_z$ on the relevant half-line and to be the identity at points not covered by the half-lines.




Then the proof that $H$ is indeed a homeomorphism requires three steps:

- $H$ is well-defined (i.e., the relevant half-lines do not intersect).
- $H$ is a self-homeomorphism of the union of the half-lines.
- $H$ remains a homeomorphism if glued with the identity.

The proofs appearing in the literature differ in the formula for $c(z)$, the choice of $v(z)$ and the definition of $h_z$.

An important part of the proof (namely of the second step) consists in using the following easy lemma.

Lemma 2. *Let $U \subset X$ be a closed convex body. Then the mapping*

$$(u, v) \mapsto w_{U-u}(v)$$

*is continuous on* $\operatorname{Int} U \times X$.

*Proof.* Let $c \in \mathbb{R}$. We will show that the sets

$$\{(u, v) \in \operatorname{Int} U \times X : w_{U-u}(v) < c\} \text{ and } \{(u, v) \in \operatorname{Int} U \times X : w_{U-u}(v) > c\}$$

are open.

If $c \leq 0$, then the first set is empty. For $c > 0$ the inequality $w_{U-u}(v) < c$ is equivalent to $v \in c \operatorname{Int}(U - u)$, so $v + cu \in \operatorname{Int} U$. It follows that in this case the first set is open.

The second set equals $\operatorname{Int} U \times X$ for $c < 0$. For $c = 0$ it equals $\operatorname{Int} U \times (X \setminus \operatorname{cc} U)$. Finally, for $c > 0$ the inequality $w_{U-u}(v) > c$ is equivalent to $v \notin c(U - u)$, i.e., $v + cu \in X \setminus cU$. In any case the second set is open as well. ∎

**3. Several variants of the proof.** In this section we collect several variants of the proof. We start by the original proof which is hidden in [1], then we continue by explaining why the proof in [2] is incorrect, and suggest two possible corrections.

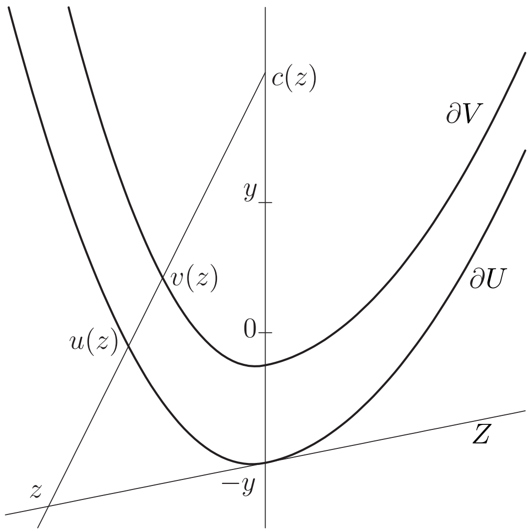
**3.1. The original proof.** As remarked above, the paper [1] in fact does not contain an explicit formulation of the theorem. But the result follows from a more general Lemma 1.3. Let us give the proof to see that it is really easy, if properly formulated.

Fix a closed convex body $V$ such that $[0, \infty)y \subset \operatorname{Int} V \subset V \subset \operatorname{Int} U$. For example, one can take $V = \frac{1}{2}U$ or $V = y/2 + U$. Set $W = V \cap (-V) \cap \operatorname{Ker} \varphi$. Then $W$ is a closed convex body in $\operatorname{Ker} \varphi$, and moreover $\operatorname{cc} W = \operatorname{cs} W = \operatorname{cs} V$.

For $z \in Z$ define $c(z) = w_W(z + y)y$ and let $v(z)$ be the last point of the segment $[c(z), z]$ contained in $V$. The homeomorphism $h_z$ is defined as the identity on the segment $(c(z), v(z)]$, on $[v(z), u(z)]$ it is the affine transformation sending this segment to $[v(z), z]$, and on the half-line $u(z) + (0, \infty)(z - c(z))$ it is a translation.

The proof that the glued mapping $H$ is a homeomorphism has three steps:

Step 1: The half-lines $c(z) + (0, \infty)(z - c(z))$, $z \in Z$, are pairwise disjoint and their union is $X \setminus (\operatorname{cs} V + [0, \infty)y)$.




Let $x \in X$. Let us find conditions under which there is $z \in Z$ such that

$$x \in c(z) + (0, \infty)(z - c(z)),$$

i.e., there are $z \in Z$ and $\alpha > 0$ such that

$$x = c(z) + \alpha(z - c(z)). \tag{3.1}$$

This equation is equivalent to

$$(x - \varphi(x)y) + \varphi(x)y = \alpha(z + y) + ((1 - \alpha)w_W(z + y) - \alpha)y. \tag{3.2}$$

Applying the functional $\varphi$ to both sides we get

$$x - \varphi(x)y = \alpha(z + y) \quad \& \quad \varphi(x) = (1 - \alpha)w_W(z + y) - \alpha. \tag{3.3}$$

More precisely, applying $\varphi$ to (3.2) we get the second equation and plugging it into (3.2) we get the first equation. If we plug $z + y = \frac{1}{\alpha}(x - \varphi(x)y)$ into the second equation, we get the quadratic equation

$$\alpha^2 + \alpha(\varphi(x) + w_W(x - \varphi(x)y)) - w_W(x - \varphi(x)y) = 0.$$

If $w_W(x - \varphi(x)y) > 0$, then this equation has one positive root and one negative root. Denote the positive root by $\alpha(x)$. If $w_W(x - \varphi(x)y) = 0$ and $\varphi(x) < 0$, then the equation has one root equal to zero and the other is $\alpha(x) = -\varphi(x) > 0$. If $w_W(x - \varphi(x)y) = 0$ and $\varphi(x) \geq 0$, the equation has no positive root.

Since the conditions $w_W(x - \varphi(x)y) = 0$ and $\varphi(x) \geq 0$ hold if and only if $x \in \operatorname{cs} V + [0, \infty)y$, we see that the $\alpha$ in (3.1) is always unique, and it follows from the first equation in (3.3) that the corresponding $z$ is also unique. We denote it by $z(x)$. This finishes the proof of Step 1. Moreover, the above calculation shows that the mappings $x \mapsto z(x)$ and $x \mapsto \alpha(x)$ are continuous on $X \setminus (\operatorname{cs} V + [0,\infty)y)$.

STEP 2: $H$ is a homeomorphism of $X \setminus (\operatorname{cs} V + [0,\infty)y)$ onto itself.

It is clear that $H$ is a bijection of $X \setminus (\operatorname{cs} V + [0,\infty)y)$ onto itself. So, it is enough to show that $H$ and $H^{-1}$ are continuous on $X \setminus (\operatorname{cs} V + [0,\infty)y)$. This can be done using Lemma 2 and continuity of $z(x)$ and $\alpha(x)$.

More precisely, let us define $F$, a function of four real variables, on the set

$$M = \{(\alpha,\beta,\gamma,\delta) \in (0,\infty)^4 : \gamma > \beta \ \&\ \delta > \beta\}$$

by the formula

$$F(\alpha,\beta,\gamma,\delta) = \begin{cases} \alpha, & 0 < \alpha \le \beta, \\ \beta + \frac{\delta-\beta}{\gamma-\beta}(\alpha-\beta), & \beta \le \alpha \le \gamma, \\ \alpha + \delta - \gamma, & \gamma \le \alpha. \end{cases}$$

This function is continuous on its domain, since all the three formulas are continuous, their domains are relatively closed and the formulas agree on the intersections of their domains.

Further,

$$\begin{aligned} H(x) &= c(z(x)) + F\big(\alpha(x), \tfrac{1}{w_{V-c(z(x))}(z(x)-c(z(x)))}, \tfrac{1}{w_{U-c(z(x))}(z(x)-c(z(x)))}, 1\big) \\ &\qquad\qquad \cdot (z(x) - c(z(x))), \\ H^{-1}(x) &= c(z(x)) + F\big(\alpha(x), \tfrac{1}{w_{V-c(z(x))}(z(x)-c(z(x)))}, 1, \tfrac{1}{w_{U-c(z(x))}(z(x)-c(z(x)))}\big) \\ &\qquad\qquad \cdot (z(x) - c(z(x))), \end{aligned}$$

so both $H$ and $H^{-1}$ are continuous.

STEP 3: $H$ is a homeomorphism of $X$ onto itself.

Since $\operatorname{cs} V + [0,\infty)y \subset \operatorname{cc} V \subset \operatorname{Int} V$ and $H$ is the identity on $V \setminus (\operatorname{cs} V + [0,\infty)y)$, the global continuity of $H$ and $H^{-1}$ follows.

REMARK 3. Lemma 1.3 of [1], mentioned above, is more general. It deals with homeomorphisms of triples, not pairs. To the set $V$ from [1] corresponds our set $U$, the sets $U$ and $P$ from [1] in our case both coincide with $V$. The 'tedious but not difficult' part skipped in [1] corresponds to our Steps 1 and 2. It is clear that the computation is not difficult, but especially Step 1 probably cannot be seen without a computation.

**3.2. The incorrect proof in [2].** On page 112 of the quoted book the authors suggest the formulas $c(z) = (w_U(z+y) - 1)y$ and $v(z) = \frac{1}{2}(u(z) + c(z))$. Further, $h_z$ is defined to be the identity on $(c(z), v(z)]$, and on the half-line $v(z) + [0,\infty)(z - c(z))$ it is an affine mapping fixing $v(z)$ and taking $u(z)$ to $z$.

We shall see that these formulas do not provide a homeomorphism. The problem is that if $z+y \in \operatorname{cc} U$, we get $c(z) = -y$. In that case $u(z)$ should be defined to be $z$, and already the mapping $z \mapsto u(z)$ may fail to be continuous.

Let us describe a counterexample. Set $X = \mathbb{R}^3$ and

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq (x_2)^+ - 1 \ \&\ x_1 \geq (x_3)^+ - 1\}.$$

Then $0 \in \operatorname{Int} U$ and one can choose $y = (1, 0, 0)$ and

$$Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = -1\}.$$

Let $x = (-1, -1, 0)$. Then $x \in Z$, $w_U(x + y) = 0$, hence $c(x) = -y$, $u(x) = x$ and so $H(x) = x$.

Further, for any $n \in \mathbb{N}$ let $x_n = (-1, -1, 1/n)$. Then $x_n \in Z$, $w_U(x_n + y) = 1/n$, hence $c(x_n) = (-1 + 1/n, 0, 0)$. Further, $u(x_n) = (-1 + 1/(2n), -1/2, 1/(2n))$ as this point belongs to the intersection of the boundary of $U$ with the segment $[c(x_n), x_n]$. Hence $v(x_n) = (-1 + 3/(4n), -1/4, 1/(4n))$ and

$$H(x_n) = v(x_n) + 3(x_n - v(x_n)) = \left(-1 - \frac{3}{2n}, -\frac{5}{2}, \frac{5}{2n}\right).$$

Since $x_n \to x$ and $H(x_n) \to (-1, -5/2, 0) \neq H(x)$, $H$ is not continuous.

**3.3. Correction of the proof: version 1.** In this section we present a possible correction of the proof from [2]. We change the formula for $c(z)$, preserving the remaining assumptions. Let us set $c(z) = (\sqrt{w_U(z + y)} - 1)y$.

In this case the equality $c(z) = -y$ remains possible, but the square root changes the order of convergence and makes the relevant mappings continuous. This version of the proof is the most complicated one but we find it interesting. So, let us give a proof.

Step 1. Set $Z' = \{z \in Z : w_U(z + y) > 0\}$. Then the half-lines $c(z) + (0, \infty)(z - c(z))$, $z \in Z'$, are disjoint and cover the set $\{x \in X : w_U(x - \varphi(x)y) > 0\}$.

Let $x \in X$. We will find conditions under which there are $z \in Z'$ and $\alpha > 0$ such that

$$x = c(z) + \alpha(z - c(z)).$$

This equation is equivalent to

$$(x - \varphi(x)y) + \varphi(x)y = \alpha(z + y) + ((1 - \alpha)(\sqrt{w_U(z + y)} - 1) - \alpha)y.$$

By applying $\varphi$ to both sides we get (as in "(3.2)⇒(3.3)" above)

$$(3.4)\quad x - \varphi(x)y = \alpha(z + y) \quad \& \quad \varphi(x) = (1 - \alpha)(\sqrt{w_U(z + y)} - 1) - \alpha.$$

From the first equation it follows that $w_U(x - \varphi(x)y) > 0$ if we want $z \in Z'$. Further, if we isolate $z + y$ from the first equation and plug the result into the second one, we get

$$\alpha\sqrt{w_U(x-\varphi(x)y)}+\sqrt{\alpha}(\varphi(x)+1)-\sqrt{w_U(x-\varphi(x)y)}=0.$$

This is a quadratic equation for $\sqrt{\alpha}$ with a unique positive root $\alpha=\alpha(x)$. Hence, by the first equation in (3.4), there is a unique $z=z(x)$.

This completes the proof of Step 1. Moreover, the computation shows that the mappings $x\mapsto\alpha(x)$ and $x\mapsto z(x)$ are continuous on $\{x\in X: w_U(x-\varphi(x)y)>0\}$.

STEP 2: *$H$ is a homeomorphism of $\{x\in X: w_U(x-\varphi(x)y)>0\}$ onto itself.*

It is clear that $H$ is a bijection of the given set onto itself. It remains to show that $H$ and $H^{-1}$ are continuous.

Let us define two functions on $\mathbb{R}\times(0,2)$ by

$$G_1(\alpha,\beta)=\begin{cases}\alpha, & \alpha\le\beta/2,\\ \frac{\beta}{2}+\frac{2-\beta}{\beta}\left(\alpha-\frac{\beta}{2}\right), & \alpha\ge\beta/2,\end{cases}$$

$$G_2(\alpha,\beta)=\begin{cases}\alpha, & \alpha\le\beta/2,\\ \frac{\beta}{2}+\frac{\beta}{2-\beta}\left(\alpha-\frac{\beta}{2}\right), & \alpha\ge\beta/2.\end{cases}$$

These functions are clearly continuous (the individual formulas are continuous, coincide on the intersection of the domains and the domains are relatively closed). Further, for $x\in\{x\in X: w_U(x-\varphi(x)y)>0\}$ we have

$$H(x)=c(z(x))+G_1\left(\alpha(x),\frac{1}{w_{U-c(z(x))}(z(x)-c(z(x)))}\right)(z(x)-c(z(x))),$$

$$H^{-1}(x)=c(z(x))+G_2\left(\alpha(x),\frac{1}{w_{U-c(z(x))}(z(x)-c(z(x)))}\right)(z(x)-c(z(x))).$$

It follows from Lemma 2 using the continuity of mappings $x\mapsto\alpha(x)$, $x\mapsto z(x)$ and $z\mapsto c(z)$ and the fact that $c(z(x))\in\operatorname{Int}U$ in this case that $H$ and $H^{-1}$ are continuous.

STEP 3: *$H$ is a homeomorphism of $X$ onto itself.*

On the set $\{x\in X: w_U(x-\varphi(x)y)=0\}$ the mapping $H$ is defined to be the identity. Since this set is closed, it is enough to show that whenever $x_\tau$ is a net in $\{x\in X: w_U(x-\varphi(x)y)>0\}$ such that $x_\tau\to x$ with $w_U(x-\varphi(x)y)=0$, then $H(x_\tau)\to x$ and $H^{-1}(x_\tau)\to x$.

So, let $(x_\tau)$ be such a net. Let us decompose the index set into two parts:

$$\Lambda_1=\left\{\tau: w_{U-c(z(x_\tau))}(z(x_\tau)-c(z(x_\tau)))\le\frac{1}{2\alpha(x_\tau)}\right\},$$

$$\Lambda_2=\left\{\tau: w_{U-c(z(x_\tau))}(z(x_\tau)-c(z(x_\tau)))>\frac{1}{2\alpha(x_\tau)}\right\}.$$

For $\tau \in \Lambda_1$ we have $H(x_\tau) = H^{-1}(x_\tau) = x_\tau$, so it remains to show that the limit along $\Lambda_2$ is also $x$, provided $\Lambda_2$ is cofinal. Without loss of generality we may assume that $\Lambda_1 = \emptyset$, i.e.

$$w_{U-c(z(x_\tau))}(z(x_\tau) - c(z(x_\tau))) > \frac{1}{2\alpha(x_\tau)} \qquad \text{for all } \tau.$$

Let us compute the limit of $c(z(x_\tau))$. We have

$$c(z(x_\tau)) = (\sqrt{w_U(z(x_\tau)+y)} - 1)y = \left(\sqrt{\frac{w_U(x_\tau - \varphi(x_\tau)y)}{\alpha(x_\tau)}} - 1\right)y.$$

Since

$$\begin{aligned}
&\lim_\tau \sqrt{\frac{w_U(x_\tau - \varphi(x_\tau)y)}{\alpha(x_\tau)}} \\
&\qquad = \lim_\tau \frac{\sqrt{w_U(x_\tau - \varphi(x_\tau)y)} \cdot 2\sqrt{w_U(x_\tau - \varphi(x_\tau)y)}}{-(\varphi(x_\tau)+1) + \sqrt{(\varphi(x_\tau)+1)^2 + 4w_U(x_\tau - \varphi(x_\tau)y)}} \\
&\qquad = \lim_\tau \tfrac{1}{2}\big((\varphi(x_\tau)+1) + \sqrt{(\varphi(x_\tau)+1)^2 + 4w_U(x_\tau - \varphi(x_\tau)y)}\big) \\
&\qquad = \tfrac{1}{2}\big((\varphi(x)+1) + \sqrt{(\varphi(x)+1)^2 + 4w_U(x - \varphi(x)y)}\big) \\
&\qquad = (\varphi(x)+1)^+,
\end{aligned}$$

we get $c(z(x_\tau)) \to ((\varphi(x)+1)^+ - 1)y = \max(\varphi(x), -1)y$.

If $\varphi(x) > -1$, then $c(z(x_\tau)) \to \varphi(x)y \in \operatorname{Int} U$, hence by Lemma 2 we get

$$w_{U-c(z(x_\tau))}(x_\tau - c(z(x_\tau))) \to w_{U-\varphi(x)y}(x - \varphi(x)y) = 0$$

and hence $w_{U-c(z(x_\tau))}(x_\tau - c(z(x_\tau))) < 1/2$ for large $\tau$. This means that for large $\tau$ we have $\tau \in \Lambda_1$, a contradiction.

Thus $\varphi(x) \le -1$. Then $c(z(x_\tau)) \to -y$. We will show that

$$w_{U-c(z(x_\tau))}(z(x_\tau) - c(z(x_\tau))) \to 1.$$

Suppose this is not the case. Since $w_{U-c(z)}(z - c(z)) \ge 1$ for each $z \in Z'$, up to passing to a subnet we may assume that there is some $d > 1$ such that

$$w_{U-c(z(x_\tau))}(z(x_\tau) - c(z(x_\tau))) > d \quad \text{for each } \tau.$$

This means that $z(x_\tau) - c(z(x_\tau)) \notin d(U - c(z(x_\tau)))$, hence

$$\frac{1}{d}z(x_\tau) + \left(1 - \frac{1}{d}\right)c(z(x_\tau)) \notin U \quad \text{for each } \tau.$$

So,

$$\begin{aligned}
&\frac{w_U(z(x_\tau)+y)}{d} \cdot \frac{z(x_\tau)+y}{w_U(z(x_\tau)+y)} + \left(1 - \frac{w_U(z(x_\tau)+y)}{d}\right) \cdot (-y) \\
&\qquad + \left(\left(1 - \frac{1}{d}\right)\sqrt{w_U(z(x_\tau)+y)} - \frac{w_U(z(x_\tau)+y)}{d}\right)y \notin U.
\end{aligned}$$

Since $w_U(z(x_\tau)+y) \to 0$, the sum of the first two terms is, for $\tau$ large enough, a convex combination of $\frac{z(x_\tau)+y}{w_U(z(x_\tau)+y)}$ and $-y$, hence it belongs to $U$. Further, the coefficient of the last term is positive for $\tau$ large enough (here the choice of the square root is essential), which yields a contradiction as $y \in \operatorname{cc} U$. Thus indeed $w_{U-c(z(x_\tau))}(z(x_\tau)-c(z(x_\tau))) \to 1$.

Now we are ready to conclude. To shorten the notation, set $\alpha_\tau = \alpha(x_\tau)$ and $w_\tau = w_{U-c(z(x_\tau))}(z(x_\tau)-c(z(x_\tau)))$. Since $\alpha_\tau > \frac{1}{2w_\tau}$, we have

$$\begin{aligned}
H(x_\tau) &= c(z(x_\tau)) + G_1\left(\alpha_\tau, \frac{1}{w_\tau}\right)(z(x_\tau)-c(z(x))) \\
&= c(z(x_\tau)) + \left(\alpha_\tau(2w_\tau-1) - 1 + \frac{1}{w_\tau}\right)(z(x_\tau)-c(z(x_\tau))) \\
&= c(z(x_\tau)) + \left(2w_\tau - 1 - \frac{1}{\alpha_\tau} + \frac{1}{\alpha_\tau w_\tau}\right)(x_\tau - c(z(x_\tau))) \\
&= 2c(z(x_\tau))(1-w_\tau) + x_\tau(2w_\tau-1) + \frac{1-w_\tau}{\alpha_\tau w_\tau}(x_\tau - c(z(x_\tau))) \to x
\end{aligned}$$

since $x_\tau \to x$, $c(z(x_\tau)) \to -y$, $w_\tau \to 1$ and $\alpha_\tau w_\tau > 1/2$.

Similarly,

$$\begin{aligned}
H^{-1}(x_\tau) &= c(z(x_\tau)) + G_2\left(\alpha_\tau, \frac{1}{w_\tau}\right)(z(x_\tau)-c(z(x))) \\
&= c(z(x_\tau)) + \left(\frac{\alpha_\tau}{2w_\tau-1} + \frac{w_\tau-1}{w_\tau(2w_\tau-1)}\right)(z(x_\tau)-c(z(x_\tau))) \\
&= c(z(x_\tau)) + \left(\frac{1}{2w_\tau-1} + \frac{w_\tau-1}{\alpha_\tau w_\tau(2w_\tau-1)}\right)(x_\tau - c(z(x_\tau))) \to x.
\end{aligned}$$

This completes the proof.

**3.4. Correction of the proof: version 2.** Another possibility of correcting the proof is to use the formula $c(z) = (w_U(z+y)+1)y$ for $z \in Z$. In this case the problem appearing in the original version and in the first correction disappears, since $c(z) \in \operatorname{Int} U$ for all $z \in Z$. Let us show that this modification works.

STEP 1: The half-lines $c(z)+(0,\infty)(z-c(z))$, $z \in Z$, are pairwise disjoint and their union is $X \setminus ((\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y)$.

Let $x \in X$. Let us find conditions under which there are $z \in Z$ and $\alpha > 0$ such that

$$x = c(z) + \alpha(z - c(z)).$$

This equation is equivalent to

$$(x - \varphi(x)y) + \varphi(x)y = \alpha(z+y) + \big((1-\alpha)(w_U(z+y)+1) - \alpha\big)y.$$

By applying $\varphi$ to both sides we see that (as in "(3.2)⇒(3.3)") the above equation is equivalent to

$$x - \varphi(x)y = \alpha(z+y) \quad \& \quad \varphi(x) = (1-\alpha)(w_U(z+y)+1) - \alpha. \tag{3.5}$$

From the first equation isolate $z+y$ and plug into the second equation. We thus get a quadratic equation for $\alpha$:

$$2\alpha^2 + \alpha\big(\varphi(x) - 1 + w_U(x - \varphi(x)y)\big) - w_U(x - \varphi(x)y) = 0.$$

If $w_U(x - \varphi(x)y) > 0$, then there is a unique positive root $\alpha = \alpha(x)$. If $w_U(x - \varphi(x)y) = 0$ and $\varphi(x) < 1$, there is a unique positive root $\alpha(x) = (1 - \varphi(x))/2$. If $w_U(x - \varphi(x)y) = 0$ and $\varphi(x) \geq 1$ (i.e., if $x \in (\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y$), then there is no positive root. This shows there is a unique $\alpha = \alpha(x)$ and it follows from the first equation in (3.5) that there is also a unique $z = z(x)$. This completes the proof of Step 1. Moreover, the computation shows that the mappings $x \mapsto \alpha(x)$ and $x \mapsto z(x)$ are continuous on $X \setminus ((\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y)$.

Step 2: $H$ is a homeomorphism of $X \setminus ((\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y)$ onto itself.

It is clear that $H$ is a bijection of the given set onto itself. Further, the formulas for $H$ and $H^{-1}$ are the same as in the previous case. Of course, $z(x)$, $\alpha(x)$ and $c(z(x))$ are given by different formulas, but since these mappings are continuous, so are $H$ and $H^{-1}$.

Step 3: $H$ is a homeomorphism of $X$ onto itself.

Since $H$ is defined as the identity on $(\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y$ and this set is closed in $X$, it is enough to show the following: Let $(x_\tau)$ be a net in $X \setminus ((\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y)$ converging to some $x \in (\operatorname{Ker}\varphi \cap \operatorname{cc} U) + [1,\infty)y$. Then $H(x_\tau) \to x$ and $H^{-1}(x_\tau) \to x$. So, let us take such a net.

We have $x_\tau = c(z(x_\tau)) + \alpha(x_\tau)(z(x_\tau) - c(z(x_\tau)))$. We will show that for $\tau$ large enough,

$$\alpha(x_\tau) \leq \frac{1}{2w_{U - c(z(x_\tau))}(z(x_\tau) - c(z(x_\tau)))}.$$

Then the proof will be complete, as it will follow that for $\tau$ large enough, $H(x_\tau) = H^{-1}(x_\tau) = x_\tau$.

The desired inequality is equivalent to $w_{U-c(z(x_\tau))}(z(x_\tau) - c(z(x_\tau))) \leq \frac{1}{2\alpha(x_\tau)}$, i.e. $c(z(x_\tau)) + 2\alpha(x_\tau)(z(x_\tau) - c(z(x_\tau))) \in U$, equivalently

$$c(z(x_\tau)) + 2(x_\tau - c(z(x_\tau))) \in U.$$

Let us analyze the limit behavior of the left-hand side. Set $t_\tau = \varphi(x_\tau)$ and $a_\tau = x_\tau - t_\tau y$. Then $t_\tau \to \varphi(x)$ and $a_\tau \to x - \varphi(x)y$, hence $w_U(a_\tau) \to 0$.

We have

$$c(z(x_\tau)) = (w_U(z(x_\tau)+y)+1)y = \left(\frac{w_U(a_\tau)}{\alpha(x_\tau)}+1\right)y.$$

If $w_U(a_\tau) = 0$, then $c(z(x_\tau)) = y$. Moreover, $t_\tau < 1$, hence $\varphi(x) \le 1$, so necessarily $\varphi(x) = 1$. It follows that $c(z(x_\tau)) = \varphi(x)y$.

If $w_U(a_\tau) > 0$, then $\alpha(x_\tau)$ is the positive root of the quadratic equation from Step 1, so

$$\frac{w_U(a_\tau)}{\alpha(x_\tau)} = \frac{1}{2}\big(\sqrt{(t_\tau-1+w_U(a_\tau))^2+8w_U(a_\tau)}+t_\tau-1+w_U(a_\tau)\big) \to \varphi(x)-1.$$

It follows that $c(z(x_\tau)) \to \varphi(x)y$, hence

$$c(z(x_\tau))+2(x_\tau-c(z(x_\tau))) \to \varphi(x)y+2(x-\varphi(x)y).$$

Since $x-\varphi(x)y \in \operatorname{cc} U$, $y \in \operatorname{cc} U$ and $\varphi(x) \ge 1$, we get $\varphi(x)y+2(x-\varphi(x)y) \in \operatorname{cc} U \subset \operatorname{Int} U$. Hence $c(z(x_\tau))+2(x_\tau-c(z(x_\tau))) \in U$ for $\tau$ large enough, and the proof is complete.

Remark 4. It took us some time to discover that the proof in [2] is incorrect. As remarked above, the error is already in the second step, since the assignment $z \mapsto u(z)$ fails to be continuous. The correction from Section 3.3 is quite complicated but we find it interesting since it uses some balance of asymptotic behavior. The correction from Section 3.4 is much simpler and now, a posteriori, we are convinced that this is the formula the authors had in mind. But it is still more complicated than the original proof, the main difference is in Step 3. While in the original version Step 3 is trivial, in the method described in Section 3.4 Step 3 requires some nontrivial computation. At least we do not see how to prove it without any computation like in the original version.

**4. Topological version of Dobrowolski's proof.** The approach of [4] is a bit different, it focuses on smooth bodies in Banach spaces and refers to the implicit function theorem. As remarked above, the proof is extremely concise and the missing computations (checking the assumptions of the implicit function theorem) are nontrivial—they would be much longer than the proof itself. In this section we give a modification of the proof from [4] which works simultaneously in the topological and the smooth cases. Our version is moreover simplified and more elementary. In particular, it uses a simpler version of the implicit function theorem (not only is its proof simpler, but the assumptions are easier to check) and our formula is simpler (although the mapping is the same) since we use the Minkowski functional related to only one convex body.

We will give the proof in the topological case and then comment why it also works in the smooth case.

First, choose two auxiliary $C^\infty$ functions $\lambda$ and $\gamma$ defined on $\mathbb{R}$ with the following properties:

- $\lambda$ is nondecreasing, $\lambda = 0$ on $(-\infty, 1/2]$, $\lambda = 1$ on $[1, \infty)$.
- $\gamma = 0$ on $(-\infty, 1/2]$, $\lim_{t\to\infty} \gamma(t) = \infty$ and $0 \le \gamma'(t) < \frac{1}{t}(\gamma(t) + 1)$ for $t > 0$.

The existence of $\lambda$ is a well-known fact. The existence of $\gamma$ is not obvious and in [4] it is just postulated. One can take, for example, $\gamma(t) = \delta\lambda(t)\ln(t + 1)$ for $t > -1$, where $\delta > 0$ is a small enough number and extend this function by zero on the rest of $\mathbb{R}$.

In the proof we will need the following version of the implicit function theorem.

Theorem 5. *Let $X$ be a topological space, $\Omega \subset X \times \mathbb{R}$ an open set, $F = F(x,t) : \Omega \to \mathbb{R}$ a function and $(x_0, t_0) \in \Omega$. Suppose that:*

- *$F$ and $\frac{\partial F}{\partial t}$ are continuous on $\Omega$.*
- *$F(x_0, t_0) = 0$.*
- *$\frac{\partial F}{\partial t}(x_0, t_0) \neq 0$.*

*Then there is a neighborhood $G$ of $x_0$ in $X$, a neighborhood $H$ of $t_0$ in $\mathbb{R}$ and a continuous function $f : G \to H$ such that $G \times H \subset \Omega$ and for $(x,t) \in G \times H$ one has $t = f(x)$ if and only if $F(x,t) = 0$.*

This theorem follows from a more general [5, Chapter III, Section 8, Theorem 25] (which deals with a normed space in place of $\mathbb{R}$). However, our version is much simpler and can be proved in the same way as the easiest version for $C^1$ functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Let us now start the construction itself. For $z \in Z$ let

$$c(z) = \gamma(w_U(y + z))y.$$

Step 1: The mapping $\Phi : (\alpha, z) \mapsto c(z) + \alpha(z - c(z))$ is a homeomorphism of $(0, \infty) \times Z$ onto $X \setminus ((\operatorname{cc} U \cap \operatorname{Ker}\varphi) + [0, \infty)y)$.

Note that in [4] there is a small error, where instead of $\operatorname{cc} U \cap \operatorname{Ker}\varphi$ the author writes $\{0\}$. We proceed as above. Fix $x \in X$ and try to find $\alpha > 0$ and $z \in Z$ such that $\Phi(\alpha, z) = x$. This equation is equivalent to

$$\alpha(z + y) + \big((1 - \alpha)\gamma(w_U(z + y)) - \alpha\big)y = (x - \varphi(x)y) + \varphi(x)y,$$

hence by applying $\varphi$ to both sides we get (as in “(3.2)⇒(3.3)”)

$$\alpha(z + y) = x - \varphi(x)y \quad \& \quad (1 - \alpha)\gamma(w_U(z + y)) - \alpha = \varphi(x).$$

If we isolate $z + y$ from the first equation and plug it into the second one, we get

$$(1 - \alpha)\gamma\left(\frac{1}{\alpha} w_U(x - \varphi(x)y)\right) - \alpha - \varphi(x) = 0.$$

Denote the left-hand side by $F(x,\alpha)$. It is clear that $F$ is defined and continuous on $X\times(0,\infty)$, and moreover

$$\begin{aligned}\frac{\partial F}{\partial\alpha}(x,\alpha) &= -\gamma\Big(\frac{1}{\alpha}w_U(x-\varphi(x)y)\Big)\\ &\qquad -\frac{1-\alpha}{\alpha^2}w_U(x-\varphi(x)y)\gamma'\Big(\frac{1}{\alpha}w_U(x-\varphi(x)y)\Big)-1,\end{aligned}$$

which is also continuous on $X\times(0,\infty)$.

Further, $\frac{\partial F}{\partial\alpha}(x,\alpha)<0$ for $(x,\alpha)\in X\times(0,\infty)$. Indeed, if $w_U(x-\varphi(x)y)=0$, then $\frac{\partial F}{\partial\alpha}(x,\alpha)=-1$. If $w_U(x-\varphi(x)y)>0$ and $\alpha\le 1$, then $\frac{\partial F}{\partial\alpha}(x,\alpha)\le -1$. Finally, if $w_U(x-\varphi(x)y)>0$ and $\alpha>1$, then by the properties of $\gamma$ we get

$$\begin{aligned}\frac{\partial F}{\partial\alpha}(x,\alpha) &< -\gamma\Big(\frac{1}{\alpha}w_U(x-\varphi(x)y)\Big)\\ &\qquad +\frac{\alpha-1}{\alpha^2}\Big(\gamma\Big(\frac{1}{\alpha}w_U(x-\varphi(x)y)\Big)+1\Big)-1\le 0.\end{aligned}$$

Let us describe the range of $\Phi$. Fix $x\in X$. There are two possibilities:

Case 1: $w_U(x-\varphi(x)y)=0$. Then $F(x,\alpha)=-\alpha-\varphi(x)$. If $\varphi(x)<0$, there is a unique positive root $\alpha=-\varphi(x)$. If $\varphi(x)\ge 0$, there is no positive root.

Case 2: $w_U(x-\varphi(x)y)>0$. Then $\lim_{\alpha\to 0+}F(x,\alpha)=\infty$ (as $\gamma$ has limit $\infty$ at $\infty$) and $\lim_{\alpha\to\infty}F(x,\alpha)=-\infty$ (as $\gamma$ vanishes in a neighborhood of zero). Further, since $\alpha\mapsto F(x,\alpha)$ is continuous and strictly decreasing on $(0,\infty)$, there is a unique root.

It follows that $\Phi$ is one-to-one and its range is $X\setminus((\operatorname{cc}U\cap\operatorname{Ker}\varphi)+[0,\infty)y)$. It is clear that $\Phi$ is continuous. By Theorem 5, the second coordinate of the inverse is continuous, the continuity of the first coordinate then follows, hence $\Phi^{-1}$ is continuous.

Step 2: The mapping $\Psi$ defined by the formula

$$\Psi(\alpha,z)=\big(\alpha\lambda(\alpha w_{U-c(z)}(z-c(z)))(w_{U-c(z)}(z-c(z))-1)+\alpha,z\big)$$

is a homeomorphism of $(0,\infty)\times Z$ onto itself.

Since $w_{U-c(z)}(z-c(z))\ge 1$ whenever $z\in Z$, $\Psi$ maps $(0,\infty)\times Z$ into itself. $\Psi$ is clearly continuous. To show that $\Psi$ is a bijection and the inverse is continuous, let us investigate the first coordinate, i.e., the mapping

$$\theta(\alpha,z)=\alpha\lambda(\alpha w_{U-c(z)}(z-c(z)))\big(w_{U-c(z)}(z-c(z))-1\big)+\alpha.$$

We have

$$\begin{aligned}\frac{\partial}{\partial\alpha}\theta(\alpha,z) &= \big(\lambda(\alpha w_{U-c(z)}(z-c(z)))\\ &\quad+\alpha\lambda'(\alpha w_{U-c(z)}(z-c(z)))w_{U-c(z)}(z-c(z))\big)\big(w_{U-c(z)}(z-c(z))-1\big)+1.\end{aligned}$$

This partial derivative is continuous and strictly positive on $(0,\infty)\times Z$. Moreover, for any $z\in Z$ we have

$$\lim_{\alpha\to 0+}\theta(\alpha,z)=0 \quad\text{and}\quad \lim_{\alpha\to\infty}\theta(\alpha,z)=\infty,$$

hence $\Psi$ is a bijection. Moreover, the continuity of $\Psi^{-1}$ follows from Theorem 5. This completes the proof of Step 2.

STEP 3: The mapping $H=\Phi\circ\Psi\circ\Phi^{-1}$ is a homeomorphism of the set $X\setminus((\operatorname{cc}U\cap\operatorname{Ker}\varphi)+[0,\infty)y)$ onto itself. Moreover, it maps each half-line $c(z)+\mathbb{R}^+(z-c(z))$ onto itself in an increasing manner so that the segment $(c(z),u(z)]$ is mapped onto $(c(z),z]$.

Indeed, $H$ is a homeomorphism as a composition of homeomorphisms. Further, from the construction it is clear that it preserves the above mentioned half-lines in an increasing manner. The last thing to show is that $H(u(z))=z$. To see this notice first that $\Phi^{-1}(u(z))=\big(\frac{1}{w_{U-c(z)}(z-c(z))},z\big)$, hence $\Psi(\Phi^{-1}(u(z))=(1,z)$, thus $H(u(z))=z$.

STEP 4: If we extend $H$ by the identity on $(\operatorname{cc}U\cap\operatorname{Ker}\varphi)+[0,\infty)y$, we get a homeomorphism of $X$ onto itself with the required properties.

Since $(\operatorname{cc}U\cap\operatorname{Ker}\varphi)+[0,\infty)y$ is closed, it is enough to check that $H$ and $H^{-1}$ are continuous at points of this set. So, fix $x$ in this set and a net $x_\tau$ in the complement converging to $x$. Let $(\alpha_\tau,z_\tau)=\Phi^{-1}(x_\tau)$. Let us first show that $\alpha_\tau\to 0$. Suppose not. Then, passing to a subnet if necessary, we may assume that $\alpha_\tau\to\alpha\in(0,\infty]$. We have

$$\varphi(x)=\lim_\tau\varphi(x_\tau)=\lim_\tau(1-\alpha_\tau)\gamma\bigg(\frac{1}{\alpha_\tau}w_U(x_\tau-\varphi(x_\tau)y)\bigg)-\alpha_\tau.$$

If $\alpha=\infty$, then the limit on the right-hand side is $-\infty$ (since $\gamma$ is zero on a neighborhood of zero), which is not possible. If $\alpha\in(0,\infty)$, then the right-hand side is $-\alpha$ (since $w_U(x_\tau-\varphi(x_\tau)y)\to w_U(x-\varphi(x)y)=0$). Thus $\varphi(x)<0$, a contradiction.

So, we have proved that $\alpha_\tau\to 0$. Further,

$$\begin{aligned}c(z_\tau)=\gamma(w_U(z_\tau+y))y&=\gamma\bigg(\frac{1}{\alpha_\tau}w_U(x_\tau-\varphi(x_\tau)y)\bigg)y\\&=\frac{\varphi(x_\tau)+\alpha_\tau}{1-\alpha_\tau}y\to\varphi(x)y.\end{aligned}$$

Hence $w_{U-c(z_\tau)}(x_\tau-c(z_\tau))\to w_{U-\varphi(x)y}(x-\varphi(x)y)$ by Lemma 2. But the latter value is zero, since $w_U(x-\varphi(x)y)=0$ and $y\in\operatorname{cc}U$, so for each $t>0$ we have $t(x-\varphi(x)y)+\varphi(x)y\in U$. So, for $\tau$ large enough we have

$$\frac{1}{\alpha}w_{U-c(z_\tau)}(z_\tau-c(z_\tau))=w_{U-c(z_\tau)}(x_\tau-c(z_\tau))<\frac{1}{2},$$

hence $\Psi(\alpha_\tau,z_\tau)=(\alpha_\tau,z_\tau)$. Finally, for those $\tau$ we have $H(x_\tau)=H^{-1}(x_\tau)=x_\tau$.

This completes the proof.

Remark 6. In case $X$ is a Banach space and $U$ is a $C^p$-smooth convex body (where $p \in \mathbb{N} \cup \{\infty\}$), the homeomorphism $H$ constructed above is a $C^p$-diffeomorphism. Indeed, first remark that if in Theorem 5 we moreover assume that $X$ is a Banach space and $F$ is $C^p$-smooth, then so is $f$. Further, in this case the function from Lemma 2 is $C^p$-smooth on the complement of its zero set by [4, Lemma 1]. (The proof of this lemma is omitted in [4], but it is an easy consequence of the definition.) Further, the function $F$ used in Step 1 is $C^p$-smooth on $X \times (0, \infty)$ (at points where $w_U(x_0 - \varphi(x_0)y) > 0$ this is a composition of $C^p$ functions mentioned above; if $w_U(x_0 - \varphi(x_0)y) = 0$, then $F(x, \alpha) = -\alpha - \varphi(x)$ on a neighborhood of $(x_0, \alpha_0)$). It follows that $\Phi$ is a $C^p$-diffeomorphism. Similarly we can see that the mapping $\Psi$ from Step 2 is a $C^p$-diffeomorphism. Finally, from the proof of Step 4 we see that each point of $(\operatorname{cc} U \cap \operatorname{Ker} \varphi) + [0, \infty)y$ has a neighborhood on which $H$ is the identity, so $H$ is a $C^p$-diffeomorphism.

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