COLLOQUIUM MATHEMATICUM

VOL. 140 2015 NO. 1

ON MODULES AND RINGS WITH THE RESTRICTED MINIMUM CONDITION

BY

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Abstract. A module M satisfies the restricted minimum condition if M/N is artinian for every essential submodule N of M. A ring R is called a right RM-ring whenever R_R satisfies the restricted minimum condition as a right module. We give several structural necessary conditions for particular classes of RM-rings. Furthermore, a commutative ring R is proved to be an RM-ring if and only if $R/\mathrm{Soc}(R)$ is noetherian and every singular module is semiartinian.

1. Introduction. Given a module M over a ring R, recall that N is an *essential submodule* of M if there is no non-zero submodule K of M such that $K \cap N = 0$. We say that M satisfies the *restricted minimum condition* (RMC) if for every essential submodule N of M, the factor module M/N is artinian. It is easy to see that the class of modules satisfying RMC is closed under taking submodules, factors and finite direct sums. A ring R is called a *right RM-ring* if R_R satisfies RMC as a right module. An integral domain R satisfying the restricted minimum condition is called an *RM-domain*, i.e. R/I is artinian for all non-zero ideals I of R (see [4]). Note that a noetherian domain has Krull dimension 1 if and only if it is an RM-domain [5, Theorem 1].

The purpose of the present paper is to continue on studies [3], [4], [5], [10] and [14], in which the basic structure theory of RM-rings and RM-domains was introduced by Albrecht and Breaz [1], which describes some properties of classes of torsion modules over RM-domains, and widely studied for corresponding classes of abelian groups. As the method of [1] appears to be fruitful, this paper focuses on the study of the structure of modules satisfying RMC, in particular singular ones. For a module M with the essential socle, we show that M satisfies RMC if and only if $M/\mathrm{Soc}(M)$ is artinian. It is also proved, among other results, that for a module M over a right RM-ring R, if M is singular, then M is semiartinian. These tools allow us to obtain ring-theoretical results for both non-commutative and commutative

2010 *Mathematics Subject Classification*: Primary 16D40; Secondary 16E50.

Key words and phrases: restricted minimum condition, singular module, semiartinian ring, semilocal ring, max ring, perfect ring.

DOI: 10.4064/cm140-1-6

rings. Namely, if R is a right RM-ring and $\operatorname{Soc}(R) = 0$, we prove that R is a non-singular ring of finite Goldie dimension. As a consequence, in Section 2 we obtain some characterizations of various classes of right RM-rings via some well-known and important rings (semiartinian, (von Neumann) regular, semilocal, max, perfect) plus some (socle finiteness) conditions: In the case when R is a semilocal right RM-ring and $\operatorname{Soc}(R) = 0$, we show that R is noetherian if and only if $J(R)$ is finitely generated if and only if the socle length of $E(R/J(R))$ is at most ω. If R is a right max right RM-ring, we prove that $R/\operatorname{Soc}(R)$ is right noetherian.

In Section 3, we focus on commutative rings. It is shown that such a ring R satisfies RMC if and only if $R/\operatorname{Soc}(R)$ is noetherian and every singular module is semiartinian.

Throughout this paper, rings are associative with unity and modules are unital right R-modules, where R denotes such a ring and M denotes such a module. We write $J(R), J(M), \operatorname{Soc}(R), \operatorname{Soc}(M)$ for the respective Jacobson radicals and socles. We also write $N \trianglelefteq M$ to indicate that N is an essential submodule of M, and $E(M)$ for the injective hull of M.

2. The structure of general right RM-rings. Firstly, we prove the following lemma which is quite useful for the study of modules and rings with the right restricted minimum condition, and then recall a useful folklore observation (see [11, Lemma 3.6]).

LEMMA 2.1. *Let K and N be submodules of M such that $K \trianglelefteq N$. If M satisfies RMC, then N/K is artinian.*

Proof. If we choose a submodule A for which $N \cap A = 0$ and $N \oplus A \trianglelefteq M$, then $K \oplus A \trianglelefteq M$. Hence $M/(K \oplus A)$ and $(N \oplus A)/(K \oplus A) \cong N/K$ are artinian modules. ∎

A non-zero module M is called *uniform* if the intersection of any two non-zero submodules of M is non-zero, or equivalently, every non-zero submodule of M is essential in M.

A module M is said to have *Goldie dimension* (or *uniform dimension*) n, denoted $\operatorname{Gdim}(M) = n$, if $E(M)$ is a direct sum of n submodules, equivalently if M has an essential submodule which is a direct sum of n uniform submodules.

LEMMA 2.2. *If a module M satisfies RMC, then $M/\operatorname{Soc}(M)$ has finite Goldie dimension.*

Proof. Set $S_0 := \operatorname{Soc}(M)$, and fix a submodule S_1 of the module M such that $S_0 \subseteq S_1$ and $S_1/S_0 = \operatorname{Soc}(M/S_0)$. By Zorn's Lemma, we may choose a maximal set of elements $m_i \in M$ such that $S_1 \cap (\bigoplus_{i\in I} m_iR) = 0$. It is easy to see that $S_1 \oplus (\bigoplus_{i\in I} m_iR) \trianglelefteq M$. Since $\bigoplus_{i\in I} m_iR \cap S_0 = 0$, every module m_iR

has zero socle. Hence m_iR is not simple, and any maximal submodule of m_iR is essential in m_iR. For every $i \in I$, let N_i be a fixed maximal submodule in m_iR. As $\bigoplus_{i\in I} N_i \trianglelefteq \bigoplus_{i\in I} m_iR$, the module $L = S_0 \oplus \bigoplus_{i\in I} N_i$ is essential in M. Since M satisfies RMC, we see that M/L is an artinian module containing an isomorphic copy of $(S_1/S_0)\oplus(\bigoplus_{i\in I} m_iR/N_i)$, which implies that I is finite and S_1/S_0 is a finitely generated semisimple module. By [12, Proposition 6.5], we conclude that the uniform dimension of $M/\mathrm{Soc}(M)$ is finite. ■

Following [7, Section 7.2], the class $\mathcal{M}_\alpha$ of modules M of *Krull dimension* α, written $\mathrm{Kdim}(M) = \alpha$, is defined as follows. The class $\mathcal{M}_{-1}$ consists of the module $M = 0$. If the class $\mathcal{M}_\beta$ of modules of Krull dimension β has been defined for every $\beta < \alpha$, then $\mathcal{M}_\alpha$ is defined as the class of all modules M such that

(i) $M \notin \bigcup_{\beta<\alpha} \mathcal{M}_\beta$,
(ii) for every decreasing chain $M_0 \supseteq M_1 \supseteq \cdots$ of submodules of M, there exists n such that $M_i/M_{i+1} \in \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ for all $i \geq n$.

We also note that:

- $\mathrm{Kdim}(M_R) = -1$ if and only if $M_R = 0$.
- $\mathrm{Kdim}(M_R) = 0$ if and only if M_R is a non-zero artinian module.
- Every module with Krull dimension has finite Goldie dimension (see [7, Proposition 7.13]).

PROPOSITION 2.3. *If a module M satisfies RMC, then* $\mathrm{Kdim}(M/\mathrm{Soc}(M))$ *is at most one.*

Proof. Let $N_0 \supseteq N_1 \supseteq \cdots$ be a descending chain of submodules of $M/\mathrm{Soc}(M)$. As $M/\mathrm{Soc}(M)$ has a finite Goldie dimension by Lemma 2.2, there exists n such that for each $i \geq n$ either $N_i = 0$ or $N_{i+1} \trianglelefteq N_i$. Since N_i/N_{i+1} is artinian by Lemma 2.1, we conclude that $M/\mathrm{Soc}(M)$ has Krull dimension at most 1. ■

A module M is called *semiartinian* if every non-zero factor of M contains a non-zero socle. A ring R is called *right semiartinian* if R_R is a right semiartinian module. Note that every non-zero right module over a right semiartinian ring is semiartinian (see [9]).

Let M be a semiartinian module. By [8] or [13], every semiartinian module contains an increasing chain of submodules $(S_\alpha \mid \alpha \geq 0)$ (called the *socle chain*) satisfying

$$S_0 = 0,$$
$$S_{\alpha+1}/S_\alpha = \mathrm{Soc}(M/S_\alpha) \quad \text{for each ordinal } \alpha,$$
$$S_\alpha = \bigcup_{\beta<\alpha} S_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Furthermore, the first ordinal σ such that $S_\sigma = M$ is said to be the *socle length* of M.

Since every semiartinian ring contains the essential socle, we obtain the following easy observation.

LEMMA 2.4. *Let R be a right semiartinian ring. Then R is a right RM-ring if and only if $R/\mathrm{Soc}(R)$ is artinian.*

Obviously, the class of right RM-rings is closed under taking factors and finite products. But, in general, this is not true of taking extensions.

EXAMPLE 2.5. Let R be a right semiartinian ring of socle length 3 and $R/\mathrm{Soc}(R)$ non-artinian. Hence R is not a right RM-ring by Lemma 2.4. Since $R_0/\mathrm{Soc}(R_0)$ is semisimple, we infer that $R_0 = R/\mathrm{Soc}(R)$ is a right RM-ring by Lemma 2.4. Clearly $\mathrm{Soc}(R)$ satisfies RMC as well. Hence the short exact sequence

$$0 \to \mathrm{Soc}(R) \to R \to R/\mathrm{Soc}(R) \to 0$$

shows that the class of all modules satisfying RMC is not closed under extensions.

In particular, using constructions of [6], we can fix a field F and take as R_1 the F-subalgebra of the F-algebra F^ω of all countable sequences over F generated by the ideal of ultimately zero sequences $F^{(\omega)}$, where ω denotes the first infinite ordinal. Note that this F-subalgebra contains exactly ultimately constant sequences. Now R_2 is defined as an F-subalgebra of a natural F-algebra R_1^ω generated by $R_1^{(\omega)}$. It is easy to see that R_2 is a right semiartinian ring of socle length 3 and $R_2/\mathrm{Soc}(R_2)$ is non-artinian.

Let us recall the following well-known observation.

LEMMA 2.6. *Let M be an artinian R-module. If $J(N) \neq N$ for every non-zero submodule N of M, then M is noetherian.*

Proof. Assume that M is not noetherian. Then it contains a semiartinian submodule of infinite socle length. As M is artinian, there is a minimal submodule N of infinite socle length. Thus N contains no maximal submodule, i.e. $J(N) = N$. ∎

Now we are able to clarify the structure of RM-rings, which is similar (and in some sense dual) to the structure of semiartinian rings.

THEOREM 2.7. *Let R be a right RM-ring, $S(R)$ the greatest right semiartinian ideal of R, and set $A := R/\mathrm{Soc}(R)$ and $S(A) := S(R)/\mathrm{Soc}(R)$. Then:*

(i) $\bigcap_{n<\omega} J(A)^n$ *is nilpotent,*
(ii) $S(A) \cap J(A)$ *is nilpotent,*
(iii) $S(A)/(S(A) \cap J(A))$ *is noetherian.*

Proof. (i) Since the Krull dimension of A is 0 or 1 by Proposition 2.3, we deduce that $\bigcap_n J(A)^n$ is a nilpotent by [7, Theorem 7.26].

(ii) Set $K := S(A) \cap \bigcap_n J(A)^n$ and $I := S(A) \cap J(A)$. Note that K is nilpotent by (i). Since $S(A)$ is artinian by Lemma 2.1, so is I. Moreover, $I^n \subseteq J(A)^n$, and so $\bigcap_n I^n \subseteq K$. Since I artinian, there exists n for which $I^n \subseteq K$, which finishes the proof.

(iii) Note that $S(A)$, and so $M = S(A)/(S(A) \cap J(A))$ is artinian and $J(M) = 0$. Hence $J(N) = 0$ for each submodule N of M. The rest follows from Lemma 2.6. ∎

Corollary 2.8. *If* $\mathrm{Soc}(R) = 0$ *and* $J(R)^2 = J(R)$ *for a ring* R*, then* R *is not a right RM-ring.*

A ring R is *regular* if for every $x \in R$ there exists $y \in R$ such that $x = xyx$.

Proposition 2.9. *The following conditions are equivalent for a regular ring* R*:*

(i) R *is a right RM-ring,*
(ii) $R/\mathrm{Soc}(R)$ *is artinian,*
(iii) R *is semiartinian of socle length* 2.

Proof. (i)⇒(ii). By Lemma 2.2, $R/\mathrm{Soc}(R)$ is of finite Goldie dimension. Since $R/\mathrm{Soc}(R)$ is a regular ring which cannot contain an infinite set of orthogonal set idempotents, we conclude that $R/\mathrm{Soc}(R)$ is artinian.

(ii)⇒(iii). This is obvious because an artinian regular ring is semisimple.

(iii)⇒(i). This follows from Lemma 2.4. ∎

Recall that the *singular submodule* $Z(M)$ of a module M is defined by

$$Z(M) = \{m \in M : mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

The module M is called *singular* if $M = Z(M)$, and *non-singular* if $Z(M) = 0$. Clearly, every regular ring is non-singular (for more properties cf. [15]).

Lemma 2.10. *Let* R *be a right RM-ring. Then* $Z(M)$ *is semiartinian for each right* R*-module* M*.*

Proof. Let $m \in Z(M)$. Clearly, $r(m)$ is an essential right ideal of R, where $r(m) = \{a \in A \mid ma = 0\}$. Hence $mR \cong R/r(m)$ is artinian and so semiartinian. ∎

Theorem 2.11. *Let* R *be a right RM-ring and* M *a right* R*-module.*

(i) *If* M *is singular, then* M *is semiartinian.*
(ii) $E(M)/M$ *is semiartinian.*
(iii) *If* M *is semiartinian, then* $E(M)$ *is semiartinian. In particular,* $E(S)$ *is semiartinian for every simple module* S*.*

Proof. Assume that M is singular. By Lemma 2.10, $Z(M) = M$ is semiartinian, hence (i) holds. Since $E(M)/M$ is a singular module by [12, Example 7.6(3)] and the class of semiartinian modules is closed under taking essential extensions, (ii) and (iii) hold. ∎

Since for a ring R with no simple submodule we obtain $Z(R) = 0$ by Lemma 2.10, we can formulate the following observation which is a consequence of Lemma 2.2.

Corollary 2.12. *If* $\mathrm{Soc}(R) = 0$ *for a right RM-ring* R*, then* R *is a non-singular ring of finite Goldie dimension.*

Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple artinian.

Lemma 2.13. *If* R *is a semilocal ring, then* $J(R) + \mathrm{Soc}(R) \trianglelefteq R$.

Proof. Assume that $J(R) + \mathrm{Soc}(R)$ is not essential in R. Then there exists a non-zero right ideal $I \subseteq R$ such that $I \cap (J(R) + \mathrm{Soc}(R)) = 0$. Since $\mathrm{Soc}(I) = \mathrm{Soc}(R) \cap I = 0$ and $R/J(R)$ contains an ideal which is isomorphic to I, we find that $\mathrm{Soc}(R/J(R)) \neq R/J(R)$. Hence R is not semilocal, a contradiction. ∎

The following example shows that the converse of Lemma 2.13 is not true.

Example 2.14. Suppose that R is a local commutative domain with maximal ideal J. It is easy to see that J^ω is the Jacobson radical of the ring R^ω and it is essential in R^ω. However R^ω is not semilocal.

Recall that $J(R/J(R)) = \{0 + J(R)\}$ for an arbitrary ring R.

Proposition 2.15. *Assume that* R *is a right RM-ring.*

- (i) *If* $\mathrm{Soc}(R) = 0$*, then* $J(R) \trianglelefteq R$ *if and only if* R *is semilocal.*
- (ii) *If* R *is a semilocal ring, then* $J(R)/\mathrm{Soc}(J(R))$ *is finitely generated as a two-sided ideal.*

Proof. (i) Since $J(R) \trianglelefteq R_R$ and R_R satisfies right RMC, we see that $R/J(R)$ is an artinian ring. On the other hand, $J(R/J(R)) = \{0 + J(R)\}$ implies that $R/J(R)$ is semisimple, and hence R is semilocal. The converse follows from Lemma 2.13.

(ii) We note that there exists a finitely generated right ideal $F \subseteq J(R)$ such that $F + (\mathrm{Soc}(R) \cap J(R)) \trianglelefteq J(R)$, since $J(R)/(\mathrm{Soc}(R) \cap J(R))$ has a finite Goldie dimension by Lemma 2.2. Thus $RF+\mathrm{Soc}(R)$ is a two-sided ideal which is essential in R as a right ideal, by Lemma 2.13. By the hypothesis, $R/(RF+\mathrm{Soc}(R))$ is a right artinian ring. Since $J(R)+\mathrm{Soc}(R)/(RF+\mathrm{Soc}(R))$

is finitely generated as a right ideal and

$$\begin{aligned}(J(R)+\mathrm{Soc}(R))/(RF+\mathrm{Soc}(R)) &\cong J(R)/(J(R)\cap(RF+\mathrm{Soc}(R)))\\ &= J(R)/(RF+(J(R)\cap\mathrm{Soc}(R)))\\ &= J(R)/(RF+\mathrm{Soc}(J(R))),\end{aligned}$$

we conclude that the ideal $J(R)/\mathrm{Soc}(J(R))$ is finitely generated. ■

Recall that every artinian module is semiartinian, and ω denotes the first infinite ordinal.

LEMMA 2.16. *The following are equivalent for an artinian R-module M:*

(i) *The socle length of M is greater than ω.*
(ii) *M contains a cyclic submodule with infinitely generated Jacobson radical.*
(iii) *M contains a cyclic submodule which is not noetherian.*

Proof. (i)⇒(ii). Let M be an artinian module of non-limit infinite socle length, and fix $x \in M$ such that xR has socle length $\omega+1$. Denote by S_α the αth member of the socle sequence of xR. Since xR is artinian, $J(xR)$ is the intersection of finitely many maximal submodules, which implies that $xR/J(xR)$ is semisimple. Because xR/S_ω is semisimple as well, we have $J(xR) \subseteq S_\omega$. Hence the socle length of $J(xR)$ is at most ω. Assume that $J(xR)$ is finitely generated. Then the socle length of $J(xR)$ is non-limit, and hence finite. This implies that xR has a finite socle length, a contradiction, i.e. $J(xR)$ is infinitely generated.

(ii)⇒(iii). This is clear.

(iii)⇒(i). As a cyclic non-noetherian artinian module is of infinite non-limit socle length, the length has to be greater than ω. ■

The next result characterizes semilocal right RM-rings further.

THEOREM 2.17. *The following conditions are equivalent for a semilocal right RM-ring R with* $\mathrm{Soc}(R)=0$*:*

(i) *R is right noetherian.*
(ii) *$J(R)$ is finitely generated as a right ideal.*
(iii) *The socle length of $E(R/J(R))$ is at most ω.*

Proof. (i)⇒(ii). This is obvious.

(ii)⇒(iii). Note that every cyclic submodule of $E(R/J(R))$ is artinian by Theorem 2.11. Suppose that the socle length of $E(R/J(R))$ is greater than ω. Hence $E(R/J(R))$ contains an artinian submodule of socle length greater than ω. By Lemma 2.16, there exists a cyclic module xR with infinitely generated Jacobson radical. Fix right ideals I_1 and I_2 such that $xR \cong R/I_1$, $I_1 \subseteq I_2$ and $I_2/I_1 = J(R/I_1)$. It is easy to see that I_2 is infinitely generated and $J(R) \subseteq I_2$. Since $I_2/J(R)$ is a right ideal of the semisimple ring $R/J(R)$, it follows that $I_2/J(R)$ is finitely generated, and hence $J(R)$ is an infinitely generated right ideal.

(iii)⇒(i). Let I be a right ideal. We show that I is finitely generated. By Lemma 2.2, there exist finitely generated right ideals F and G such that $F \trianglelefteq I$, $I \cap G = 0$ and $F + G \trianglelefteq R$. First we note that $R/(F + G)$ is an artinian module with a submodule isomorphic to I/F. It is also easy to see that $R/(F+G)$ is isomorphic to a submodule of $\bigoplus_{i\leq n} E(S_i)$ for some simple modules $S_1, \dots, S_n$. Since each $E(S_i)$ is isomorphic to some submodule of $E(R/J(R))$, the socle length of $\bigoplus_{i\leq n} E(S_i)$ and so of $R/(F + G)$ is at most ω. As $R/(F + G)$ is a cyclic module, it is an artinian module of finite socle length, which implies that $R/(F + G)$ is also a noetherian module. Therefore I/F and so I are finitely generated modules. ∎

Recall that a ring R is called *right max* if every non-zero right module has a maximal proper submodule.

THEOREM 2.18. *If R is a right max right RM-ring, then $R/\mathrm{Soc}(R)$ is right noetherian.*

Proof. Let I be a right ideal of $R/\mathrm{Soc}(R)$. It is enough to show that I is finitely generated. If we apply Lemma 2.2 to I, we see that there exists a finitely generated right ideal F such that $F \trianglelefteq I$ and I/F is artinian. Since R is a right max ring, every non-zero submodule of I/F contains a maximal submodule, and so I/F is noetherian. By Lemma 2.6, it is finitely generated. Thus I is finitely generated as well. ∎

As right perfect rings are right max, we get

COROLLARY 2.19. *If R is a right perfect right RM-ring, then $R/\mathrm{Soc}(R)$ is right noetherian.*

The following example shows that a perfect right RM-ring need not be a (right) noetherian ring.

EXAMPLE 2.20. Let F be a commutative field and V be a vector space over F. Consider the trivial extension $R = F \times V$. Then R is a local ring, hence it is perfect. The proper ideals of R are the $0 \times W$, where W is an F-subspace of V. Hence the only essential ideals of R are R and the maximal ideal $0 \times V$. Then R_R satisfies the right RMC. We note that if V is infinite-dimensional, then R is not noetherian.

Since every left perfect ring is right artinian, the following observation follows from Lemma 2.4.

COROLLARY 2.21. *If R is a left perfect right RM-ring, then $R/\mathrm{Soc}(R)$ is right artinian.*

3. Characterizations of commutative RM-rings. We recall the terminology that we need in this section. Let P be a maximal ideal of a domain R. For every R-module M, the symbol $M_{[P]}$ denotes the sum of all finite length submodules U of M such that all composition factors of U are isomorphic to R/P.

A module M is *self-small* if the functor $\operatorname{Hom}(M, -)$ commutes with all direct powers of M. Recall that M is not self-small if and only if there exists a chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ of submodules such that $\bigcup_n M_n = M$ and $\operatorname{Hom}(M/M_n, M) \neq 0$ for each n.

Let $\operatorname{Max}(M)$ denote the set of all maximal submodules of M.

First, let us formulate some results of [1] in the following observation.

Theorem 3.1 ([1, Theorem 6, Lemma 3(2), Theorem 9]). *The following conditions are equivalent for a commutative domain R:*

(i) *R is an RM-domain,*
(ii) *$M = \bigoplus_{P\in\operatorname{Max}(R)} M_{[P]}$ for all torsion modules M,*
(iii) *R is noetherian and every non-zero (cyclic) torsion R-module has an essential socle,*
(iv) *R is noetherian and every self-small torsion module is finitely generated.*

The following is, maybe, well-known.

Lemma 3.2. *Every cyclic artinian module over a commutative ring is noetherian.*

The following example shows that the assumption of commutativity in Lemma 3.2 is not superfluous.

Example 3.3. Let F be a field and $I = \mathbb{N}\cup\{\omega\}$ be a countable set (I consists of all natural numbers plus a further index ω). The ring R is the ring of non-commutative polynomials with coefficients in F and in the non-commutative indeterminates x_i, $i \in I$. The cyclic module will be a vector space V over F of countable dimension, with basis v_i, $i \in I$, over the field F.

We must say how R acts on V. For every $n \in \mathbb{N}$, set $x_n v_i = v_n$ if $i \geq n$ and $i \in \mathbb{N}$, $x_n v_i = 0$ if $i < n$ and $i \in \mathbb{N}$, and $x_n v_\omega = v_n$. Moreover, set $x_\omega v_i = 0$ for every $i \in \mathbb{N}$, and $x_\omega v_\omega = v_\omega$. Thus we obtain a left R-module ${}_RV$. Now ${}_RV$ is cyclic generated by v_ω (because $x_n v_\omega = v_n$).

The R-submodules of ${}_RV$ are

$$Rv_0 \subset Rv_1 \subset \cdots \subset \bigcup_{i\in\mathbb{N}} Rv_i \subset Rv_\omega = V.$$

Thus the lattice of R-submodules of ${}_RV$ is isomorphic to $\mathbb{N}\cup\{\omega\}$, that is, is order-isomorphic to the cardinal $\omega + 1$. Thus the cyclic R-module ${}_RR$ is artinian but not noetherian.

The following observation generalizes [1, Lemma 3(2)].

THEOREM 3.4. *Let R be a commutative ring. Then R is an RM-ring if and only if $R/\mathrm{Soc}(R)$ is noetherian and every singular module is semiartinian.*

Proof. (:⇒) Let R be an RM-ring, and let A be the greatest semiartinian ideal in R. Then R/A has zero socle and $\mathrm{Soc}(R) \trianglelefteq A$. By Lemma 2.1, $A/\mathrm{Soc}(R)$ is artinian, and so is noetherian by Lemma 3.2. It remains to show that R/A is noetherian. Without loss of generality, we may suppose that $\mathrm{Soc}(R) = 0$. Let I be an ideal of R. We show that it is finitely generated. Repeating the argument for (iii)⇒(i) in the proof of Theorem 2.17, we can find finitely generated ideals F and G such that $F \trianglelefteq I$, $I \cap G = 0$ and $F + G \trianglelefteq R$. Hence $R/(F+G)$ is artinian and it has a submodule which is isomorphic to I/F. Since $R/(F+G)$ is noetherian by Lemma 3.2, I/F as well as I are finitely generated. The rest follows from Lemma 2.10.

(⇐:) Suppose $R/\mathrm{Soc}(R)$ is noetherian and every singular module is semiartinian. Fix an ideal $I \trianglelefteq R$. By Lemma 2.10, R/I is singular and so semiartinian. Moreover, R/I is noetherian and semiartinian, and hence it is artinian, which finishes the proof. ∎

In light of Theorem 3.4, we ask the following.

QUESTION 3.5. *Is $R/\mathrm{Soc}(R)$ noetherian for each non-commutative right RM-ring R?*

Recall (Theorem 3.1) that R is an RM-domain if and only if

$$M = \bigoplus_{P \in \mathrm{Max}(R)} M_{[P]}$$

for all torsion modules M.

LEMMA 3.6. *If M is a singular module over a commutative RM-ring R, then $M = \bigoplus_{P \in \mathrm{Max}(R)} M_{[P]}$.*

Proof. Assume that $M \neq \bigoplus_{P \in \mathrm{Max}(R)} M_{[P]}$ and fix $m \in M \setminus \bigoplus_{P \in \mathrm{Max}(R)} M_{[P]}$. Since M is singular, mR is artinian and

$$mR \cong R/r(m) \cong \prod_{r(m) \subseteq I} A_I,$$

where each A_I is a local commutative artinian ring with maximal ideal I. As $A_I \subseteq M_{[I]}$ and there are only finitely many $I \in \mathrm{Max}(R)$, we get a contradiction. ∎

We finish this paper with the following observation.

THEOREM 3.7. *The following conditions are equivalent for a commutative ring R:*

(i) *R is an RM-ring,*
(ii) *$M = \bigoplus_{P \in \mathrm{Max}(R)} M_{[P]}$ for all singular modules M,*
(iii) *$R/\mathrm{Soc}(R)$ is noetherian and every self-small singular module is finitely generated.*

Proof. (i)⇒(ii). This follows from Lemma 3.6.

(ii)⇒(i). We follow the proof of [1, Theorem 6]. Let I be an essential ideal of R. Then R/I is a cyclic singular module, and hence $R/I \cong \bigoplus_{P \in \mathrm{Max}(R)} A_{[P]}$ where each $A_{[P]}$ is cyclic and only finitely many $A_{[P]}$ are non-zero. Since every cyclic module $A_{[P]}$ is a submodule of a sum of finite-length modules, it is artinian. Thus R/I is artinian and R is an RM-ring.

(i)⇒(iii). By Theorem 3.4 and Lemma 2.16, $R/\mathrm{Soc}(R)$ is noetherian and every singular module is semiartinian of socle length less than or equal to ω. Let M be a self-small singular module. Then $M = \bigoplus_{P \in \mathrm{Max}(R)} M_{[P]}$ by Lemma 3.6, and hence $M_{[P]} \neq 0$ for only finitely many $[P]$. Since $\mathrm{Hom}(M_{[P]}, M_{[Q]}) = 0$ for all $P \neq Q$, we may suppose that $M = M_{[P]}$ for a single maximal ideal P by [16, Proposition 1.6]. Let M_i denote the ith member of the socle sequence of M. It is easy to see that $M_i = \{m \in M \mid mP^i = 0\}$. Assume that the socle length of M is infinite, i.e. $M_i \neq M_{i+1}$ and $M = \bigcup_{i<\omega} M_i$. Then for each $i < \omega$, there exist $m_i \in M_{i+1} \setminus M_i$ and $p_i \in P^i$ such that $0 \neq m_i p_i \in \mathrm{Soc}(M)$. Then multiplication by p_i is a non-zero endomorphism on M for which $M_i \subseteq \ker p_i$, a contradiction because M is self-small. We have proved that there exists n such that $M_n = M$ and so M has a natural structure of a self-small module over the commutative artinian ring R/P^n. Hence M is finitely generated by [2, Proposition 2.9].

(iii)⇒(i). We follow the proof of [1, Theorem 9]. If I is an essential ideal of R, then $\mathrm{Soc}(R) \subseteq I$, hence R/I is noetherian. Moreover, every self-small module over R/I is singular as an R-module, and so it is finitely generated. Now, the conclusion follows immediately from [2, Proposition 3.17]. ■

REMARK 3.8. Note that Theorem 3.1 is a direct consequence of Theorems 3.4 and 3.7 since singular modules over commutative domains are exactly torsion modules.

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Received 19 June 2014;
revised 5 December 2014 (6292)