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RANGE OF A CONTRACTIVE STRONGLY POSITIVE PROJECTION IN A C^*-ALGEBRA

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Abstract. We generalize a result of Choi and Effros on the range of a contractive completely positive projection in a C^*-algebra to the case when this projection is only strongly positive using, moreover, an elementary argument instead of a 2×2-matrix technique.

Let $\mathcal{A}$ be a C^*-algebra, and let $E\colon \mathcal{A}\to\mathcal{A}$ be a linear contractive projection. In [1] it was shown that if E is *completely positive*, then $E(\mathcal{A})$ is a C^*-algebra with the product

$$a*b=E(ab),\qquad a,b\in E(\mathcal{A}).$$

The crucial point in the proof was the relation

$$E(ab)=E(aEb)\qquad \text{for all } a\in E(\mathcal{A}),\ b\in\mathcal{A},$$

which yielded the associativity of the product. This relation was proven by using a 2×2-matrix technique showing that the conclusion remains true if we only assume 2-positivity of E. In this note, we show by a simple elementary argument that the assumption on positivity of E can be further weakened to *strong positivity* (or in other words to E being a so-called *Schwarz* or $1\frac{1}{2}$ *positive* map), by which is meant that the following Kadison–Schwarz inequality holds:

$$E(a^*a)\geq Ea^*Ea\qquad \text{for all } a\in\mathcal{A}.$$

This seems to be of some interest since such maps appear frequently in ergodic theory of operator algebras (cf. [2, 3, 4]).

Let us explain this statement in more detail. Suppose that $\mathcal{M}$ is a von Neumann algebra, and that $\mathfrak{T}=(T_t : t\in\mathbb{G})$ is an abelian semigroup of linear positive normal unital maps on $\mathcal{M}$, i.e. a *quantum dynamical semigroup*. Denote by $\operatorname{Fix}\mathfrak{T}$ the fixed-point space of the semigroup $\mathfrak{T}$, and by $\operatorname{Fix}\mathfrak{T}_*$ the fixed-point space of the semigroup $\mathfrak{T}_*$, i.e.

$$\operatorname{Fix}\mathfrak{T}_*=\{\varphi\in\mathcal{M}_* : \varphi\circ T_t=\varphi \text{ for all } t\in\mathbb{G}\}.$$

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Now, if the positive elements of $\operatorname{Fix}\mathfrak{T}_*$ form a *faithful family*, by which is meant that for any $0 \le x \in \mathcal{M}$ the equality $\varphi(x) = 0$ for each $\varphi \in \operatorname{Fix}\mathfrak{T}_*$ yields $x = 0$, then the structure of $\operatorname{Fix}\mathfrak{T}$ depends on the degree of positivity of the T_t: if they are strongly positive, then $\operatorname{Fix}\mathfrak{T}$ is a von Neumann algebra, while if they are just positive, then $\operatorname{Fix}\mathfrak{T}$ is a JW*-algebra (cf. [6]). In both cases, there exists a linear normal unital projection E from $\mathcal{M}$ *onto* $\operatorname{Fix}\mathfrak{T}$, and, depending on the degree of positivity of the T_t, E is either a conditional expectation or a Jordan homomorphism (cf. [4, 6]). However, if we have $\operatorname{Fix}\mathfrak{T}_* \neq \{0\}$ but the positive elements of $\operatorname{Fix}\mathfrak{T}_*$ *do not* form a faithful family, then there still exists a linear normal unital projection E from $\mathcal{M}$ *into* $\operatorname{Fix}\mathfrak{T}$, but now, again depending on the degree of positivity of the T_t, E is either strongly positive or simply positive (cf. [3, 5]). An important notion of *mean ergodicity* of the semigroup $\mathfrak{T}$ amounts to the relation

$$E(\mathcal{M}) = \operatorname{Fix}\mathfrak{T},$$

thus the question of the range of the projection is the question of the structure of the fixed-point space of a mean ergodic quantum dynamical semigroup.

Our main result is the following.

PROPOSITION. *Let $\mathcal{A}$ be a C^*-algebra, and let $E\colon \mathcal{A} \to \mathcal{A}$ be a contractive strongly positive projection. Then*

$$E((Ea)b) = E(EaEb) = E(aEb) \quad \textit{for all } a, b \in \mathcal{A}. \tag{1}$$

Proof. Take an arbitrary positive functional $\varphi \in \mathcal{A}^*$ and define a sesquilinear form $[\cdot,\cdot]_\varphi$ on $\mathcal{A}$ by

$$[a, b]_\varphi = \varphi(E(ab^*) - E(EaEb^*)).$$

Then

$$[a, a]_\varphi = \varphi(E(aa^*) - E(EaEa^*)) = \varphi(E(E(aa^*) - EaEa^*)) \ge 0,$$

since $E^2 = E$ and E is strongly positive. Consequently, from the Schwarz inequality we obtain

$$\begin{aligned}|\varphi(E((Ea)b) - E(EaEb))|^2 &= |[Ea, b^*]_\varphi|^2 \le [Ea, Ea]_\varphi [b, b^*]_\varphi \\ &= \varphi(E(EaEa^*) - E(EaEa^*))[b, b^*]_\varphi = 0,\end{aligned}$$

and since φ was arbitrary, we get

$$E((Ea)b) - E(EaEb) = 0.$$

The second equality in (1) is obtained by taking adjoints. ∎

As a corollary we obtain

COROLLARY. *For arbitrary $a \in E(\mathcal{A})$ and $b \in \mathcal{A}$, we have*

$$E(ab) = E(aEb), \qquad E(ba) = E((Eb)a).$$

Now we have the following generalization of a result in [1].

THEOREM. *Let $\mathcal{A}$ and E be as before. Define on $E(\mathcal{A})$ the product $*$ by the formula*

$$a * b = E(ab), \qquad a, b \in E(\mathcal{A}).$$

Then $(E(\mathcal{A}), *)$ is a C^*-algebra.

Proof. The proof goes exactly as in [1]. Since E is a projection, we deduce that $E(\mathcal{A})$ is norm-closed. For any $a, b \in E(\mathcal{A})$, the contractivity of E yields

$$\|a * b\| = \|E(ab)\| \leq \|ab\| \leq \|a\| \, \|b\|.$$

For any $a \in E(\mathcal{A})$, on account of the Kadison–Schwarz inequality, and the equality $Ea = a$, we have

$$\|a^* * a\| = \|E(a^*a)\| \geq \|Ea^*Ea\| = \|a^*a\| = \|a\|^2,$$

which shows that the C^*-condition for the norm holds.

As for the associativity of the product, take arbitrary $a, b, c \in E(\mathcal{A})$. Then the corollary yields

$$a * (b * c) = a * E(bc) = E(aE(bc)) = E(a(bc)) = E(abc)$$

and

$$(a * b) * c = E(ab) * c = E(E(ab)c) = E((ab)c) = E(abc),$$

showing the claim. ∎

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