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SOME NEW LIGHT ON A FEW CLASSICAL RESULTS

BY

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Abstract. The purpose of this paper is to describe a unified approach to proving vector-valued inequalities without relying on the full strength of weighted theory. Our applications include the Fefferman–Stein and Córdoba–Fefferman inequalities, as well as the vector-valued Carleson operator. Using this approach we also produce a proof of the boundedness of the classical bi-parameter multiplier operators, which does not rely on product theory. Our arguments are inspired by the vector-valued restricted type interpolation used by Bateman and Thiele (2013).

1. The General Principle. In this paper we describe an alternative approach to a few well known vector-valued inequalities. One of them leads to an alternative way to estimate bi-parameter linear operators. This approach has already played a crucial role in recent work in the linear setting [1] but also in the context of bilinear operators [14], where weighted estimates were not available. At its core lies restricted type vector-valued interpolation as encoded by the following principle:

THEOREM 1.1 (The General Principle [1]). *Let $p_0, p_1 \in (1,\infty)$ be such that $p_0 < p_1$ and let $\{T_j\}_j$ be a (possibly finite) sequence of sublinear operators on $\mathbb{R}^n$ which are uniformly bounded on L^2. Assume that for $p \in \{p_0, p_1\}$ there is $C_p > 0$ with the following property:*

(P) *for any finite non-zero measure sets $H, G \subset \mathbb{R}^n$ there exist subsets $H' \subset H$ and $G' \subset G$ with*

$$|H'| \geq \tfrac{1}{2}|H|, \qquad |G'| \geq \tfrac{1}{2}|G| \tag{1.1}$$

such that

$$\int |T_j(f1_{H'})|^2 1_{G'} \leq C_p \left(\frac{|G|}{|H|}\right)^{1-2/p} \int |f|^2 \tag{1.2}$$

for each j and each $f \in L^2(\mathbb{R}^n)$.

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Then

$$\Big\| \Big(\sum_j |T_j f_j|^2 \Big)^{1/2} \Big\|_q \lesssim_q \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_q \tag{1.3}$$

for each $p_0 < q < p_1$ *and any* f_j.

It is important to note that the choice of the subsets H', G' as well as the constant C_p are independent of j. In our applications we can always work with either $G' = G$ or $H' = H$ for a given value of p. However we believe that this more general form of the principle may one day find applications.

The hypothesis that $\sup_j \|T_j\|_{L^2 \to L^2} < \infty$ can be easily relaxed, but works fine with our applications.

We use Theorem 1.1 to obtain vector-valued estimates for a family of operators by proving uniform L^2 estimates for a related family of operators. Indeed, if we define

$$S_{j,G',H'}(f) = T_j(f 1_{H'}) 1_{G'},$$

then the estimate (1.2) can be written as

$$\|S_{j,G',H'}\|_{L^2 \to L^2} \lesssim_p (|G|/|H|)^{1/2 - 1/p}.$$

We prove Theorem 1.1 in Section 2. Note that when $p \neq 1/2$, one of the exponents of $|H|$ and $|G|$ becomes negative, so using L^p estimates for T_j with Hölder's inequality does not give the above L^2 estimates for $S_{j,G',H'}$.

In Sections 4, 5 and 6 we give new proofs for three classical results using Theorem 1.1 and elements of the approach described in Section 3. Sections 5 and 6 contain proofs for two classical bi-parameter problems: boundedness of bi-parameter multiplier operators and the Córdoba–Fefferman inequality. We reduce both these problems to vector-valued estimates for single scale operators and then use Theorem 1.1 to prove these vector-valued estimates. The key advantage of this approach is that we avoid product theory or explicit weighted theory and reduce bi-parameter problems to essentially single-parameter problems.

Our first application in Section 4 is a proof of the Fefferman–Stein inequality that avoids explicit use of weighted theory. This proof follows the line of argument from Section 3 in a much simpler setting. In Section 7 we give a similar proof for the vector-valued estimates for the Carleson operator.

Our proofs are in general not easier than the classical ones. This is mostly due to technicalities associated with various decompositions. To keep the exposition as transparent as possible, we choose to focus mainly on how the General Principle 1.1 works in each case, and less on various other technicalities. We caution the reader that various parts of the argument need to be worked out in more detail and draw attention to the large body of literature where most of these details are explained in related contexts.

Our hope is that the approach relying on Theorem 1.1 described in this paper will find further applications. We point out that the employment of this method was critical to the theorems proved in [1] and [14].

2. Proof of Theorem 1.1. A proof appears in [1], but we include it here too, for the reader's convenience.

Using generalized restricted type interpolation in the vector-valued setting, to obtain (1.3) it is enough to show that the l^2-valued sublinear operator $\mathbf{T}$ defined by

$$\mathbf{T}(\mathbf{f}) = (T_j(f_j))_j$$

for each $\mathbf{f} = (f_j)_j$ is *restricted weak-type* (p, p) for $p \in \{p_1, p_2\}$. By this we mean that given any positive measure sets $G, H \subset \mathbb{R}^n$, we have

$$\int_G \|\mathbf{T}(\mathbf{f})(x)\|_{l^2}\, dx \le A_p |H|^{1/p} |G|^{1/p'},$$

with a constant A_p independent of G and H, whenever

$$\|\mathbf{f}(x)\|_{l^2} \le 1_H(x), \qquad \text{a.e. } x.$$

Note that to prove this for a fixed p it suffices to prove the following superficially weaker statement: Let $\gamma = \gamma(p) = 6^{\max(p,p')}$. Then given any positive measure sets $G, H \subset \mathbb{R}^n$, there exist subsets $H' \subset H$ and $G' \subset G$ with $|H'| \ge \frac{\gamma-1}{\gamma}|H|$ and $|G'| \ge \frac{\gamma-1}{\gamma}|G|$ such that

$$\int_{G'} \|\mathbf{T}(\mathbf{f})(x)\|_{l^2}\, dx \le B_p |H|^{1/p} |G|^{1/p'}, \tag{2.1}$$

with a constant B_p independent of G and H, whenever

$$\|\mathbf{f}(x)\|_{l^2} \le 1_{H'}(x), \qquad \text{a.e. } x.$$

Indeed, note first that

$$\begin{aligned}\int_G \|\mathbf{T}(\mathbf{f})(x)\|_{l^2}\, dx \le{}& \int_{G'} \|\mathbf{T}(\mathbf{f}1_{H'})(x)\|_{l^2}\, dx + \int_{G\setminus G'} \|\mathbf{T}(\mathbf{f}1_{H'})(x)\|_{l^2}\, dx \\ &+ \int_{G'} \|\mathbf{T}(\mathbf{f}1_{H\setminus H'})(x)\|_{l^2}\, dx + \int_{G\setminus G'} \|\mathbf{T}(\mathbf{f}1_{H\setminus H'})(x)\|_{l^2}\, dx.\end{aligned}$$

The first term on the right hand side can be bounded by $B_p|H|^{1/p}|G|^{1/p'}$. For the remaining three terms we iterate the decomposition. Note that after k iterations the error term is the sum of 3^k integrals of the form $\int_{G^*} \|\mathbf{T}(\mathbf{f}1_{H^*})(x)\|_{l^2}\, dx$ with $|G^*|\,|H^*| \le (\gamma^{-1})^k |G|\,|H|$. Since the T_j are uniformly bounded on L^2, each of these integrals is bounded by $C|G^*|^{1/2}|H^*|^{1/2}$. The choice of γ forces the error term to go to zero.

By repeating the argument we are led to the upper bound

$$\sum_{k=0}^{\infty} B_p \gamma^{-k\min(1/p,1/p')} 3^k |H|^{1/p}|G|^{1/p'} \le A_p |H|^{1/p}|G|^{1/p'}.$$

Let now G', H' be the subsets provided by property (P) in Theorem 1.1. Using Hölder's inequality, to verify (2.1) it is enough to show that

$$\Big\| \Big(\sum_j |T_j(f_j)|^2 1_{G'} \Big)^{1/2} \Big\|_2 \lesssim_p |H|^{1/p}|G|^{1/p'-1/2}. \tag{2.2}$$

Then using the fact that $\{f_j\}_j$ satisfy $\sum_j |f_j|^2 \le 1_{H'}$, in order to get (2.2) it is enough to show that

$$\sum_j \|T_j(f_j) 1_{G'}\|_2^2 \lesssim_p \left(\frac{|G|}{|H|}\right)^{1-2/p} \sum_j \|f_j\|_2^2. \tag{2.3}$$

But this follows from (1.2).

3. Some results from time frequency analysis. In this section we briefly recall the main tools used in the proof in [8] of Carleson's theorem. These tools will be used in simpler settings to prove our results in the following sections. Let us start by recalling that the *Carleson operator* is defined by

$$Cf(x) = \int_{\mathbb{R}} f(x+t) \frac{e^{iN(x)t}}{t}\, dt, \tag{3.1}$$

where $N : \mathbb{R} \to \mathbb{R}$ is an arbitrary measurable choice function.

The approach developed in [8] relies on a size lemma, a mass lemma, a single tree estimate, as well as on arguments involving obtaining better control over size and mass by removing exceptional sets.

We refer the reader to [8] and [16] for the proofs of these results in the Fourier case. The discussion of the simpler Walsh case can be found in [3].

A first crucial idea in [8] is to decompose the Carleson operator into discrete model operators, where wave packets are used to capture both frequency and spatial localizations.

Definition 3.1 (Tiles and bi-tiles). A *tile* s is a product of two dyadic intervals with area 1, that is, $s = I_s \times \omega_s$ with $|I_s| \times |\omega_s| = 1$. The dyadic intervals I_s and ω_s are respectively called the *spatial interval* and the *frequency interval* of s. A *bi-tile* $P = (P_1, P_2)$ is a pair of tiles P_1, P_2 with $I_{P_1} = I_{P_2} = I_P$ and such that the intervals ω_{P_1} and ω_{P_2} have the same dyadic parent. We denote the frequency interval of the bi-tile P by $\omega_P = \omega_{P_1} \cup \omega_{P_2}$.

Given a finite interval $I \subset \mathbb{R}$ we denote its center by $c(I)$ and by $\tilde{\chi}_I$ the cutoff function given by

$$\tilde{\chi}_I(x) = \left(1 + \left(\frac{x - c(I)}{|I|}\right)^2\right)^{-1/2}. \tag{3.2}$$

Definition 3.2 (Wave packet associated to a tile). Let $s = I_s \times \omega_s$ be a tile. A *wave packet* on s is a smooth function φ_s which has Fourier support in ω_s and obeys the spatial decay estimates

$$\left|\frac{d^\alpha}{dx^\alpha}[e^{-ic(\omega_s)x}\varphi_s(x)]\right| \lesssim_{M,\alpha} |I_s|^{-1/2-\alpha}\tilde{\chi}_{I_s}^M(x), \qquad x \in \mathbb{R},$$

for all $M > 0$ and all non-negative integers α.

The following Fefferman ordering of bi-tiles is used in combinatorial arguments involving organizing the collection of bi-tiles.

Definition 3.3 (Partial ordering of bi-tiles). Given a pair of bi-tiles P, P' we define $P < P'$ to mean

$$I_P \subset I_{P'}, \qquad \omega_{P'} \subset \omega_P.$$

Remark 3.4. One may similarly define product tiles and wave packets in higher dimensions. The combinatorics in that context is much more difficult, in part due to the fact that there is no good substitute for the above partial ordering. To avoid this difficulty, in our forthcoming analysis of the bi-parameter operators we first use Littlewood–Paley theory to reduce matters to one-dimensional vector-valued estimates.

It turns out that the Carleson operator can be written as a superposition of discrete operators of the type

$$\sum_P \langle f, \varphi_{P_1}\rangle \varphi_{P_2}(x) 1_{N^{-1}(\omega_{P_2})}(x), \tag{3.3}$$

where $N^{-1}(\omega_{P_2}) = \{x \in \mathbb{R} : N(x) \in \omega_{P_2}\}$.

To prove the boundedness of Carleson's operator it suffices to prove uniform weak type bounds for the model operator in (3.3), where one can restrict attention to a finite, convex collection of bi-tiles (see [8]). "$\mathbb{P}_0$ is a convex collection of bi-tiles" means that $P' \in \mathbb{P}_0$ whenever $P \le P' \le P''$ and $P, P'' \in \mathbb{P}_0$. Let us fix such a finite collection $\mathbb{P}_0$.

Next we define a tree. This is a collection of tiles whose associate model sum plays the role of the Hilbert transform in the discrete setting.

Definition 3.5 (Tree). A *tree* T with top data (ξ_T, I_T) is a convex collection of bi-tiles such that for all $P \in T$ we have $I_P \subset I_T$ and $\xi_T \in \omega_P$.

Next we recall the notion of *size*.

DEFINITION 3.6 (Size). Let $\mathbf{S} \subset \mathbb{P}_0$ be a collection of bi-tiles and let $f \in L^2(\mathbb{R})$. The *size* of $\mathbf{S}$ with respect to f is defined by

$$\operatorname{size}(\mathbf{S}, f) = \sup_T \left(\frac{1}{|I_T|} \sum_{P \in T} |\langle f, \varphi_{P_1} \rangle|^2 \right)^{1/2},$$

where sup is taken over all the trees T in $\mathbf{S}$ with top data (ξ_T, I_T) that satisfy $\xi_T \in \omega_{P_2}$ for each $P \in \mathbf{T}$.

The following lemma is used to partition a collection of bi-tiles into further subcollections with good control over size.

LEMMA 3.7 (Size Lemma [8]). *Given a convex collection* $\mathbf{S}$ *of bi-tiles and* $f \in L^2(\mathbb{R})$ *there exists a decomposition* $\mathbf{S} = \mathbf{S}_{\text{big}} \cup \mathbf{S}_{\text{small}}$ *such that* $\mathbf{S}_{\text{small}}$ *is convex,* $\operatorname{size}(\mathbf{S}_{\text{small}}) \leq \frac{1}{2}\operatorname{size}(\mathbf{S})$ *and* $\mathbf{S}_{\text{big}} = \bigcup_{T \in \mathcal{F}} T$*, where* $\mathcal{F}$ *is a collection of trees* (*a forest*) *with*

$$\sum_{T \in \mathcal{F}} |I_T| \lesssim (\operatorname{size}(\mathbf{S}))^{-2} \|f\|_2.$$

Next we recall the concept of mass. Given a measurable function $N : \mathbb{R} \to \mathbb{R}$, a bi-tile P and a set $E \subset \mathbb{R}$, define

$$E_P = E \cap \{x : N(x) \in \omega_P\}.$$

DEFINITION 3.8 (Mass). The *mass* of a convex collection of bi-tiles $\mathbf{S}$ is given by

$$\operatorname{mass}(\mathbf{S}, E) = \sup_{P \in \mathbf{S}} \frac{1}{|I_P|} \int_{E_P} \tilde{\chi}_{I_P}^{100}(x)\, dx,$$

Similar to the size decomposition lemma, we have a mass decomposition lemma.

LEMMA 3.9 (Mass Lemma [8]). *Given a convex collection* $\mathbf{S}$ *of bi-tiles there exists a decomposition* $\mathbf{S} = \mathbf{S}_{\text{big}} \cup \mathbf{S}_{\text{small}}$ *such that* $\mathbf{S}_{\text{small}}$ *is convex,* $\operatorname{mass}(\mathbf{S}_{\text{small}}) \leq \frac{1}{2}\operatorname{mass}(\mathbf{S})$ *and* $\mathbf{S}_{\text{big}} = \bigcup_{T \in \mathcal{F}} T$*, where* $\mathcal{F}$ *is a collection of trees with*

$$\sum_{T \in \mathcal{F}} |I_T| \lesssim (\operatorname{mass}(\mathbf{S}))^{-1} |E|.$$

The Mass Lemma and the Size Lemma can be iterated to decompose a given convex collection $\mathbf{S}$ of bi-tiles as

$$\mathbf{S} = \bigcup_{2^{-n} \leq \operatorname{size}(\mathbf{S})} \bigcup_{2^{-m} \leq \operatorname{mass}(\mathbf{S})} \mathbf{S}_{n,m}, \tag{3.4}$$

where $\mathbf{S}_{n,m}$ consists of a collection $\mathcal{F}_{n,m}$ of trees whose size and mass are bounded by 2^{-n} and 2^{-m} respectively and such that

$$\sum_{T \in \mathcal{F}_{n,m}} |I_T| \lesssim \min(2^{2n} \|f\|_2^2, 2^m |E|).$$

We also recall

LEMMA 3.10 (Tree Estimate [8]). *For every tree T,*

$$\sum_{P\in T} |\langle f,\varphi_{P_1}\rangle\langle 1_E,\varphi_{P_2}1_{N^{-1}(\omega_{P_2})}\rangle| \lesssim |I_T|\,\mathrm{size}(T,f)\,\mathrm{mass}(T,E).$$

The final ingredient of the proof of the Carleson theorem is the argument involving removing exceptional sets to get better bounds for the size and mass of a collection of bi-tiles. We need the following estimate on size. Note first that we have the trivial estimate $\mathrm{size}(\mathbf{S},f)\lesssim \|f\|_\infty$.

LEMMA 3.11 (Size Estimate). *If $\mathbf{S}$ is a convex collection of bi-tiles then*

$$\mathrm{size}(\mathbf{S},f)\lesssim \sup_{P\in\mathbf{S}}\inf_{x\in I_P} M(f)(x),$$

where M is the Hardy–Littlewood maximal function.

When we remove exceptional sets to obtain a better bound for mass, we use the following estimate. Note also that we always have the trivial estimate $\mathrm{mass}(\mathbf{S})\lesssim 1$.

LEMMA 3.12 (Mass Estimate). *Let $\mathbf{S}$ be a collection of bi-tiles and $E\subset\mathbb{R}$. Then*

$$\mathrm{mass}(\mathbf{S})\lesssim \sup_{P\in\mathbf{S}}\inf_{x\in I_P} M(1_E)(x).$$

The argument of removing exceptional sets involves decomposing the collection $\mathbf{S}$ of bi-tiles into further subcollections $\mathbf{S}_k$, $k\ge 0$, based on the position relative to the exceptional sets and using Lemma 3.11 or Lemma 3.12 to obtain better estimates for the size and mass of those subcollections. In the arguments from our paper we simply state the bounds we get for the subcollection $\mathbf{S}_0$, and we refer the reader to [11, p. 442] for details on how to deal with $\mathbf{S}_k$, $k>0$.

The proof of the boundedness in Carleson's theorem will follow by combining the decomposition (3.4) with the Tree, Size and Mass Lemmas which become effective outside certain small exceptional sets. The result is a convergent double geometric sum. We refer the reader to [3, Section 6] for the details.

4. The Fefferman–Stein inequality. In this section we give a proof for the Fefferman–Stein inequality in the dyadic case, using Theorem 1.1 and a very rudimentary version of the time-frequency tools recalled in the previous section. Let

$$Mf(x) := \sup_{x\in I}\frac{1}{|I|}\int_I |f(y)|\,dy$$

be the dyadic maximal function, where I runs over all dyadic intervals containing x.

THEOREM 4.1 (Fefferman–Stein [5]). *For each* $1 < p < \infty$ *and any* f_j,

$$\Big\|\Big(\sum_j |Mf_j|\Big)^{1/2}\Big\|_p \lesssim \Big\|\Big(\sum_j |f_j|\Big)^{1/2}\Big\|_p.$$

Proof. We first give the proof in the range $p > 2$, where the classical argument relies on elementary weighted theory. We prove (1.2) for fixed G, H. Since $p > 2$ it suffices to consider the case $|G| \lesssim |H|$. Define

$$H' := H \setminus \bigcup_{I\colon |I\cap G|/|I|\geq c|G|/|H|} I,$$

for sufficiently large c, so that (1.1) holds. Fix an arbitrary measurable

$$\kappa : \mathbb{R} \to \{2^n : n \in \mathbb{Z}\}.$$

For a dyadic interval I define $V_I = \{x \in I : |I| = \kappa(x)\}$. It suffices to check the General Principle with

$$T_j f(x) = \sum_I \frac{1}{|I|}\langle f, 1_I\rangle 1_{V_I}(x),$$

where the sum runs over all dyadic I. Note that in this case all T_j are the same. We will prove (1.2) for each $p > 2$ using restricted interpolation. More precisely, we show that

$$\sum_I \frac{1}{|I|}|\langle 1_{E\cap H'}, 1_I\rangle\langle 1_{F\cap G}, 1_{V_I}\rangle| \lesssim \left(\frac{|G|}{|H|}\right)^{1/s}|E|^{1/s}|F|^{1/s'}$$

for each $1 < s < \infty$. Note that we can restrict the sum to the collection $\mathcal{I}$ of intervals I which intersect H'.

In this case tiles are indexed by intervals and we have the following analogs of size and mass:

$$\text{size}(I) := \frac{1}{|I|}|\langle 1_{E\cap H'}, 1_I\rangle|, \quad \text{mass}(I) := \frac{1}{|I|}|\langle 1_{F\cap G}, 1_{V_I}\rangle|.$$

Of course $\text{size}(I) \lesssim 1$ and moreover $\text{mass}(I) \lesssim |G|/|H|$ for each $I \in \mathcal{I}$. The latter inequality is due to the fact that $I \cap H' \neq \emptyset$, an instance of Lemma 3.12.

Let $\mathcal{I}^*_{n,m}$ be the collection of the maximal intervals in

$$\mathcal{I}_{n,m} := \{I \in \mathcal{I} : \text{size}(I) \sim 2^{-n}, \text{mass}(I) \sim 2^{-m}\}.$$

For $J \in \mathcal{I}^*_{n,m}$ the collection $\{I \in \mathcal{I}_{n,m} : I \subset J\}$ plays the role of a tree, while the collections $\mathcal{I}^*_{n,m}$ play the role of forests from Section 3. Next we prove the following analogue of the single tree estimate (see Lemma 3.10). First note that for each $J \in \mathcal{I}^*_{n,m}$,

$$\sum_{I\subset J:\, I\in\mathcal{I}_{n,m}} \frac{1}{|I|}|\langle 1_{E\cap H'}, 1_I\rangle\langle 1_{F\cap G}, 1_{V_I}\rangle| = \int_{\mathbb{R}} \sum_{I\subset J:\, I\in\mathcal{I}_{n,m}} \frac{1}{|I|}\langle 1_{E\cap H'}, 1_I\rangle 1_{F\cap G} 1_{V_I}.$$

The support of the integral is actually a subset of $J\cap F\cap G$, and since each x receives contribution from only one I, we see that the function we integrate is bounded by 2^{-n}. Thus, the integral is bounded by

$$2^{-n}\,\mathrm{mass}(J)|J| \lesssim 2^{-n-m}|J|.$$

We easily get estimates similar to the ones in the Size Lemma and the Mass Lemma from the previous section. Since each such $J \in \mathcal{I}^*_{n,m}$ is a subset of $\{M1_E > 2^{-n}\}$ and of $\{M1_F > 2^{-m}\}$ we have

$$\sum_{J\in\mathcal{I}^*_{n,m}} |J| \lesssim \min\{2^n|E|, 2^m|F|\}.$$

Note the improvement $2^n|E|$ versus $2^{2n}|E|$ in the Size Lemma. This comes from exploiting disjointness of supports (L^1 orthogonality) versus L^2 orthogonality.

Combining all these estimates we get

$$\sum_{2^{-n}\lesssim 1}\ \sum_{2^{-m}\lesssim |G|/|H|}\ \sum_{I\in\mathcal{I}_{n,m}} \frac{1}{|I|}|\langle 1_{E\cap H'}, 1_I\rangle\langle 1_{F\cap G}, 1_{V_I}\rangle|$$
$$\lesssim \sum_{2^{-n}\lesssim 1}\ \sum_{2^{-m}\lesssim |G|/|H|} 2^{-n-m}\min\{2^n|E|, 2^m|F|\} \lesssim \left(\frac{|G|}{|H|}\right)^{1/s}|E|^{1/s}|F|^{1/s'}$$

for each $1 < s < \infty$. This shows that the operator $M_{H',G}(f) := M(f1_{H'})1_G$ satisfies restricted weak type bounds on L^s. Using the log convexity of the implicit constants in restricted type interpolation, we immediately get $\|M_{H',G}\|_{2\to 2} \lesssim (|G|/|H|)^{1/2}$. This implies (1.2) for each $p > 2$ and as a result the Fefferman–Stein inequality follows in the $2 < p < \infty$ range.

To get the $1 < p < 2$ range one has to repeat the above argument with

$$G' := G \setminus \bigcup_{I:\, |H\cap I|/|I| \ge c|H|/|G|} I$$

for sufficiently large c. We can of course assume $|H| \lesssim |G|$ in this case. By restricting attention to the intervals I which intersect G' we get an improved estimate for the size:

$$\mathrm{size}(I) \lesssim |H|/|G|.$$

The previous computations will give $\|M_{H,G'}\|_{2\to 2} \lesssim (|H|/|G|)^{1/2}$. This implies (1.2) for each $p < 2$ and as a result the Fefferman–Stein inequality follows in the $1 < p < 2$ range. ■

5. Bi-parameter multipliers. Consider multipliers m defined on $\mathbb{R}^2$ which satisfy the following bi-parameter Hörmander–Mikhlin multiplier condition:

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \lesssim \frac{1}{|\xi|^\alpha |\eta|^\beta}$$

for $\xi, \eta \neq 0$ and sufficiently many α, β. It is known that the bilinear multiplier operator associated to m given by

$$Tf(x,y) := \int \widehat{f}(\xi,\eta) m(\xi,\eta) e^{2\pi i(x\xi + y\eta)}\, d\xi\, d\eta$$

is bounded on L^p for $1 < p < \infty$. The classical proofs rely on product BMO and product H^1. Here we give a different proof, whose only use of product theory is via the boundedness of the strong maximal function.

The boundedness of the operator T can be reduced to that of model sums of the form

$$\sum_R \langle f, \phi_R\rangle \psi_R, \tag{5.1}$$

the sum being taken over all dyadic rectangles $R = I_R \times J_R \subset \mathbb{R}^2$. Here

$$|\partial_x^{\alpha_1} \partial_y^{\alpha_2} F_R(x,y)| \lesssim_{M,\alpha} |I_R|^{-1/2-\alpha_1} |J_R|^{-1/2-\alpha_2} \tilde{\chi}_{I_R}^M(x) \tilde{\chi}_{J_R}^M(y) \tag{5.2}$$

for all $M > 0$ and sufficiently many non-negative integers α_i, where $F_R \in \{\phi_R, \psi_R\}$. Moreover, both $\widehat{\phi_R}$ and $\widehat{\psi_R}$ are supported in rectangles of the form $\omega_{R,1} \times \omega_{R,2}$ with $|\omega_{R,1}|\,|I_R| \sim 1$, $|\omega_{R,2}|\,|J_R| \sim 1$ and $\operatorname{dist}(\omega_{R,i}, 0) \sim |\omega_{R,i}|$. See [10] for details.

We prove the following

Theorem 5.1. *For each $2 < p < \infty$ and each f_j we have*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} \Big| \sum_{R = I\times J:\, |J| = 2^j} \langle f_j, \phi_R\rangle \psi_R \Big|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p.$$

The boundedness of the model sums in (5.1) for $p > 2$ will follow by using the Littlewood–Paley square function estimate in the second coordinate, that is, by applying the above vector-valued inequality to $f_j := S_j^2 f$, where

$$\widehat{S_j^2 f}(\xi,\eta) = \widehat{f}(\xi,\eta) 1_{\omega_j}(\eta),$$

and $\operatorname{dist}(\omega_j, 0) \sim |\omega_j| \sim 2^{-j}$. The boundedness for $1 < p < 2$ will follow by duality.

Proof of Theorem 5.1. We apply Theorem 1.1 to

$$T_j f := \sum_{R = I\times J:\, |J| = 2^j} \langle f, \phi_R\rangle \psi_R.$$

Since the scale of J is fixed, this is essentially a one-parameter multiplier and the bound $\|T_j\|_{p\to p}$ for each $1 < p < \infty$ follows via classical one-dimensional theory. It remains to check (1.2) for $p \in (2, \infty)$.

Given G and H, we note that (1.2) with any $G' \subset G$ and $H' \subset H$ follows from the bound $\|T_j\|_{p\to p} \lesssim 1$ in the case $|G| \gtrsim |H|$. Thus it suffices to assume $|G| \lesssim |H|$. In this case define

$$H' := H \setminus \bigcup_{R:\, |R\cap G|/|R| > c_\epsilon (|G|/|H|)^{1-\epsilon}} R,$$

where $\epsilon > 0$ is small enough (it will depend on p) while c_ϵ is large enough so that (1.1) holds. This can be achieved since the strong maximal function

$$M^* f(x, y) := \sup_{(x,y)\in R} \frac{1}{|R|} \int_R |f|$$

maps L^p to L^p for $p > 1$. Also note that the choice of the set H' is independent of j, as desired.

Now consider the operator

$$S_{j,G,H'}(f) = S_j(f) = \sum_{R=I_R\times J_R:\, |J_R|=2^j} \langle f 1_{H'}, \phi_R\rangle \psi_R 1_G. \tag{5.3}$$

Recall that we have to prove that for each $\delta > 0$,

$$\|S_{j,G,H'}(f)\|_2 \lesssim_\delta \left(\frac{|G|}{|H|}\right)^{1/2-\delta} \|f\|_2. \tag{5.4}$$

Using the log convexity of the implicit constants in restricted type interpolation, to prove (5.4) it is enough to show that for $E, F \subset \mathbb{R}$ with finite measure and functions f, g with $|f| \le 1_E, |g| \le 1_F$,

$$\sum_{R=I_R\times J_R:\, |J_R|=2^j} |\langle f 1_{H'}, \phi_R\rangle|\, |\langle \psi_R, g 1_G\rangle| \lesssim_{\epsilon,p} \left(\frac{|G|}{|H|}\right)^{1-\epsilon/p} |E|^{1/p} |F|^{1/p'} \tag{5.5}$$

for each $2 < p < \infty$. Indeed, it will suffice to interpolate this with the following consequence of the one-dimensional type bound $\|T_j\|_{q\to q} \lesssim 1$ (use $q < 2 < p$ with p much closer to 2 than q):

$$\sum_{R=I_R\times J_R:\, |J_R|=2^j} |\langle f 1_{H'}, \phi_R\rangle|\, |\langle \psi_R, g 1_G\rangle| \lesssim_q |E|^{1/q} |F|^{1/q'}. \tag{5.6}$$

The proof of (5.5) follows a simpler version of the approach described in Section 3. We will briefly sketch the details here.

The nice feature of the operators S_j is that they are essentially one-dimensional. In particular, the rectangles $\mathcal{R}_j := \{R : |J_R| = 2^j\}$ are nicely ordered with respect to inclusion. Note that since ϕ_R is mostly concentrated in R, the term $\langle f 1_{H'}, \phi_R\rangle$ will be small if $R \cap H' = \emptyset$. This can be made precise, as described for example in [11]. To keep technicalities to a minimum we will focus only on the contribution coming from the collection $\mathcal{R}_j$ of rectangles such that $R \cap H' \neq \emptyset$.

We have the following versions of the definitions and lemmas from Section 3.

DEFINITION 5.2. A collection $\mathcal{R} \subset \mathcal{R}_j$ is called *convex* if $R \subset R' \subset R''$ and $R, R'' \in \mathcal{R}$ imply that $R' \subset \mathcal{R}$. A *tree* $\mathbf{T}$ with top $R_{\mathbf{T}}$ is a convex collection of rectangles in $\mathcal{R}_j$ such that $R \subset R_{\mathbf{T}}$ for each $R \in \mathbf{T}$.

DEFINITION 5.3. The *size* of a finite collection $\mathcal{R} \subset \mathcal{R}_j$ is defined by

$$\operatorname{size}(\mathcal{R}) := \max_{\mathbf{T} \subset \mathcal{R}} \left(\frac{1}{|R_{\mathbf{T}}|} \sum_{R \in \mathbf{T}} |\langle f 1_{H'}, \phi_R \rangle|^2 \right)^{1/2},$$

where the maximum is taken over all the trees $\mathbf{T}$ in $\mathcal{R}$.

DEFINITION 5.4. The *mass* of a convex collection $\mathcal{R} \subset \mathcal{R}_j$ is

$$\operatorname{mass}(\mathcal{R}) := \max_{R \in \mathcal{R}} \frac{1}{|R|} \int_{F \cap G} \tilde{\chi}_R^{100}.$$

LEMMA 5.5. *For each tree* $\mathbf{T}$ *we have*

$$\sum_{R \in \mathbf{T}} |\langle f 1_{H'}, \phi_R \rangle| \, |\langle \psi_R, g 1_G \rangle| \lesssim |R_{\mathbf{T}}| \operatorname{size}(\mathbf{T}) \operatorname{mass}(\mathbf{T}). \tag{5.7}$$

By using limiting arguments we can and will assume that the sum in (5.5) is over a finite convex collection $\mathcal{R}$ of rectangles.

A key element of our construction of H' is that we have the following improvement over the trivial $O(1)$ bound on the mass:

$$\operatorname{mass}(\mathcal{R}_j) \lesssim_\epsilon \left(\frac{|G|}{|H|} \right)^{1-\epsilon}.$$

The functions ϕ_R are almost orthogonal. As a consequence, Littlewood–Paley theory immediately implies that

$$\operatorname{size}(\mathcal{R}_j) \lesssim 1.$$

Iterate Lemmas 3.7 and 3.9 (see again [8] for details) to decompose

$$\mathcal{R} = \bigcup_{2^{-n} \lesssim 1} \; \bigcup_{2^{-m} \lesssim_\epsilon (|G|/|H|)^{1-\epsilon}} \mathcal{R}_{n,m}, \qquad \operatorname{size}(\mathcal{R}_{n,m}) \le 2^{-n}, \operatorname{mass}(\mathcal{R}_{n,m}) \le 2^{-m},$$

where each $\mathcal{R}_{n,m}$ is the union of a family $\mathcal{F}_{n,m}$ of trees satisfying

$$\sum_{\mathbf{T} \in \mathcal{F}_{n,m}} |R_{\mathbf{T}}| \lesssim \min\{2^{2n}|E|, 2^m |F|\}.$$

The final computations are as follows:

$$\begin{aligned}
&\sum_{R\in\mathcal{R}} |\langle f1_{H'},\phi_R\rangle|\,|\langle\psi_R, g1_G\rangle| \\
&\quad= \sum_{2^{-n}\lesssim 1}\ \sum_{2^{-m}\lesssim_\epsilon(|G|/|H|)^{1-\epsilon}}\ \sum_{\mathbf{T}\in\mathcal{F}_{n,m}}\ \sum_{R\in\mathbf{T}} |\langle f1_{H'},\phi_R\rangle|\,|\langle\psi_R, g1_G\rangle| \\
&\quad\lesssim \sum_{2^{-n}\lesssim 1}\ \sum_{2^{-m}\lesssim_\epsilon(|G|/|H|)^{1-\epsilon}}\ \sum_{\mathbf{T}\in\mathcal{F}_{n,m}} |R_{\mathbf{T}}|2^{-n}2^{-m} \\
&\quad\lesssim \sum_{2^{-n}\lesssim 1}\ \sum_{2^{-m}\lesssim_\epsilon(|G|/|H|)^{1-\epsilon}} 2^{-n}2^{-m}(2^{2n}|E|)^{1/p}(2^m|F|)^{1/p'} \\
&\quad\lesssim_{\epsilon,p}\left(\frac{|G|}{|H|}\right)^{(1-\epsilon)/p}|E|^{1/p}|F|^{1/p'}. \ \blacksquare
\end{aligned}$$

A similar approach can extend the range in Theorem 5.1 to $1<p<\infty$; the details are left to the reader.

6. The Córdoba–Fefferman inequality. For a direction $v\in\mathbb{R}^2$ and $f:\mathbb{R}^2\to\mathbb{C}$ define

$$\widehat{H_v f}(\xi,\eta) := \widehat{f}(\xi,\eta)1_{S_v}(\xi,\eta),$$

where S_v is the half-plane through the origin with normal vector v.

For any collection $\Sigma := \{v_j : j\in\mathbb{Z}\}\in\mathbb{R}^2$ of directions define

$$M_\Sigma f(x,y) = \sup_{(x,y)\in R}\frac{1}{|R|}\int_R |f|,$$

where the supremum is taken over all rectangles R containing (x,y) and whose axes point in the directions $(v,v^\perp)$ with $v\in\Sigma$. In this section we reprove the following result due to Córdoba and Fefferman.

THEOREM 6.1 ([2]). *Consider a collection $\Sigma := \{v_j : j\in\mathbb{Z}\}$ of directions such that $\|M_\Sigma\|_{L^p\to L^{p,\infty}}<\infty$ for some fixed $1<p<\infty$. Then*

$$\Big\|\Big(\sum_j |H_{v_j}f_j|^2\Big)^{1/2}\Big\|_q \lesssim \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_q \tag{6.1}$$

for q in the range

$$\left|1-\frac{2}{q}\right| < \frac{1}{p}.$$

The implicit constant depends on $\|M_\Sigma\|_{L^p\to L^{p,\infty}}$.

Proof. To simplify the exposition we work with the case $p=2$. The result is immediate when $q=2$. Using the fact that H_v is self-dual and the fact that the dual of $L^q(l^2)$ is $L^{q'}(l^2)$, it will suffice to assume $2<q<4$.

Let S_k be appropriate smooth annular truncations supported in the set $\{(\xi,\eta)\in\mathbb{R}^2 : |(\xi,\eta)|\sim 2^k\}$ and such that

$$\sum_{k\in\mathbb{Z}} S_k f = f.$$

In particular, by Littlewood–Paley theory

$$\Big\|\Big(\sum_k |S_k f|^2\Big)^{1/2}\Big\|_q \sim \|f\|_q \tag{6.2}$$

for each $1<q<\infty$.

We next remark that when $q>2$ we also have, for arbitrary f_j,

$$\Big\|\Big(\sum_j\sum_k |S_k f_j|^2\Big)^{1/2}\Big\|_q \sim \Big\|\Big(\sum_j |f_j|\Big)^{1/2}\Big\|_q.$$

This follows from a standard argument based on randomization with a doubly indexed Rademacher sequence. Indeed, on the one hand (see [15, Appendix D]),

$$\int\Big|\sum_k\sum_j r_k(\omega_1)r_j(\omega_2)S_k f_j(x)\Big|^q dx\,d\omega_1\,d\omega_2 \sim \int\Big(\sum_{k,j}|S_k f_j(x)|^2\Big)^{q/2}dx.$$

On the other hand, using (6.2) we get

$$\begin{aligned}
&\int\Big|\sum_k\sum_j r_k(\omega_1)r_j(\omega_2)S_k f_j(x)\Big|^q dx\,d\omega_1\,d\omega_2\\
&\qquad= \int\Big|\sum_k r_k(\omega_1)S_k\Big(\sum_j r_j(\omega_2)f_j\Big)(x)\Big|^q d\omega_1\,dx\,d\omega_2\\
&\qquad\sim\int\Big(\sum_k\Big|S_k\Big(\sum_j r_j(\omega_2)f_j\Big)(x)\Big|^2\Big)^{q/2}dx\,d\omega_2\\
&\qquad\sim\int\Big|\sum_j r_j(\omega_2)f_j(x)\Big|^q d\omega_2\,dx\\
&\qquad\sim\int\Big(\sum_j|f_j(x)|^2\Big)^{q/2}dx.
\end{aligned}$$

Next we prove that

$$\Big\|\Big(\sum_j\sum_k |T_{j,k}f_{j,k}|\Big)^{1/2}\Big\|_q \lesssim \Big\|\Big(\sum_j\sum_k |f_{j,k}|\Big)^{1/2}\Big\|_q,$$

where $T_{j,k}=S_kH_{v_j}$. Note that (6.1) follows from this if we take $f_{j,k}:=S'_kf_j$, with S'_k an appropriate modification of S_k.

We will apply the General Principle to the operators $T_{j,k}$, with $p_0 = 2$ and $p_1 = 4 - \epsilon$. Note that the multiplier $m_{j,k}$ of $T_{j,k}$ satisfies ([1])

$$|\partial_\xi^\alpha \partial_\eta^\beta m_{j,k}(\mathrm{Rot}_{v_j}(\xi,\eta))| \lesssim \frac{1}{|\xi|^\alpha |\eta|^\beta},$$

where Rot_v is the rotation around the origin which maps the x-axis to the line with direction v. As explained in Section 5, we can assume

$$T_{j,k} f = \sum_{R \in \mathbf{S}_{j,k}} \langle f, \phi_R \rangle \psi_R,$$

where $\mathbf{S}_{j,k}$ consists of rectangles in the direction of v_j, with one side of fixed length 2^{-k}. Also $\phi_R(\mathrm{Rot}_{v_j}(x))$ and $\psi_R(\mathrm{Rot}_{v_j}(x))$ will satisfy the same properties as the functions F_R from the previous section.

Let $L := \|M_\Sigma\|_{L^2 \to L^{2,\infty}}$. Given H and G such that $|G| \lesssim |H|$ define

$$H' := H \setminus \bigcup_{j,k} \bigcup_{R \in \mathbf{S}_{j,k} \colon |R \cap G|/|R| \geq c(n|G|/|H|)^{1/2} L} R.$$

It is easy to see that (1.1) is satisfied if c is large enough. Define

$$C_{j,k} f := T_{j,k}(f 1_{H'}) 1_G.$$

We will prove (1.2) with p_1 arbitrarily close to (but less than) 4. That is,

$$\|C_{j,k}\|_{2 \to 2} \lesssim \left(\frac{|G|}{|H|} \right)^\alpha$$

for each $0 < \alpha < 1/4$. We rely on restricted type interpolation.

The argument described in Section 5 will apply here too. Fix $E, F \subset \mathbb{R}$ with finite measure and functions f, g with $|f| \leq 1_E$, $|g| \leq 1_F$. We focus again only on those $R \in \mathbf{S}_{j,k}$ which intersect H'. Thus, for each $2 < p < \infty$,

$$\begin{aligned}
&\sum_{R \in \mathbf{S}_{j,k}} |\langle f 1_{H'}, \phi_R \rangle| \, |\langle \psi_R, g 1_G \rangle| \\
&\qquad \lesssim \sum_{2^{-m} \lesssim (|G|/|H|)^{1/2} L} \; \sum_{2^{-n} \lesssim 1} 2^{-n} 2^{-m} \min\{2^m |F|, 2^{2n} |E|\} \\
&\qquad \lesssim |E|^{1/p} |F|^{1/p'} \left(\frac{|G|}{|H|} \right)^{1/(2p)} L^{1/p.}
\end{aligned}$$

Using restricted type interpolation we get the following for each $p > 2$:

$$\|C_{j,k}\|_{p \to p} \lesssim \left(\frac{|G|}{|H|} \right)^{1/(2p)} L^{1/p}.$$

([1]) In reality the multiplier is only singular with respect to the v axis, but to make the argument more symmetric we pretend it is also singular with respect to the $v^\perp$ axis.

Interpolating the above bound for p very close to 2 with the easy one-dimensional bound $\|C_{j,k}\|_{s\to s} \lesssim 1$ for (any) fixed $s \in (1,2)$ one gets, for each $\epsilon > 0$ small enough,

$$\|C_{j,k}\|_{2\to 2} \lesssim_\epsilon \left(\frac{|G|}{|H|}\right)^{1/4-\delta_1(\epsilon)} L^{1/2-\delta_2(\epsilon)},$$

where $\delta_i(\epsilon) \to 0$ as $\epsilon \to 0$. ∎

The refinement from [6, Section 6.8] of the proof of the Córdoba–Fefferman result, combined with the sharp estimate for the Hilbert transform in weighted spaces [12], proves the following stronger result. In particular it recovers the endpoints $|1-2/q| = 1/p$, but the bound depends on the strong rather than the weak L^p norm of the maximal function. Since our approach relies on interpolation, it does not recover this stronger form of the result. We present this argument for the reader's convenience.

Theorem 6.2. *Consider a collection $\Sigma := \{v_j : j \in \mathbb{Z}\}$ of vectors such that $\|M_\Sigma\|_{L^p\to L^p} < \infty$ for some fixed $1 < p < \infty$. Then for any functions f_j and each q such that*

$$\left|1 - \frac{2}{q}\right| \le \frac{1}{p}$$

we have

$$\Big\|\Big(\sum_j |H_{v_j} f_j|^2\Big)^{1/2}\Big\|_q \lesssim \|M_\Sigma\|_{p\to p}^{p|1-2/q|} \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_q. \tag{6.3}$$

Proof. First, recall a general result about weights u on $\mathbb{R}$. Assume that u is an A_1 weight, that is,

$$Mu(x) \le \|u\|_{A_1} u(x), \qquad \text{a.e. } x.$$

Then

$$\begin{aligned}\|u\|_{A_2} &:= \sup_{I\subset\mathbb{R}:\, I \text{ interval}} \left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I u^{-1}(x)\,dx\right) \\ &\le \sup_{I\subset\mathbb{R}:\, I \text{ interval}} 2 \inf_{x\in I} Mu(x) \sup_{x\in I} \frac{1}{u(x)} \le 2\|u\|_{A_1}.\end{aligned}$$

Using this and the sharp result in [12] we deduce that

$$\int |Hf|^2 u \le C\|u\|_{A_1}^2 \int |f|^2 u, \tag{6.4}$$

where H is the Hilbert transform, and C is independent of f and u.

By duality it suffices to assume $q \ge 2$. Take $g \in L^p(\mathbb{R}^2)$. Define

$$w(x) = \sum_{k=0}^{\infty} \frac{1}{(2\|M_\Sigma\|_{L^p\to L^p})^k} M_\Sigma^k g(x),$$

where M_{Σ}^k is the composition of M_{Σ} with itself k times. Note that

$$M_{\Sigma}w(x) \le 2\|M_{\Sigma}\|_{L^p\to L^p}w(x), \qquad g(x) \le w(x), \qquad \|w\|_p \le 2\|g\|_p.$$

Using these and also (6.4) we get

$$\int |H_{v_j}f_j|^2 g \le \int |H_{v_j}f_j|^2 w \le 4C\|M_{\Sigma}\|_{L^p\to L^p}^2 \int |f_j|^2 w.$$

Next note that by interpolation it is enough to consider the endpoint $q = 2p'$. This case follows from

$$\Big\|\Big(\sum_j |H_{v_j}f_j|^2\Big)^{1/2}\Big\|_{2p'}^2 = \Big\|\sum_j |H_{v_j}f_j|^2\Big\|_{p'} = \sup_{g\in L^p,\, \|g\|_p\le 1} \int \sum_j |H_{v_j}f_j|^2 g$$
$$\lesssim \|M_{\Sigma}\|_{L^p\to L^p}^2 \sup_{g\in L^p,\, \|g\|_p=1} \int \sum_j |f_j|^2 w \lesssim \|M_{\Sigma}\|_{L^p\to L^p}^2 \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{2p'}^2,$$

where in the last inequality we have used the fact that $\|w\|_p \le 2\|g\|_p \le 2$. ■

7. Vector-valued estimates for the Carleson operator. In this section we sketch the proof of the vector-valued estimates for the Carleson operator defined in (3.1).

THEOREM 7.1 ([13]). *Let $1 < p < \infty$. Then for any f_j,*

$$\Big\|\Big(\sum_j |Cf_j|^2\Big)^{1/2}\Big\|_p \lesssim \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_p.$$

The classical proof from [13] relies on weighted estimates for the Carleson operator; see also [7] and [9, Section 8, Remark 1]. Our approach relies on the fact that Carleson's operator is bounded and on standard refinements of the proof of its boundedness. We again refer the reader to Section 3 for the relevant tools.

Proof of Theorem 7.1. For two sets A, B the operator $S_{A,B}$ is defined by

$$S_{A,B}(f) = T(f1_B)1_A.$$

Here T is the model sum operator in (3.3). First consider the case when $p > 2$. Given sets G, H with $|G| \lesssim |H|$ define $H' = H \setminus \{M1_G \gtrsim |G|/|H|\}$. It is enough to prove that for $\epsilon > 0$,

$$\|S_{G,H'}\|_{L^2\to L^2} \lesssim_{\epsilon} \left(\frac{|G|}{|H|}\right)^{1/2-\epsilon}. \tag{7.1}$$

As before, to keep the argument as non-technical as possible we only focus on the main contribution, the one coming from bi-tiles whose spatial intervals intersect H'.

Let $t > 2$. Fix $|f| \le 1_E$, $|g| \le 1_F$. Then using the bound for the mass $2^{-m} \lesssim |G|/|H|$ guaranteed by the definition of H' and Lemma 3.12, the trivial bound on the size $2^{-n} \lesssim 1$ and the machinery described in Section 3 we get

$$\begin{aligned}|\langle S_{G,H'}(f), g\rangle| &= |\langle T(f1_{H'}), g1_G\rangle| \\ &\lesssim \sum_{2^{-n}\lesssim 1} \sum_{2^{-m}\lesssim |G|/|H|} 2^{-n-m}(2^m|F|)^{1/t'}(2^{2n}|E|)^{1/t} \\ &\lesssim \left(\frac{|G|}{|H|}\right)^{1/t} |E|^{1/t}|F|^{1/t'}.\end{aligned}$$

Interpolate this bound for $t > 2$ very close to 2 with the classical $O(1)$ restricted bound below 2 for the Carleson operator to get the desired estimate.

Assume now $p < 2$. In this case we remove an exceptional set from G. Given $|G| \gtrsim |H|$ define $G' = G \setminus \{M1_H \gtrsim |H|/|G|\}$. It will suffice to prove the operator norm estimate

$$\|S_{G',H}\|_{L^2\to L^2} \lesssim_\epsilon \left(\frac{|H|}{|G|}\right)^{1/2-\epsilon}$$

for each $\epsilon > 0$. We only focus on the bi-tiles whose spatial intervals intersect G'.

Let $t > 2$. Fix $|f| \le 1_E$, $|g| \le 1_F$. Then using the bound for the size $2^{-n} \lesssim |H|/|G|$ guaranteed by the definition of G' and Lemma 3.11, the trivial bound on the mass $2^{-m} \lesssim 1$ and the machinery described in Section 3 we get

$$\begin{aligned}|\langle S_{G',H}(f), g\rangle| &= |\langle C(f1_H), 1_{G'}g\rangle| \\ &\lesssim \sum_{2^{-m}\lesssim 1} \sum_{2^{-n}\lesssim |H|/|G|} 2^{-n-m}(2^m|F|)^{1/t'}(2^{2n}|E|)^{1/t} \\ &\lesssim \left(\frac{|H|}{|G|}\right)^{1-2/t} |E|^{1/t}|F|^{1/t'}.\end{aligned}$$

Interpolate this bound for $t \to \infty$ with the classical $O(1)$ restricted bound for T near L^1 to get again the desired estimate. ∎

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