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TREES OF MANIFOLDS WITH BOUNDARY

BY

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Abstract. We introduce two new classes of compacta, called trees of manifolds with boundary and boundary trees of manifolds with boundary. We establish their basic properties.

1. Introduction. In this paper we describe two classes of metric compacta. Each space in each of the two classes is uniquely determined by a fixed, countable (finite or infinite) family $\mathcal{M}$ of compact topological manifolds with boundary, all having the same dimension n. The spaces in the first class, denoted by $\mathcal{X}(\mathcal{M})$, are called *regular trees of manifolds with boundary*, while those in the second class, denoted by $\mathcal{X}_{\mathrm{bd}}(\mathcal{M})$, are called *regular boundary trees of manifolds with boundary*. The spaces in each class are typically "wild", e.g. they are not ANRs and, as we show in this paper, their topological dimension is equal to $n-1$. To show that the corresponding spaces are uniquely determined by $\mathcal{M}$ and to calculate their dimension, we derive a few not very well known properties of topological manifolds with boundary.

Trees of manifolds with boundary are analogues of spaces which are called trees of closed manifolds. In the orientable case they were examined by W. Jakobsche [J1], [J2], who described them in terms of inverse limits of certain inverse systems of closed oriented manifolds. Every regular tree of closed manifolds depends on a closed oriented n-manifold L and a family $\mathcal{M}$ of closed oriented n-manifolds, and we denote it, following W. Jakobsche, by $X(L,\mathcal{M})$. Similar constructions were considered earlier in different contexts by L. S. Pontryagin [P], R. F. Williams [W] and F. D. Ancel and L. C. Siebenmann [AS]. Jakobsche's construction was extended to the nonorientable case by P. R. Stallings [St].

Trees of closed manifolds occur as boundaries of nonpositively curved groups. H. Fisher [F] claimed that the boundary of a right-angled Coxeter group, whose nerve is a flag PL-triangulation of a closed oriented manifold N,

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is homeomorphic to $X(N, \{N\})$. J. Świątkowski [S] corrected a minor mistake in Fisher's argument and showed that this boundary is actually homeomorphic to $X(N, \{N, \bar{N}\})$, where $\bar{N}$ denotes the oppositely oriented copy of N. P. Przytycki and J. Świątkowski [PS] showed that for any closed 3-manifold N there is a flag-no-square PL-triangulation of N, and thus trees of 3-manifolds occur as boundaries of hyperbolic right-angled Coxeter groups. The present author [Z] showed that Pontryagin spheres, both orientable and nonorientable, which are trees of tori and projective planes respectively, occur as boundaries of some systolic groups.

In view of the foregoing a natural question arises if (boundary) trees of manifolds with boundary occur as boundaries of nonpositively curved groups. The author expects that, after an appropriate extension, the ideas of [F] can be used to show that boundaries of some right-angled Coxeter groups, whose nerves are PL-triangulations of manifolds with boundary, are homeomorphic to appropriate boundary trees of manifolds with boundary. This conjecture, suggested to the author by J. Świątkowski, is explained and formulated more precisely in Section 6.

The paper is organized as follows. In Section 2 we recall a construction introduced by J. Świątkowski, called the *limit of a tree system of compact metric spaces*. We also recall some properties of spaces resulting from this construction. Trees of closed manifolds and (boundary) trees of manifolds with boundary are special cases of this construction. In Section 3 we examine some properties of topological manifolds which may be well known but whose proofs seem to be missing in the literature. The main results of this section are Facts 3.1 and 3.3, which we use later in Section 5. Section 4 contains the proofs of isotopic versions of Toruńczyk's Lemma (Lemmas 4.4 and 4.5), which, together with Fact 3.1, are crucial to the proofs of uniqueness of the spaces $\mathcal{X}(\mathcal{M})$ and $\mathcal{X}_{\mathrm{bd}}(\mathcal{M})$. In Section 5 we describe the construction of (boundary) trees of manifolds with boundary and we show that for any family $\mathcal{M}$ of compact n-manifolds with boundary there is exactly one (up to homeomorphism) regular (boundary) tree of manifolds from $\mathcal{M}$. We also calculate the topological dimension of these spaces. Section 6 contains a precise formulation of the conjecture mentioned above, concerning the appearance of boundary trees of manifolds with boundary as ideal boundaries of certain groups.

2. Trees of metric compacta. In this section we recall the notion and basic properties of trees of metric compacta—spaces resulting from the construction introduced by J. Świątkowski. The proofs of all results in this section can be found in [S].

We start by recalling some terminology and notation concerning trees. Let T denote a tree (usually countable and locally infinite). We denote by V_T

and O_T the sets of vertices and oriented edges of T respectively. For an edge $e \in O_T$ let $\alpha(e)$ and $\omega(e)$ denote its initial and terminal vertex respectively, and $\bar{e}$ the opposite edge.

For a subtree $S \subset T$ set $N_S = \{e \in O_T : \alpha(e) \in V_S \text{ and } \omega(e) \notin V_S\}$, i.e. N_S is the set of oriented edges of T starting at some vertex of S and terminating outside S. For simplicity we write N_t instead of $N_{\{t\}}$.

An (*embedded*) *combinatorial path* is a sequence of consecutive vertices (denoted by $[v_0, v_1, \ldots, v_n]$ or briefly $[s, t]$, where s and t are the initial and final vertices respectively) or a sequence of consecutive oriented edges (denoted by $[e_1, \ldots, e_m]$). A *ray* ρ is an infinite combinatorial path, and $e_1(\rho)$ denotes the first edge of ρ. We denote by R_T the set of (properly embedded) rays in T, and by E_T the set of *ends* of T (i.e. equivalence classes of rays, two rays being *equivalent* if the corresponding vertices remain a bounded distance apart). For a ray ρ we denote by $[\rho]$ its end.

Suppose now that for every $t \in V_T$ we are given a compact metric space K_t. For every $e \in O_T$ let $\Sigma_e \subset K_{\alpha(e)}$ be a compact subset and let $\phi_e : \Sigma_e \to \Sigma_{\bar{e}}$ be a homeomorphism such that $\phi_{\bar{e}} = \phi_e^{-1}$. Suppose additionally that $\{\Sigma_e : e \in N_t\}$ is a null family of pairwise disjoint subsets of K_t for every $t \in V_T$ (a family $\mathcal{A}$ of subsets of a metric space is *null* if for every $\epsilon > 0$ only finitely many $A \in \mathcal{A}$ satisfy $\operatorname{diam}(A) \geq \epsilon$).

Definition 2.1. A *tree system of metric compacta* is a tuple

$$\Theta = (T, \{K_t : t \in V_T\}, \{\Sigma_e : e \in O_T\}, \{\phi_e : e \in O_T\})$$

satisfying the conditions stated above.

To every tree system Θ of metric compacta we associate a topological space $\lim \Theta$, called the *limit* of Θ, in the following way. As a set,

$$\lim \Theta = \#\Theta \cup E_T$$

where $\#\Theta = (\bigcup_{t \in V_T} K_t)/\sim$ is the set of equivalence classes of the relation $\sim$ induced by $x \sim \phi_e(x)$ for all $e \in O_T$ and $x \in \Sigma_e$.

We will need some terminology and notation to describe the topology of $\lim \Theta$.

For a family $\mathcal{A}$ of subsets of X and a subset $U \subset X$ we say that U is $\mathcal{A}$-*saturated* if for every $A \in \mathcal{A}$ we have either $A \subset U$ or $A \cap U = \emptyset$.

For a finite subtree $F \subset T$ let

$$\Theta_F = (F, \{K_t : t \in V_F\}, \{\Sigma_e : e \in O_F\}, \{\phi_e : e \in O_F\})$$

denote the restriction of Θ to F and let $K_F = \#\Theta_F$ (equipped with the quotient topology). Set $\mathcal{A}_F = \{\Sigma_e : e \in N_F\}$ and note that $\mathcal{A}_F$ is a null family of pairwise disjoint compact subsets of K_F.

For any $\mathcal{A}_F$-saturated subset $U \subset K_F$ let $N_U = \{e \in N_F : \Sigma_e \subset U\}$ and $D_U = \{t \in V_T : [t, \omega(e)] \cap V_F = \emptyset \text{ for some } e \in N_U\}$. Denote by R_F the set of rays in T with initial vertex in V_F and all other vertices outside V_F; set also $R_U = \{\rho \in R_F : e_1(\rho) \in N_U\}$ and $E_U = \{[\rho] : \rho \in R_U\}$. Finally, set

$$G(U) = \Big(U \cup \bigcup_{t \in D_U} K_t\Big)/_{\sim} \cup E_U.$$

The topology on $\lim \Theta$ is described in the following:

FACT 2.2.

(i) ([S, Section 1.C]) *The family*

$$\{G(U) : U \subset K_F \text{ open and } \mathcal{A}_F\text{-saturated, } F \subset T \text{ a finite subtree}\}$$

is a basis for a topology on $\lim \theta$.

(ii) ([S, Proposition 1.C.1]) *For any tree system* Θ *of metric compacta,* $\lim \Theta$ *equipped with the above topology is a compact metrizable space.*

We now recall the definition of isomorphism of tree systems of metric compacta and the main property of limits of isomorphic tree systems. We will use this property in Section 5.

Let $\Theta = (T, \{K_t : t \in V_T\}, \{\Sigma_e : e \in O_T\}, \{\phi_e : e \in O_T\})$ and $\Theta' = (T', \{K'_{t'} : t' \in V_{T'}\}, \{\Sigma_{e'} : e' \in O_{T'}\}, \{\phi_{e'} : e' \in O_{T'}\})$ be two tree systems of metric compacta. An *isomorphism of tree systems*, $F : \Theta \to \Theta'$, is a tuple $F = (\lambda, \{f_t : t \in V_T\})$ such that:

- $\lambda : T \to T'$ is an isomorphism of trees,
- for each $t \in V_T$ the map $f_t : K_t \to K'_{\lambda(t)}$ is a homeomorphism,
- for each $e \in N_t$ we have $f[\Sigma_e] = \Sigma'_{\lambda(e)}$,
- for each $e \in N_t$ we have $\phi'_{\lambda(e)} \circ (f_{\alpha(e)} \restriction \Sigma_{\alpha(e)}) = f_{\omega(e)} \circ \phi_e$.

The property of limits of isomorphic tree systems mentioned before the above definition is described in the following:

FACT 2.3 ([S, Lemma 1.E.1]). *If* Θ *and* Θ' *are isomorphic tree systems of metric compacta, then their limits* $\lim \Theta$ *and* $\lim \Theta'$ *are homeomorphic.*

We now recall some procedure of changing one tree system into another without affecting its limit (called *consolidation*), which roughly speaking consists in gluing constituent spaces together into bigger ones. We will use this procedure in Section 5.

Let $\Theta = (T, \{K_t : t \in V_T\}, \{\Sigma_e : e \in O_T\}, \{\phi_e : e \in O_T\})$ be a tree system of metric compacta and let Π be a partition of T into subtrees (i.e. elements of Π are disjoint subtrees of T such that $V_T = \bigcup\{V_S : S \in \Pi\}$). We allow some of the subtrees $S \in \Pi$ to be *trivial*, i.e. to consist of a single vertex.

Let T_Π be a tree such that $V_{T_\Pi} = \Pi$ and $O_{T_\Pi} = \{e \in O_T : e \notin \bigcup_{S \in \Pi} O_S\}$. For every $e \in O_{T_\Pi}$ there is exactly one $S \in \Pi$ such that

$\alpha(e) \in V_S$ and we set $\alpha_\Pi(e) = S$. Similarly, for every $e \in O_{T_\Pi}$ there is exactly one $S \in \Pi$ such that $\omega(e) \in V_S$, and we set $\omega_\Pi(e) = S$.

For every $S \in \Pi$ set $K_S = \lim \Theta_S$ and note that, up to canonical inclusions, $\{\Sigma_e : e \in N_S\}$ is a null family of pairwise disjoint compact subsets of $K_S = K_{\alpha_\Pi(e)}$. Thus the tuple

$$\Theta_\Pi = (T_\Pi, \{K_S : S \in \Pi\}, \{\Sigma_e : e \in O_{T_\Pi}\}, \{\phi_e : e \in O_{T_\Pi}\})$$

is a tree system of metric compacta. We call it the *consolidation of Θ with respect to Π*.

FACT 2.4 ([S, Theorem 3.A.1]). *Let Θ be a tree system of metric compacta and let Π be a partition of its underlying tree. Then* $\lim \Theta$ *and* $\lim \Theta_\Pi$ *are canonically homeomorphic.*

We now focus on the topological dimension of trees of metric compacta. Recall from [J2] that the dimension of a tree of closed n-manifolds is equal to n. Using Fact 2.5 below we will show in Section 5 that the dimension of a regular (boundary) tree of n-manifolds with boundary is equal to $n-1$.

Before formulating Fact 2.5 we recall some terminology and notation. Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta. For every $e \in O_T$ set $\hat{K}_e = K_{\omega(e)}$ and suppose that we are given a map $\delta_e : \hat{K}_e \to \Sigma_e$ such that $\delta_e\lceil_{\Sigma_{\bar{e}}} = \phi_{\bar{e}}$. The reason for introducing this new symbol $\hat{K}_e$ is to be consistent with [S].

A tuple $\mathcal{E} = (\{\hat{K}_e : e \in O_T\}, \{\delta_e : e \in O_T\})$ is called a *trivial associated family of extended spaces and maps for Θ* (see [S] for more details concerning families of extended spaces).

Let now $\gamma = (e_1, \dots, e_m)$ be any finite combinatorial path in T of length $m \geq 2$. Consider the maps

$$\Sigma_{e_m} \subset \hat{K}_{e_{m-1}} \xrightarrow{\delta_{e_{m-1}}} \Sigma_{e_{m-1}} \subset \cdots \xrightarrow{\delta_{e_2}} \Sigma_{e_2} \subset \hat{K}_{e_1} \xrightarrow{\delta_{e_1}} \Sigma_{e_1}$$

and let $\delta_\gamma : \Sigma_{e_m} \to \Sigma_{e_1}$ be the composition $\delta_\gamma = \delta_{e_1} \circ \delta_{e_2} \circ \cdots \circ (\delta_{e_{m-1}}\lceil_{\Sigma_{e_m}})$.

We say that the asociated family $\mathcal{E}$ of extended spaces and maps is *fine* if for each $e \in O_T$ the family of images $\delta_\gamma[\Sigma_{e_m}]$ (where e_m is the terminal edge of γ) for all combinatorial paths in T of length ≥ 2 starting with $e_1 = e$ is a null family of subsets of Σ_e.

FACT 2.5 ([S, Proposition 2.D.2]). *Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta such that* $\sup\{\dim(K_t) : t \in V_T\} = k < \infty$. *Suppose additionally that Θ admits a trivial associated family $\mathcal{E}$ of extended spaces and maps which is fine. Then* $\dim(\lim \Theta) = k$.

3. A few properties of topological manifolds. In this section we study some basic properties of topological manifolds. In particular we show that (modulo orientation) there is exactly one (up to isotopy) flat embedding of an n-cell into a connected n-manifold (Fact 3.1) and that a manifold with disconnected boundary can be retracted onto its spherical boundary component (Fact 3.3). These two statements will be used in Section 5.

Although the results from this section seem to be well known, the author could not find their proofs in the literature and hence decided to prove them.

First, we state some terminology and notation. We denote by $\mathbb{D}^n$ the standard n-dimensional disc and by $\mathbb{S}^n$ the standard n-dimensional sphere, which is the boundary of $\mathbb{D}^{n+1}$. A space homeomorphic to $\mathbb{D}^n$ is called an *n-cell*. For a subset A of a topological space X we denote by $\mathrm{cl}(A)$ and $\mathrm{bd}(A)$ its closure and boundary respectively. For a manifold M we denote by ∂M its boundary and by $\mathrm{int}(M)$ its interior.

Recall that an embedding $f : A \to X$ is *bicollared* if there is an embedding $F : A \times [-1, 1] \to X$ such that $F(a, 0) = f(a)$. We say that $A \subset X$ is *bicollared* (*in* X) if the inclusion map is bicollared. An n-cell D (resp. $(n-1)$-sphere S) embedded in an n-manifold M is called *flat* (*in* M) if there is an open neighbourhood $U \subset M$ of D (resp. S) homeomorphic to $\mathbb{R}^n$ such that the pair (U, D) (resp. (U, S)) is homeomorphic to $(\mathbb{R}^n, \mathbb{D}^n)$ (resp. $(\mathbb{R}^n, \mathbb{S}^{n-1})$). An n-cell D embedded in an n-manifold M with boundary is called *flat in* $(M, \partial M)$ if there is an open neighbourhood $U \subset M$ of D homeomorphic to $\mathbb{R}^n_+$ such that (U, D) is homeomorphic to $(\mathbb{R}^n_+, \mathbb{R}^n_+ \cap \mathbb{D}^n)$.

Note that a flat $(n-1)$-sphere bounds a flat n-cell, and the boundary of a flat n-cell is a flat $(n-1)$-sphere. By [B] every $(n-1)$-sphere bicollared in $\mathbb{R}^n$ is flat. It is obvious that an $(n-1)$-sphere flat in $\mathbb{R}^n$ is bicollared.

We are now ready to sketch the proof of the first main result of this section.

FACT 3.1. *Let D and D' be two flat n-cells embedded in a connected n-manifold M. Let $f : D \to D'$ be a homeomorphism. If M is orientable, suppose additionally that f preserves orientation. Then there is a homeomorphism $\bar{f} : M \to M$ isotopic to the identity such that $\bar{f}\lceil_D = f$.*

Proof. Choose a sequence $D = D_0, D_1, \ldots, D_m = D'$ of flat n-cells in M such that $\mathrm{int}(D_i) \cap \mathrm{int}(D_{i+1}) \neq \emptyset$ for $i = 0, 1, \ldots, m-1$ and let $E_i \subset \mathrm{int}(D_i) \cap \mathrm{int}(D_{i+1})$ be a flat n-cell for $i = 0, 1, \ldots, m-1$. We want to choose orientations on D_i's and E_i's so that:

(1) the inclusions $E_i \subset D_i$ and $E_i \subset D_{i+1}$ are orientation preserving,
(2) $f : D \to D'$ is orientation preserving.

If M is orientable, we can choose an orientation on it and take the induced orientations on D_i's and E_i's. If M is non-orientable, an arbitrary choice of orientation on D_1 will induce orientations on the remaining D_i's and E_i's so that (1) holds. However, (2) may not hold: $f : D \to D'$ may be orientation reversing. In this case, since M is nonorientable, it contains orientation reversing loops and therefore we can reroute the sequence $D = D_0, D_1, \ldots, D_m = D'$ along such a loop. This will result in reversing the orientation of D', making $f : D \to D'$ orientation preserving. Thus, in either case, we can assume the D_i's and E_i's are oriented so that properties (1) and (2) hold.

The Annulus Theorem ([R], [Mo], [K], [Q], [E]) implies that the spaces $D_i \setminus \mathrm{int}(E_i)$ and $D_{i+1} \setminus \mathrm{int}(E_i)$ are homeomorphic to $\mathbb{S}^{n-1} \times [0,1]$. Use the product structures (together with exterior collars on the ∂D_i's and interior collars on the ∂E_i's) to produce a sequence $h_1, \ldots, h_{2m-2}$ of isotopies of M, each starting at the identity, so that at time 1: h_1 moves D_1 to E_1, h_2 moves E_1 to D_2, h_3 moves D_2 to $E_2, \ldots, h_{2m-3}$ moves D_{m-1} to E_{m-1} and h_{2m-2} moves E_{m-1} to D_m. Each of these time 1 maps is clearly orientation preserving. Stack these isotopies to obtain an isotopy of M such that H_0 is the identity and $H_1[D] = D'$. Thus $H_1\lceil_D\colon D \to D'$ preserves orientation. Therefore, $f \circ H_1^{-1}\lceil_{D'}\colon D' \to D'$ and $f \circ H_1^{-1}\lceil_{\partial D'}\colon \partial D' \to \partial D'$ preserve orientation.

The Annulus Theorem implies that all orientation preserving homeomorphisms of $\mathbb{R}^{n-1}$ are stable, and hence isotopic to the identity (see [BG, Theorem 9.4]). It follows that all orientation preserving homeomorphisms of $\mathbb{S}^{n-1}$ are isotopic to the identity. Indeed, a given orientation preserving homeomorphism of $\mathbb{S}^{n-1}$ can be isotoped to make it fix a point of $\mathbb{S}^{n-1}$. Then apply the preceding statement in the complement of that point to isotope the homeomorphism of $\mathbb{S}^{n-1}$ to the identity.

Hence, $f \circ H_1^{-1}\lceil_{\partial D'}\colon \partial D' \to \partial D'$ is isotopic to the identity. Using a bicollar on $\partial D'$, we can run this isotopy tapering it off near the ends of the bicollar to obtain an isotopy G_t of M such that G_0 is the identity, $G_1[D'] = D'$ and $G_1\lceil_{\partial D'} = f \circ H_1^{-1}\lceil_{\partial D'}$. Thus, $G_t \circ H_t$ is an isotopy of M such that $G_0 \circ H_0$ is the identity, $G_1 \circ H_1[D] = D'$ and $G_1 \circ H_1\lceil_{\partial D} = f\lceil_{\partial D}$. Let $k = f \circ (G_1 \circ H_1)^{-1} : D' \to D'$. Then k is a homeomorphism of D' such that $k\lceil_{\partial D'}$ is the identity. Identify D' with $\mathbb{D}^n$ and define the *Alexander isotopy* A_t of D' by $A_t(x) = x$ for $t \le \|x\| \le 1$ and $A_t(x) = tk(x/t)$ for $0 \le \|x\| \le t$. Then A_0 is the identity, $A_t\lceil_{\partial D'}$ is the identity and $A_1 = k$. We can extend A_t to an isotopy of M by declaring A_t to be the identity on $M \setminus D'$. Now $A_t \circ G_t \circ H_t$ is an isotopy of M such that $A_0 \circ G_0 \circ H_0$ is the identity and $A_1 \circ G_1 \circ H_1\lceil_D = f$. To finish the proof, let $\bar{f} = A_1 \circ G_1 \circ H_1$. ■

In the proof of Fact 3.3 below, which is the second main result of this section, to be used in Section 5.3, we will need the following technical lemma. The idea of its proof is due to Krzysztof Omiljanowski.

Lemma 3.2. *Let M be a connected n-manifold with nonempty boundary and $x_1, x_2 \in \partial M$ be different. Then there is an embedding $f\colon \mathbb{D}^{n-1} \times [0,1] \to M$*

such that:

(i) $f(\bar{0},0) = x_1$ *and* $f(\bar{0},1) = x_2$,
(ii) $f^{-1}[\partial M] = \mathbb{D}^{n-1} \times \{0,1\}$.

Proof. Let $U_1, U_2 \subset M$ be open neigbourhoods of x_1 and x_2 respectively homeomorphic to $\mathbb{R}^n_+$ such that $\mathrm{cl}(U_1)$ and $\mathrm{cl}(U_2)$ are disjoint n-cells. For $i = 1,2$ choose $x'_i \in U_i \cap \mathrm{int}(M)$ and let $D \subset \mathrm{int}(M)$ be a flat n-cell containing x'_1 and x'_2 in its interior.

For $i = 1,2$ let $\alpha_i : [0,1] \to U_i$ be an arc connecting x_i to x'_i and let $[x_i, x'_i]$ denote its image. We can assume that $[x_i, x'_i] \cap \partial(\mathrm{cl}(U_i)) = \{x_i\}$. Since ∂D disconnects M, it follows that there is a minimal $t_i \in [0,1]$ such that $\alpha_i(t_i) \in \partial D$. Set $x''_i = \alpha_i(t_i)$ and note that $\alpha_i\lceil_{[0,t_i]}$ is an arc such that $[x_i, x''_i] \cap D = \{x''_i\}$.

Let $D''_i \subset \partial D \cap U_i$ be a flat $(n-1)$-cell containing x''_i in its interior. Since D''_i is flat in ∂D and ∂D is bicollared in M, there is an embedding $g_i : \mathbb{D}^{n-1} \times [0,1] \to U_i$ disjoint from ∂M such that $g_i[\mathbb{D}^{n-1} \times [0,1]] \cap \partial D = g_i[\mathbb{D}^{n-1} \times \{1/2\}] = D''_i$ and $g_i[\mathbb{D}^{n-1} \times [0,1]] \cap D = g_i[\mathbb{D}^{n-1} \times [1/2,1]]$. Denote the last intersection by D_i and note that D_i is an n-cell contained in D such that $\partial D_i \cap \partial D = D''_i$. Decreasing D''_i if necessary we can assume that D_i is flat in $\mathrm{int}(\mathrm{cl}(U_i))$. Therefore, by the Annulus Theorem, $\mathrm{cl}(U_i) \setminus \mathrm{int}(D_i) \simeq \mathbb{S}^{n-1} \times [0,1]$. Denote this annulus by A_i.

To finish the proof note that $D \setminus (\mathrm{int}(D_i) \cup \mathrm{int}(D''_i))$ is compact and thus $F_i = A_i \cap (D \setminus (\mathrm{int}(D_i) \cup \mathrm{int}(D''_i)))$ is closed in A_i. Since $[x_i, x''_i] \subset A_i \setminus F_i$, we can choose a chain $V^i_1, \dots, V^i_{n_i}$ of base open subsets of A_i, disjoint from F_i, such that $V^i = V^i_1 \cup \dots \cup V^i_{n_i}$ covers $[x_i, x''_i]$. Using the smooth structure on A_i we can choose a smooth arc $\beta_i : [0,1] \to V^i$ connecting x_i to x''_i. Thus there is a smooth embedding (a tubular neighbourhood) $f_i : \mathbb{D}^{n-1} \times [0,1] \to V^i$ such that $f_i[\mathbb{D}^{n-1} \times [0,1]] \cap \partial A_i = f_i[\mathbb{D}^{n-1} \times \{0,1\}]$, $f_i((\bar{0},0)) = x_i$ and $f_i((\bar{0},1)) = x''_i$. Using f_1, f_2 and a homeomorphism $g : \mathbb{D}^{n-1} \times [0,1] \to D$ such that $g[\mathbb{D}^{n-1} \times \{0\}] = D''_1$ and $g[\mathbb{D}^{n-1} \times \{1\}] = D''_2$, we get the required embedding. ■

FACT 3.3.

(i) *Let M be an n-manifold with nonempty boundary and let $D \subset \partial M$ be an $(n-1)$-cell. Then there is a retraction $r_D : M \to D$.*
(ii) *Let M be a connected n-manifold with nonempty disconnected boundary such that there is a connected component $S \subset \partial M$ homeomorphic to $\mathbb{S}^{n-1}$. Then there is a retraction $r_S : M \to S$.*

Proof. (i) follows from the fact that discs are ARs.

To prove (ii) let N be a connected component of ∂M different from S and choose $x \in S$ and $y \in N$. Let $f : \mathbb{D}^{n-1} \times [0,1] \to M$ be an embedding given by Lemma 3.2. We can assume that $f(\bar{0},0) = x$ and $f(\bar{0},1) = y$.

Set $T = f[\mathbb{D}^{n-1} \times [0,1]]$, $\dot{T} = f[\mathrm{int}(\mathbb{D}^{n-1}) \times [0,1]]$ and $\dot{M} = M \setminus \dot{T}$. Set also $D_i = f[\mathbb{D}^{n-1} \times \{i\}]$ and $\dot{D}_i = f[\mathrm{int}(\mathbb{D}^{n-1}) \times \{i\}]$ for $i = 1, 2$.

Now $\dot{M}$ is an n-manifold with boundary consisting of connected components of ∂M different from S and N, and one additional connected component N' (homeomorphic to N). Note that $D' = (S \setminus \dot{D}_0) \cup f[\mathbb{S}^{n-1} \times [0,1]]$ is an $(n-1)$-cell contained in N'. Thus there is a retraction $\dot{r}_{D'} : \dot{M} \to D'$.

To finish the proof consider the retraction $r'_S : D' \cup T \to S$ given by $r'_S(f(\bar{x}, t)) = f(\bar{x}, 0)$ for all $f(\bar{x}, t) \in T$ and note that the map $r_S : M \to S$ given by $r_S(x) = r'_S(x)$ for $x \in T$ and $r_S(x) = r'_S \circ \dot{r}_{D'}(x)$ for $x \in \dot{M}$ is a well defined retraction. ∎

4. Extensions of Toruńczyk's Lemma. In this section we prove two extensions of Toruńczyk's Lemma. This lemma was proved in [J1, Lemma 4] in dimension 3 and later in [J2, Lemma (3.1)] in the general case. The original statement says that there is only one, up to homeomorphism, dense and null family of mutually disjoint flat n-cells embedded in the interior of a compact n-manifold. In Section 5 we need stronger variants. Namely, such a family is unique up to a homeomorphism isotopic to any prescribed homeomorphism of the underlying manifold (Lemma 4.4), and the same holds for two such families, one for the boundary and the other for the interior of the underlying manifold (Lemma 4.5).

We start by recalling some definitions.

Definition 4.1. Let M be an n-manifold, let $C \subset M$ be closed and let $\mathcal{D}$ be a family of n-cells contained in $\mathrm{int}(M) \setminus C$. We say that $\mathcal{D}$ is *good for the pair* (M, C) if

- elements of $\mathcal{D}$ are flat and mutually disjoint,
- $\mathcal{D}$ is a null family,
- $\mathcal{S}(\mathcal{D}) = \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}\}$ is dense in $M \setminus C$.

For $C = \emptyset$ we say that $\mathcal{D}$ is *good for M*. We will often just say "a good family" if M and C are fixed and no misunderstanding is possible.

Definition 4.2. Let $\mathcal{D}$ be a good family of n-cells for a pair (M, C). We say that a partition $\mathcal{D} = \bigcup_{i \in I} \mathcal{D}_i$ is a *good stratication* of $\mathcal{D}$ if

- the set I is countable (finite or not) and each subfamily $\mathcal{D}_i$ is countable infinite,
- $\mathcal{S}(\mathcal{D}_i)$ is dense in $(M \setminus C) \setminus \mathcal{S}(\bigcup_{j \neq i} \mathcal{D}_j)$ for every $i \in I$.

Definition 4.3. Let M and N be topological spaces, and let $\mathcal{Y}$ and $\mathcal{Z}$ be families of pairwise disjoint subsets of M and N respectively. Suppose that partitions $\mathcal{Y} = \bigcup_{i \in I} \mathcal{Y}_i$ and $\mathcal{Z} = \bigcup_{i \in I} \mathcal{Z}_i$ are given. We say that a homeomorphism $f : M \to N$ *respects the stratifications* if $f[Y] \in \mathcal{Z}_i$ for every $i \in I$ and every $Y \in \mathcal{Y}_i$ and moreover the map $f_i : \mathcal{Y}_i \to \mathcal{Z}_i$ given by $f_i[Y] = f[Y]$ is bijective for every $i \in I$.

The following lemma is our first extension of Toruńczyk's Lemma. We will use it in the proofs of Lemma 4.5 and Theorem 5.2.

LEMMA 4.4. *Let M and N be compact n-manifolds with (possibly empty) boundary, let $C_M \subset M$ and $C_N \subset N$ be closed subsets and let $\mathcal{Y} = \bigcup_{i\in I} \mathcal{Y}_i$ and $\mathcal{Z} = \bigcup_{i\in I} \mathcal{Z}_i$ be good stratified families of n-cells for (M, C_M) and (N, C_N) respectively, consisting of the same number, finite or not, of subfamilies. Let $h : (M, C_M) \to (N, C_N)$ be a homeomorphism. Then there exists a homeomorphism $h' : (M, C_M) \to (N, C_N)$ isotopic to h, coinciding with h on $\partial M \cup C_M$ and respecting the stratifications.*

Proof. The proof is adapted from [J1]. More precisely, first note that without loss of generality we can assume that $M = N$, $C_M = C_N$, $h = \mathrm{id}_M$ and $\mathrm{diam}(M) < 1$. Let $H(M, C_M \cup \partial M)$ denote the group of homeomorphisms of M which are the identity on $\partial M \cup C_M$ and let $H_0(M, C_M \cup \partial M)$ denote its subgroup consisting of homeomorphisms isotopic to the identity.

Choose $\epsilon > 0$ such that if $\mathrm{dist}(h, \mathrm{id}) < \epsilon$, then h is isotopic to the identity. Such an ϵ exists, since the homeomorphism group of M is locally arcwise connected (see [EK]).

For $i \in I$ and $n = 1, 2, \ldots$ let $\mathcal{Y}_i^n = \{Y \in \mathcal{Y}_i : \mathrm{diam}(Y) \geq \epsilon 2^{-n}\}$ and $\mathcal{Z}_i^n = \{Z \in \mathcal{Z}_i : \mathrm{diam}(Z) \geq \epsilon 2^{-n}\}$. Set $\mathcal{Y}^n = \bigcup_{i\in I} \mathcal{Y}_i^n$ and $\mathcal{Z}^n = \bigcup_{i\in I} \mathcal{Z}_i^n$. For a homeomorphism $f : M \to M$ and a family $\mathcal{T}$ of subsets of M let $f(\mathcal{T}) = \{f[T] : T \in \mathcal{T}\}$.

We inductively construct homeomorphisms $f_n, g_n \in H_0(M, C_M \cup \partial M)$ satisfying the following conditions:

- (a_n) for every $Y \in \mathcal{Y}_i^n$ there is $Z_Y \in \mathcal{Z}_i$ such that $f_n[Y] = g_n[Z_Y]$,
- (b_n) for every $Z \in \mathcal{Z}_i^n$ there is $Y_Z \in \mathcal{Y}_i$ such that $f_n[Y_Z] = g_n[Z]$,
- (c_n) $\mathrm{diam}(f_n[Y]) < \epsilon 2^{-n}$ for every $Y \in \mathcal{Y} \setminus (\mathcal{Y}_n \cup f_n^{-1} \circ g_n(\mathcal{Z}_n))$,
- (d_n) $\mathrm{diam}(g_n[Z]) < \epsilon 2^{-n}$ for every $Z \in \mathcal{Z} \setminus (\mathcal{Z}_n \cup g_n^{-1} \circ f_n(\mathcal{Y}_n))$,
- (e_n) $f_n\lceil_Y = f_{n-1}\lceil_Y$ for $Y \in \mathcal{Y}_{n-1} \cup f_{n-1}^{-1} \circ g_{n-1}(\mathcal{Z}_{n-1})$,
- (f_n) $g_n\lceil_Z = g_{n-1}\lceil_Z$ for $Z \in \mathcal{Z}_{n-1} \cup g_{n-1}^{-1} \circ f_{n-1}(\mathcal{Y}_{n-1})$,
- (g_n) $\mathrm{dist}(f_n, f_{n-1}) \leq \epsilon 2^{-n+2}$ and $\mathrm{dist}(f_n^{-1}, f_{n-1}^{-1}) \leq \epsilon 2^{-n+2}$,
- (h_n) $\mathrm{dist}(g_n, g_{n-1}) \leq \epsilon 2^{-n+3}$ and $\mathrm{dist}(g_n^{-1}, g_{n-1}^{-1}) \leq \epsilon 2^{-n+3}$.

The construction is almost the same as in [J1, proof of Lemma 4]. There are only few differences. Namely, elements Z_Y and Y_Z from the above mentioned proof can be chosen from the families $\mathcal{Z}_i$ and $\mathcal{Y}_i$, the Annulus Theorem holds in every dimension and all the homeomorphisms considered there can be chosen from $H_0(M, C_M \cup \partial M)$. We omit further details.

Let $f = \lim f_n$ and $g = \lim g_n$. These two maps are in $H(M, C_M \cup \partial M)$ due to conditions (g_n) and (h_n) (see [BP, Propositions 1.1 and 1.2]); moreover, they are both in $H_0(M, C_M \cup \partial M)$, since they are both within ϵ of the identity. The composition $h' = g^{-1} \circ f$ has the required properties. ■

The following is another extension of Toruńczyk's Lemma. We will use it in the proof of Theorem 5.5.

LEMMA 4.5. *Let M and N be compact n-manifolds with boundary. Let $C_M \subset \partial M$ and $C_N \subset \partial N$ be closed subsets. Let $\mathcal{Y}^1 = \bigcup_{i\in I} \mathcal{Y}_i^1$ and $\mathcal{Z}^1 = \bigcup_{i\in I} \mathcal{Z}_i^1$ be good stratified families of $(n-1)$-cells for the pairs $(\partial M, C_M)$ and $(\partial N, C_N)$ respectively, consisting of the same number, finite or not, of subfamilies. Similarly, let $\mathcal{Y}^0 = \bigcup_{j\in J} \mathcal{Y}_j^0$ and $\mathcal{Z}^0 = \bigcup_{j\in J} \mathcal{Z}_j^0$ be good stratified families of n-cells for M and N respectively, consisting of the same number, finite or not, of subfamilies. Let $h : (M, C_M) \to (N, C_N)$ be a homeomorphism. Then there exists a homeomorphism $h' : (M, C_M) \to (N, C_N)$ isotopic to h, coinciding with h on C_M and respecting the stratifications.*

Proof. First we apply Lemma 4.4 to the pairs $(\partial M, C_M)$ and $(\partial N, C_N)$, to the homeomorphism $h\lceil_{\partial M}: \partial M \to \partial N$ and to the good stratified families $\mathcal{Y}^1 = \bigcup_{i\in I} \mathcal{Y}_i^1$ and $\mathcal{Z}^1 = \bigcup_{i\in I} \mathcal{Z}_i^1$ to get a homeomorphism $h'' : \partial M \to \partial N$ isotopic to $h\lceil_{\partial M}$ such that $h''\lceil_{C_M} = h\lceil_{C_M}$ and h'' respects the stratifications. Since h'' is isotopic to the restriction of $h : M \to N$, and since ∂M and ∂N are collared in M and N respectively (see [D, p. 40]), there is an extension $\tilde{h''} : M \to N$ of h'' isotopic to h. Applying again Lemma 4.4 to $\tilde{h''} : M \to N$, to $C_M = C_N = \emptyset$ and to the good stratified families $\mathcal{Y}^0 = \bigcup_{j\in J} \mathcal{Y}_j^0$ and $\mathcal{Z}^0 = \bigcup_{j\in J} \mathcal{Z}_j^0$ we get a homeomorphism $h' : M \to N$ isotopic to $\tilde{h''}$ such that $h'\lceil_{\partial M} = \tilde{h''}\lceil_{\partial M}$ and which respects the stratifications. ■

5. Trees of manifolds with boundary. In this section we give a precise description of spaces which we call trees of manifolds with boundary and boundary trees of manifolds with boundary. We also study their basic properties.

The constructions are described in Sections 5.1 and 5.2, where we also show that, for any countable (finite or not) family $\mathcal{M}$ of compact n-manifolds with boundary, there is exactly one (up to homeomorphism) regular (boundary) tree of manifolds from $\mathcal{M}$.

In Section 5.3 we prove that the topological dimension of a saturated (boundary) tree of n-manifolds with boundary is equal to $n-1$. Thus these spaces differ from trees of closed n-manifolds, which have dimension n (see [J2] and [St]).

It is an open question whether a (boundary) tree of n-manifolds with boundary can be homeomorphic to a tree of closed $(n-1)$-manifolds, but the author expects that some cohomological properties can be used to distinguish these classes.

5.1. Trees of manifolds with boundary: description and uniqueness. Let T be a countable (finite or not) tree. Let $\mathcal{M}$ be a countable (finite or not) family of compact, connected n-manifolds and suppose that $\partial M_0 \neq \emptyset$ for some $M_0 \in \mathcal{M}$.

For $t \in V_T$ let M_t be a manifold homeomorphic to some $M \in \mathcal{M}$, let $\mathcal{D}_t = \{D_e : e \in N_t\}$ be a null family of pairwise disjoint flat n-cells in $\operatorname{int}(M_t)$ and let $K_t = M_t \setminus \mathcal{S}(\mathcal{D}_t)$.

For $e \in O_T$ set $\Sigma_e = \operatorname{bd}(D_e) \subseteq K_{\alpha(e)}$ and let $\phi_e : \Sigma_e \to \Sigma_{\bar{e}}$ be a homeomorphism such that $\phi_{\bar{e}} = \phi_e^{-1}$.

Let $c : V_T \to \mathcal{M}$ be a function such that M_t is homeomorphic to $c(t)$. Such a map is called a *colouring* and T is called a *coloured tree*. For $t \in V_T$ and $M \in \mathcal{M}$ set $\mathcal{D}_{t,M} = \{D_e \in \mathcal{D}_t : c(\omega(e)) = M\}$.

Definition 5.1. A *tree system of manifolds from* $\mathcal{M}$ is a tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ as described above. The system is called *dense* if additionally $\mathcal{D}_t$ is a good family for M_t for every $t \in V_T$, and *saturated* if it is dense and $\{t \in V_T : \partial M_t \neq \emptyset\}$ spans all of T (i.e. the minimal subtree containing this subset of vertices is T). We say that Θ is *regular* if it is dense and:

- $\mathcal{D}_t = \bigcup_{M \in \mathcal{M}} \mathcal{D}_{t,M}$ is a good stratification for every $t \in V_T$ (in particular $\mathcal{D}_{t,M}$ is infinite for all t and M),
- if elements of $\mathcal{M}$ are orientable, then each M_t is oriented so that the corresponding homeomorphism $M_t \to c(t)$ preserves orientation and the homeomorphisms ϕ_e reverse the induced orientations.

The limit $\lim \Theta$ is called a (resp. *dense, saturated, regular*) *tree of manifolds from* $\mathcal{M}$.

Observe that if Θ is regular, then for each vertex t of T, c maps the set of vertices of T that are adjacent to t ∞-to-1 onto $\mathcal{M}$.

The main result of this section is the following:

Theorem 5.2. *Let* $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ *and* $\Theta' = (T', \{K'_{t'}\}, \{\Sigma'_{e'}\}, \{\phi'_{e'}\})$ *be any two regular tree systems of manifolds from* $\mathcal{M}$. *Then these tree systems are isomorphic and hence their limits are homeomorphic.*

We denote by $\mathcal{X}(\mathcal{M})$ any regular tree of manifolds from $\mathcal{M}$. In view of Theorem 5.2, this space is unique up to homeomorphism and depends only on the family $\mathcal{M}$.

Before we start the proof of Theorem 5.2, we prove the following:

Lemma 5.3. *Let* Θ_1 *be a regular tree system of manifolds from* $\mathcal{M} = \{M_i : i \in \mathbb{I}\}$ *and let* $i_0 \in \mathbb{I}$. *Then there is a regular tree system* Θ_2 *of* *manifolds from* $\{M_i \# M_{i_0} : i \in \mathbb{I} \setminus \{i_0\}\}$ *such that* $\lim \Theta_1$ *and* $\lim \Theta_2$ *are homeomorphic. More precisely,* Θ_2 *can be obtained from* Θ_1 *by consolidation as described before Fact* 2.4.

Proof. Let $\Theta_1 = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$. We will construct a partition Π of T into subtrees such that every $S \in \Pi$ has exactly two vertices, one with colour M_{i_0} and the other with colour M_i for some $i \neq i_0$. The consolidation Θ_Π of Θ_1 with respect to Π satisfies the desired conditions.

To construct Π let $\{t_1, t_2, \ldots\}$ be an enumeration of all vertices t of T with $c(t) = M_{i_0}$, and let $\{t'_1, t'_2, \ldots\}$ be an enumeration of all other vertices of T. Note that since Θ is regular, for every i the set $\{j : t'_j \in N_{t_i}\}$ is infinite, and similarly for every j the set $\{i : t_i \in N_{t'_j}\}$ is infinite.

We inductively construct sequences $i_0, i_1, \ldots$ and $j_0, j_2, \ldots$ and sequence $S_0, S_1, \ldots$ such that:

(i) $S_k = (t_{i_k}, t'_{j_k})$ is a subtree of T consisting of two vertices, which is disjoint from $S_0, S_1, \ldots, S_{k-1}$,
(ii) $c(t_{i_k}) = M_{i_0}$ and $c(t'_{j_k}) \neq M_{i_0}$,
(iii) $\{t_{i_k} : k = 0, 1, \ldots\} \cup \{t'_{j_k} : k = 0, 1, \ldots\} = V_T$.

Let $i_0 = 0$, $j_0 = \min\{j : t'_j \in N_{t_{i_0}}\}$ and let $S_0 = (t_{i_0}, t'_{j_0})$. Now suppose that we have constructed $S_0 = (t_{i_0}, t'_{j_0}), S_1 = (t_{i_1}, t'_{j_1}), \ldots, S_{n-1} = (t_{i_{n-1}}, t'_{j_{n-1}})$ satisfying (i) and (ii). We will construct i_n, j_n and S_n such that (i) and (ii) are satisfied.

Suppose first that $n = 2k+1$. Let $j_n = \min\{j : t'_j \notin S_0 \cup S_1 \cup \cdots \cup S_{n-1}\}$ and let $i_n = \min\{i : t_i \in N_{t'_{j_n}},\ t_i \notin S_0 \cup S_1 \cup \cdots \cup S_{n-1}\}$. Set $S_n = (t_{i_n}, t'_{j_n})$. The construction for $n = 2k$ is similar.

To finish the proof note that (iii) is satisfied, and so $\Pi = \{S_0, S_1, \ldots\}$ is the required partition. ∎

Proof of Theorem 5.2. By Lemma 5.3 we can assume that all manifolds from $\mathcal{M}$ have nonempty boundary and, by a similar argument, that either all manifolds from $\mathcal{M}$ are oriented, or all are nonorientable.

For a tree T, a vertex $t \in V_T$ and a natural number k let $T_k(t)$ denote the subtree of T spanned by all vertices at distance less than or equal to k from t.

Select $t_0 \in V_T$. Inductively we construct the following data:

(i) isomorphisms $\lambda_k : T_k(t_0) \to T'_k(\lambda_0(t_0))$ of coloured trees such that $\lambda_{k+1}\lceil_{T_k(t)} = \lambda_k$,
(ii) homeomorphisms $f_t : M_t \to M'_{\lambda_k(t)}$ for $t \in V_{T_k(t_0)} \setminus V_{T_{k-1}(t_0)}$ (orientation preserving if all manifolds from $\mathcal{M}$ are oriented) respecting the stratifications such that $f_t[\Sigma_e] = \Sigma'_{\lambda_k(e)}$ and $\phi'_{\lambda_k(e)} \circ (f_{\alpha(e)}\lceil_{\Sigma_{\alpha(e)}}) = f_{\omega(e)} \circ \phi_e$ for the unique edge $e \in N_t \cap O_{T_k(t_0)}$.

Let $t' \in V_{T'}$ be a vertex such that $c(t_0) = c'(t')$ and let $h : M_{t_0} \to M'_{t'}$ be an (orientation preserving) homeomorphism. We apply Lemma 4.4 to

$h : (M_{t_0}, \partial M_{t_0}) \to (M'_{t'}, \partial M'_{t'})$ and to the good stratified families $\mathcal{D}_{t_0,M}$ and $\mathcal{D}'_{t',M}$ to get a homeomorphism $f_{t_0} : M_{t_0} \to M'_{t'}$ isotopic to h (and thus orientation preserving if necessary) which respects the stratifications. Set $\lambda_0(t_0) = t'$.

Suppose now that for $k = 0, 1 \ldots, m$ we have constructed isomorphisms λ_k and homeomorphisms f_t satisfying the required conditions.

For $t \in V_{T_{m+1}(t_0)} \setminus V_{T_m}(t_0)$ let e be the unique edge such that $\alpha(e) = t$ and $\omega(e) \in V_{T_m(t_0)}$. Consider the cell $D_{\bar{e}} \subset M_{\omega(e)}$. There is a unique $M \in \mathcal{M}$ such that $D_{\bar{e}} \in \mathcal{D}_{\omega(e),M}$. Let $e' \in O_{T'}$ be the unique edge such that $f_{\omega(e)}[D_{\bar{e}}] = D'_{\bar{e'}}$ and let $t' = \alpha'(e')$. Set $\lambda_{m+1}(t) = t'$, $\lambda_{m+1}(e) = e'$ and $\lambda_{m+1}(\bar{e}) = \bar{e'}$.

Note that $D'_{\bar{e'}} \in \mathcal{D}'_{\lambda_m(\omega(e)),M}$ since $f_{\omega(e)}$ respects the stratifications. Consider now the homeomorphism $(\phi'_{e'})^{-1} \circ f_{\omega(e)} \circ \phi_e : \Sigma_e \to \Sigma'_{e'}$ and note that we can extend it to an (orientation preserving) homeomorphism $f_e : D_e \to D'_{e'}$. Since D_e and $D'_{e'}$ are flat in M_t and $M'_{t'}$ respectively and since $c(t) = c'(t')$, by Fact 3.1 there is an (orientation preserving) homeomorphism $h_t : M_t \to M'_{t'}$ extending f_e. We now apply Lemma 4.4 to $h_t : (M_t, D_e \cup \partial M_t) \to (M'_{t'}, D'_{e'} \cup \partial M'_{t'})$ and to the good stratified families $\mathcal{D}_{t,M} \setminus \{D_e\}$ and $\mathcal{D}'_{t',M} \setminus \{D'_{e'}\}$ to get an (orientation preserving) homeomorphism $f_t : M_t \to M'_{t'}$ extending f_e and preserving the stratifications.

Finally, set $\lambda_{m+1}(t) = \lambda_m(t)$ for $t \in V_{T_m(t_0)}$, $\lambda_{m+1}(e) = \lambda_m(e)$ for $e \in O_{T_m(t_0)}$ and note that conditions (i)–(ii) are satisfied with m replaced by $m + 1$.

To finish the proof set $\lambda = \bigcup_m \lambda_m$ and note that $F = (\lambda, \{f_t \lceil K_t\})$ is an isomorphism of the tree systems Θ and Θ'. ■

5.2. Boundary trees of manifolds with boundary: description and uniqueness. Let T be a countable tree (finite or not). Let $\mathcal{M}$ be a a countable (finite or not) family of compact, connected n-manifolds and suppose that $\partial M_0 \neq \emptyset$ for some $M_0 \in \mathcal{M}$.

For $t \in V_T$ let M_t be a manifold homeomorphic to some $M \in \mathcal{M}$. Let $O_T = O_T^{\text{int}} \cup O_T^{\text{bd}}$ be a partition such that if $e \in O_T^{\text{bd}}$ then $\bar{e} \in O_T^{\text{bd}}$ and $\partial M_{\alpha(e)} \neq \emptyset$.

Set $N_t^{\text{int}} = N_t \cap O_T^{\text{int}}$ and $N_t^{\text{bd}} = N_t \cap O_T^{\text{bd}}$. Let $\mathcal{D}_t^0 = \{D_e : e \in N_t^{\text{int}}\}$ be a null family of pairwise disjoint flat n-cells in M_t, let $\mathcal{D}_t^1 = \{D_e : e \in N_t^{\text{bd}}\}$ be a null family of pairwise disjoint flat $(n-1)$-cells in ∂M_t and set $K_t = M_t \setminus \mathcal{S}(\mathcal{D}_t^0)$.

For $e \in O_T^{\text{int}}$ set $\Sigma_e = \partial D_e \subset K_{\alpha(e)}$, for $e \in O_T^{\text{bd}}$ set $\Sigma_e = D_e \subset K_{\alpha(e)}$ and for every $e \in O_T$ let $\phi_e : \Sigma_e \to \Sigma_{\bar{e}}$ be a homeomorphism such that $\phi_{\bar{e}} = \phi_e^{-1}$.

For $M \in \mathcal{M}$ let $\partial M = N_{M,1}^+ \sqcup \cdots \sqcup N_{M,k_M}^+ \sqcup N_{M,1}^- \sqcup \cdots \sqcup N_{M,l_M}^-$ be a decomposition of ∂M into connected components, where the $N_{M,j}^+$ are orientable and the $N^-_{M,j}$ are nonorientable (it may happen that all components are orientable and then $l_M = 0$, or all are nonorientable, and then $k_M = 0$; it may also happen that $\partial M = \emptyset$ and then $k_M = l_M = 0$, but recall that $\partial M_0 \neq \emptyset$ for some $M_0 \in \mathcal{M}$).

Let $c : V_T \to \mathcal{M}$ be a colouring such that M_t is homeomorphic to $c(t)$. Thus, for $t \in V_T$ with $c(t) = M$ we have a decomposition $\partial M_t = N^+_{t,M,1} \sqcup N^+_{t,M,2} \sqcup \cdots \sqcup N^+_{t,M,k_M} \sqcup N^-_{t,M,1} \sqcup N^-_{t,M,2} \sqcup \cdots \sqcup N^-_{t,M,l_M}$ where $N^+_{t,M,j}$ (resp. $N^-_{t,M,j}$) is a connected component of ∂M_t corresponding to $N^+_{M,j}$ (resp. $N^-_{M,j}$).

For $M \in \mathcal{M}$ let $\mathcal{D}^0_{t,M} = \{D_e \in \mathcal{D}^0_t : c(\omega(e)) = M\}$. For $M \in \mathcal{M}$ and $j \in \{1, \ldots, k_M\}$ let $\mathcal{D}^{1+}_{t,M,j} = \{D_e \in \mathcal{D}^1_t : c(\omega(e)) = M$ and $D_{\bar{e}} \subset N^+_{\omega(e),M,j}\}$. Similarly, for $M \in \mathcal{M}$ and $j \in \{1, \ldots, l_M\}$ let $\mathcal{D}^{1-}_{t,M,j} = \{D_e \in \mathcal{D}^1_t : c(\omega(e)) = M$ and $D_{\bar{e}} \subset N^-_{\omega(e),M,j}\}$. Set $\mathcal{D}^{1+}_t = \bigcup_{M\in\mathcal{M},\, j\in\{1,\ldots,k_M\}} \mathcal{D}^{1+}_{t,M,j}$ and $\mathcal{D}^{1-}_t = \bigcup_{M\in\mathcal{M},\, j\in\{1,\ldots,l_M\}} \mathcal{D}^{1-}_{t,M,j}$.

Definition 5.4. A *boundary tree system of manifolds from $\mathcal{M}$* is a tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ as described above. It is called *dense* if additionally $\mathcal{D}^0_t$ is a good family for M_t and $\mathcal{D}^1_t$ is a good family for ∂M_t for every $t \in V_T$, and *saturated* if it is dense and the set $\{t \in V_T : \partial M_t \neq \emptyset\}$ spans all of T. We say that Θ is *regular* if it is dense and

- $\mathcal{D}^0_t = \bigcup_{M\in\mathcal{M}} \mathcal{D}^0_{t,M}$ is a good stratification for M_t for every $t \in V_T$,
- $\mathcal{D}^{1+}_t = \bigcup_{M\in\mathcal{M},\, j\in\{1,\ldots,k_M\}} \mathcal{D}^{1+}_{t,M,j}$ is a good stratification for the orientable part of ∂M_t for every $t \in V_T$,
- $\mathcal{D}^{1-}_t = \bigcup_{M\in\mathcal{M},\, j\in\{1,\ldots,l_M\}} \mathcal{D}^{1-}_{t,M,j}$ is a good stratification for the nonorientable part of ∂M_t for every $t \in V_T$,
- if elements of $\mathcal{M}$ are all orientable, they are oriented and the homeomorphisms ϕ_e reverse orientation (the orientation of the boundary is consistent with the orientation of the manifold); otherwise, we assume that all orientable components of the boundaries are oriented and ϕ_e reverses orientation for $e \in O^{\mathrm{bd}}_T$ such that Σ_e is contained in some oriented component of $\partial M_{\alpha(e)}$.

The limit $\lim \Theta$ is called a (resp. *dense, saturated, regular*) *boundary tree of manifolds from $\mathcal{M}$*.

The main result of this section is the following:

Theorem 5.5. *Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ and $\Theta' = (T', \{K'_{t'}\}, \{\Sigma'_{e'}\}, \{\phi'_{e'}\})$ be any two regular boundary tree systems of manifolds from $\mathcal{M}$. Then these systems are homeomorphic and hence their limits are homeomorphic.*

We denote by $\mathcal{X}_{\mathrm{bd}}(\mathcal{M})$ any regular boundary tree of manifolds from $\mathcal{M}$. Again, this space is unique up to homeomorphism and depends only on $\mathcal{M}$.

An argument very similar to that from the proof of Lemma 5.3 can be used to show the following lemma. We omit the details.

LEMMA 5.6. *Let Θ_1 be a regular boundary trees system of manifolds from $\mathcal{M} = \{M_i : i \in \mathbb{I}\}$ and let $i_0 \in \mathbb{I}$. Then there is a regular boundary tree system Θ_2 of manifolds from $\{M_i \# M_{i_0} : i \in \mathbb{I} \setminus \{i_0\}\}$ such that $\lim \Theta_1$ and $\lim \Theta_2$ are homeomorphic.*

Proof of Theorem 5.5. By Lemma 5.6 we can assume that all manifolds from $\mathcal{M}$ have nonempty boundary. We can also assume that either they are all oriented, or all are nonorientable.

Fix $t_0 \in T$. As in the proof of Theorem 5.2, we inductively construct the following data:

(i) isomorphisms $\lambda_k : T_k(t_0) \to T'_k(\lambda_0(t_0))$ of coloured trees such that $\lambda_{k+1}\lceil_{T_k(t)} = \lambda_k$,

(ii) homeomorphisms $f_t : M_t \to M'_{\lambda_k(t)}$ for $t \in V_{T_k(t_0)} \setminus V_{T_{k-1}(t_0)}$ respecting the stratifications such that $f_t[\Sigma_e] = \Sigma'_{\lambda_k(e)}$ and $\phi'_{\lambda_k(e)} \circ (f_{\alpha(e)}\lceil_{\Sigma_{\alpha(e)}}) = f_{\omega(e)} \circ \phi_e$ for $e \in N_t \cap O_{T_k(t_0)}$ (if all manifolds from $\mathcal{M}$ are oriented, we demand that every f_t preserves orientation, in the oposite case every f_t should preserve the orientation of the oriented parts of the boundary).

Let $t' \in T'$ with $c'(t') = c(t_0)$ and let $h : M_{t_0} \to M'_{t'}$ be an (orientation preserving or boundary orientation preserving if necessary) homeomorphism. We apply Lemma 4.5 to $h : M_{t_0} \to M'_{t'}$ and to the good stratified families $\mathcal{D}^0_{t_0,M}$, $(\mathcal{D}')^0_{t',M}$, $\mathcal{D}^{1+}_{t_0,M,j}$, $(\mathcal{D}')^{1+}_{t',M,j}$ and $\mathcal{D}^{1-}_{t_0,M,j}$, $(\mathcal{D}')^{1-}_{t',M,j}$, to get a homeomorphism $f_{t_0} : M_{t_0} \to M_{t'}$ isotopic to h (and thus orientation preserving or boundary orientation preserving if necessary) which respects the stratifications. Set $\lambda_0(t_0) = t'$.

Suppose now that for $k = 0, 1, \ldots, m$ we have constructed isomorphisms λ_k and homeomorphisms f_t satisfying the required conditions.

For $t \in V_{T_{m+1}(t_0)} \setminus V_{T_m(t_0)}$ let e be the unique edge such that $\alpha(e) = t$ and $\omega(e) \in V_{T_m(t_0)}$.

Suppose first that $e \in O^{\mathrm{int}}_T$. Then there is a unique $M \in \mathcal{M}$ such that $D_{\bar{e}} \in \mathcal{D}^0_{\omega(e),M}$. Let $e' \in O^{\mathrm{int}}_{T'}$ be the unique edge such that $f_{\omega(e)}[D_{\bar{e}}] = D'_{\bar{e'}}$ and let $t' = \alpha'(e')$. Set $\lambda_{m+1}(t) = t'$, $\lambda_{m+1}(e) = e'$ and $\lambda_{m+1}(\bar{e}) = \bar{e'}$.

Note that $D'_{\bar{e'}} \in (\mathcal{D}')^0_{\lambda_m(\omega(e)),M}$ since $f_{\omega(e)}$ respects the stratifications. Consider now the homeomorphism $(\phi')^{-1}_{e'} \circ f_{\omega(e)} \circ \phi_e : \Sigma_e \to \Sigma'_{e'}$ and note that we can extend it to an (orientation preserving) homeomorphism $f_e : D_e \to D'_{e'}$. Since D_e and $D'_{e'}$ are flat in M_t and $M'_{t'}$ respectively and since $c(t) = c'(t')$, by Fact 3.1 there is an (orientation preserving or boundary orientation preserving if necessary) homeomorphism $h_t : M_t \to M'_{t'}$ extending f_e.

We now apply Lemma 4.5 to $h_t\lceil_{M_t \setminus \mathrm{int}(D_e)}$: $(M_t \setminus \mathrm{int}(D_e), \Sigma_e) \to (M'_{t'} \setminus \mathrm{int}(D'_{e'}), \Sigma_{e'})$ and to the good stratified families $\mathcal{D}^0_{t,M} \setminus \{D_e\}$, $(\mathcal{D}')^0_{t',M} \setminus \{D'_{e'}\}$, $\mathcal{D}^{1+}_{t,M,j}$, $(\mathcal{D}')^{1+}_{t',M,j}$ and $\mathcal{D}^{1-}_{t,M,j}$, $(\mathcal{D}')^{1-}_{t',M,j}$ to get an (orientation preserving or boundary orientation preserving if necessary) homeomorphism $h'_t : M_t \setminus \mathrm{int}(D_e) \to M'_{t'} \setminus \mathrm{int}(D'_{e'})$ extending $(\phi')^{-1}_{e'} \circ f_{\omega(e)} \circ \phi_e$ which respects the stratifications. Set $f_t = h'_t \cup f_e\lceil_{D_e}$.

Suppose now that $e \in O^{\mathrm{bd}}_T$. Then there is a unique pair (M, j) such that $D_{\bar{e}} \in \mathcal{D}^{1+}_{\omega(e),M,j}$ or $D_{\bar{e}} \in \mathcal{D}^{1-}_{\omega(e),M,j}$. Suppose the former case holds (the latter is similar). Let $e' \in O^{\mathrm{bd}}_T$ be the unique edge such that $f_{\omega(e)}[D_{\bar{e}}] = D'_{\bar{e}'}$ and let $t' = \alpha'(e')$. Set $\lambda_{m+1}(t) = t'$, $\lambda_{m+1}(e) = e'$ and $\lambda_{m+1}(\bar{e}) = \bar{e}'$.

Note that $D'_{\bar{e}'} \in (\mathcal{D}')^{1+}_{\lambda_m(\omega(e)),M,j}$ since $f_{\omega(e)}$ respects the stratifications. Moreover, $D_e \subset N^+_{t,M,j}$ and $D'_{e'} \subset (N')^+_{t',M,j}$. Consider the orientation preserving homeomorphism $(\phi')^{-1}_{e'} \circ f_{\omega(e)} \circ \phi_e : D_e \to D'_{e'}$. Since D_e and $D'_{e'}$ are flat in ∂M_t and $\partial M'_{t'}$ (and thus in $N^+_{t,M,j}$ and $(N')^+_{t',M,j}$) respectively, and since $c(t) = c'(t')$, by Fact 3.1 there is an orientation preserving homeomorphism $h'_t : N^+_{t,M,j} \to (N')^+_{t',M,j}$ extending $(\phi')^{-1}_{e'} \circ f_{\omega(e)} \circ \phi_e$. We can extend h'_t to a homeomorphism $h''_t : \partial M_t \to \partial M'_{t'}$. Moreover, since h'_t is isotopic to a homeomorphism which is extendable, we can extend it to an (orientation preserving or boundary orientation preserving if necessary) homeomorphism $h_t : M_t \to M'_{t'}$.

We now apply Lemma 4.5 to $h_t : (M_t, D_e) \to (M'_{t'}, D'_{e'})$ and to the good stratified families $\mathcal{D}^0_{t,M}$, $(\mathcal{D}')^0_{t',M}$, $\mathcal{D}^{1+}_{t,M,j} \setminus \{D_e\}$, $(\mathcal{D}')^{1+}_{t',M,j} \setminus \{D'_{e'}\}$ and $\mathcal{D}^{1-}_{t,M,j}$, $(\mathcal{D}')^{1-}_{t',M,j}$ to get a homeomorphism $f_t : M_t \to M'_{t'}$ coinciding with h_t on D_e (and thus extending $(\phi')^{-1}_{e'} \circ f_{\omega(e)} \circ \phi_e$) and isotopic to h_t (orientation preserving or boundary orientation preserving if necessary), which respects the stratifications.

Finally, set $\lambda_{m+1}(t) = \lambda_m(t)$ for $t \in V_{T_m(t_0)}$, $\lambda_{m+1}(e) = \lambda_m(e)$ for $e \in O_{T_m(t_0)}$ and note that conditions (i)–(ii) are satisfied with m replaced by $m + 1$.

To finish the proof set $\lambda = \bigcup_m \lambda_m$ and note that $F = (\lambda, \{f_t\lceil_{K_t}\})$ is an isomorphism of the tree systems Θ and Θ'. ∎

Remark 5.7. Note that in our construction, operations of boundary sum are performed separately along oriented and nonorientable parts of boundaries.

This obstruction can be omitted using Stalling's notion of *dense orientation* (see [St]), but this notion is very technical and complicated, thus we have decided to restrict our construction to the case described before Definition 5.4.

5.3. Topological dimension. In this section we calculate the topological dimension of (boundary) trees of manifolds. The main result of this section is the following:

THEOREM 5.8.

(i) *Let Θ be a saturated tree system of n-manifolds with boundary from $\mathcal{M}$. Then* $\dim(\lim\Theta) = n-1$.
(ii) *Let Θ be a saturated boundary tree system of n-manifolds with boundary from $\mathcal{M}$. Then* $\dim(\lim\Theta) = n-1$.

Before the proof we need some preparation. We will need the following:

LEMMA 5.9.

(i) *Let M be an n-manifold with nonempty boundary and let $\mathcal{D}$ be a good family of n-cells for M. Then for every $D \in \mathcal{D}$ there is a retraction $r_{\partial D} : M \setminus \mathrm{int}(D) \to \partial D$ such that $r_{\partial D}[D']$ is a point for every $D' \in \mathcal{D} \setminus \{D\}$.*
(ii) *Let M be an n-manifold with nonempty boundary, let $\mathcal{D}^0$ be a good family of n-cells for M and let $\mathcal{D}^1$ be a good family of $(n-1)$-cells for ∂M. Then:*
    (a) *for every $D \in \mathcal{D}^0$ there is a retraction $r_{\partial D} : M \setminus \mathrm{int}(D) \to \partial D$ such that $r_{\partial D}[D']$ is a point for every $D' \in (\mathcal{D}^0 \setminus \{D\}) \cup \mathcal{D}^1$,*
    (b) *for every $D \in \mathcal{D}^1$ there is a retraction $r_D : M \to D$ such that $r_D[D']$ is a point for every $D' \in \mathcal{D}^0 \cup (\mathcal{D}^1 \setminus \{D\})$.*

In the proof of this lemma we will use the notion of a shrinkable decomposition of a topological space, which we recall below.

A *decomposition* $\mathcal{G}$ of a topological space X is a partition of X. A decomposition $\mathcal{G}$ of X is *upper semicontinuous* (usc) if it consists of compact elements and for each $g \in \mathcal{G}$ and every open neighbourhood $U \subset X$ of g there is an open neighbourhood $V \subset X$ of g such that every $g' \in \mathcal{G}$ intersecting V is contained in U. An usc decomposition $\mathcal{G}$ of X is *shrinkable* if for every open, $\mathcal{G}$-saturated cover $\mathcal{U}$ of X (i.e. every $U \in \mathcal{U}$ is $\mathcal{G}$-saturated) and every open cover $\mathcal{V}$ of X there is a homeomorphism $h : X \to X$ satisfying:

- for every $x \in X$ there is $U \in \mathcal{U}$ such that $x, h(x) \in U$,
- for every $g \in \mathcal{G}$ there is $V \in \mathcal{V}$ such that $h[g] \subset V$.

For a decomposition $\mathcal{G}$ of X let $H_{\mathcal{G}} = \{g \in \mathcal{G} : g$ contains more than one point$\}$ and let $N_{\mathcal{G}} = \bigcup H_{\mathcal{G}}$.

A compact subset $C \subset X$ is *locally shrinkable in* X if for every open neighbourhood $U \subset X$ of C and every open cover $\mathcal{V}$ of X there are a homeomorphism $h : X \to X$ and $V \in \mathcal{V}$ such that $h[C] \subset V$ and $h\lceil_{X\setminus U} = \mathrm{id}$.

In the proof of Lemma 5.9 we will use the following two lemmas.

LEMMA 5.10. *Let $\mathcal{D}$ be a good family of n-cells for $\mathbb{D}^n$. Then for every $\epsilon > 0$ there is a homeomorphism $h : \mathbb{D}^n \to \mathbb{D}^n$ isotopic to the identity such that $h\lceil_{\mathbb{S}^{n-1}} = \mathrm{id}$ and $\mathrm{diam}(h[D]) < \epsilon$ for every $D \in \mathcal{D}$.*

The proof is almost the same as the proof of Lemma 2 from [M], so we omit it.

LEMMA 5.11.

(i) *Let M be an n-manifold with (possibly empty) boundary and let $\mathcal{D}$ be a good family of n-cells for M. Then for every $D \in \mathcal{D}$ and for every open neighbourhood U of D there is another open neighbourhood U_D of D contained in U such that $\mathrm{cl}(U_D)$ is an n-cell flat in M and $\partial(\mathrm{cl}(U_D))$ is disjoint from elements of $\mathcal{D}$.*

(ii) *Let M be an n-manifold with nonempty boundary, let $\mathcal{D}^0$ be a good family of n-cells for M and let $\mathcal{D}^1$ be a good family of $(n-1)$-cells for ∂M. Then:*

  (a) *for every $D \in \mathcal{D}^0$ and every open neighbourhood U of D there is another open neighbourhood U_D of D contained in U such that $\mathrm{cl}(U_D)$ is an n-cell flat in M and $\partial(\mathrm{cl}(U_D))$ is disjoint from elements of $\mathcal{D}^0$,*

  (b) *for every $D \in \mathcal{D}^1$ and every open neighbourhood U of D there is another open neighbourhood U_D of D contained in U such that $\mathrm{cl}(U_D)$ is an n-cell flat in $(M, \partial M)$, $\mathrm{cl}(U_D) \cap \partial M$ is a flat $(n-1)$-cell in ∂M with boundary disjoint from elements of $\mathcal{D}^1$, and $\partial(\mathrm{cl}(U_D))$ is disjoint from elements of $\mathcal{D}^0$.*

Proof. For (i) we consider a decomposition $\mathcal{G}$ of M such that $H_{\mathcal{G}} = \mathcal{D}$. By [Fr, Theorem 7.2] this decomposition is shrinkable (for $M = \mathbb{R}^3$ this result is due to [Bi] and [M], and actually its proof easily adapts to any manifold of any dimension), and thus by [D, Theorem 2, p. 23] the quotient map $\Pi : M \to M/\mathcal{G}$ can be approximated by homeomorphisms. In particular $M/\mathcal{G}$ is homeomorphic to M. Consider now an n-cell $\bar{D}$ flat in $M/\mathcal{G}$ containing $\Pi[D]$ such that $\partial\bar{D} \cap \Pi[\mathcal{D}] = \emptyset$. Its preimage clearly satisfies the required conditions.

The proof of (ii)(a) is the same. To prove (ii)(b) we first consider a decomposition $\mathcal{G}_1$ of ∂M such that $H_{\mathcal{G}_1} = \mathcal{D}^1$ to get an $(n-1)$-cell $\tilde{D}$ flat in ∂M containing D with boundary disjoint from elements of $\mathcal{D}^1$. Denote this boundary by S. We then consider a decomposition $\mathcal{G}_0$ of M such that $H_{\mathcal{G}_0} = \mathcal{D}^0$. Again $M/\mathcal{G}_0$ is homeomorphic to M. Let $\bar{D}$ be an n-cell flat in

$(M/\mathcal{G}, \partial[M/\mathcal{G}])$ such that $\bar{D} \cap \partial M = \tilde{D}$ and $\partial\bar{D}$ is disjoint from $\Pi[\mathcal{D}^0]$. Its preimage satisfies the required conditions. ∎

Proof of Lemma 5.9. For (i), let $M' = M \setminus \operatorname{int}(D)$. Consider a decomposition $\mathcal{G}$ of M' such that $H_{\mathcal{G}} = \mathcal{D} \setminus \{D\}$. By [D, Proposition 3, p. 14], $\mathcal{G}$ is usc, and by [Fr, Theorem 7.2] it is shrinkable. Therefore, by [D, Theorem 2, p. 23], the decomposition map $\pi : M' \to M'/\mathcal{G}$ can be approximated by homeomorphisms. In particular $M'/\mathcal{G}$ is homeomorphic to M' and thus is an n-manifold with disconnected boundary. Note also that $\pi\lceil_{\partial D}: \partial D \to \partial D$ is a homeomorphism. By Fact 3.3.ii, there is a retraction $r'_{\pi[\partial D]} : M'/\mathcal{G} \to \pi[\partial D]$. The composition $r_{\partial D} = (\pi\lceil_{\partial D})^{-1} \circ r'_{\pi[\partial D]} \circ \pi$ satisfies the desired conditions.

For (ii)(a), consider a decomposition $\mathcal{G}$ of $M' = M \setminus \operatorname{int}(D)$ such that $H_{\mathcal{G}} = (\mathcal{D}^0 \setminus \{D\}) \cup \mathcal{D}^1$, which is again usc.

For $\epsilon > 0$ and $g \in \mathcal{G}$ set $N(g,\epsilon) = \{x \in M' : d(x,g) < \epsilon\}$. We will show that for every $\epsilon > 0$ and every $g_0 \in H_{\mathcal{G}}$ there is a homeomorphism $h : M' \to M'$ such that:

- $h\lceil_{M'\setminus N(g_0,\epsilon)} = \mathrm{id}$,
- $\operatorname{diam}(h[g_0]) < \epsilon$,
- for every $g \in \mathcal{G}$ we have $\operatorname{diam}(h[g]) < \epsilon$ or $h[g] \subset N(g,\epsilon)$.

To see this, for $g \in \mathcal{D}^0 \setminus \{D\}$ consider an open neighbourhood U_g of g given by Lemma 5.11(ii)(a). By Lemma 5.10 there is a homeomorphism $h' : \operatorname{cl}(U_g) \to \operatorname{cl}(U_g)$ such that $\operatorname{diam}(h'[g']) < \epsilon$ for every $g' \in \mathcal{G}$ with $g' \subset U_g$ and $h'\lceil_{\partial(\operatorname{cl}(U_g))} = \mathrm{id}$. It can be extended via id to a homeomorphism $h : M' \to M'$ which clearly satisfies the required conditions.

For $g \in \mathcal{D}^1$ consider an open neighbourhood U_g of g given by Lemma 5.11(ii)(b). Set $D_g = \operatorname{cl}(U_g) \cap \partial M$; this is an $(n-1)$-cell flat in ∂M. By Lemma 5.10 there is a homeomorphism $f' : D_g \to D_g$ such that $f'\lceil_{\partial D_g} = \mathrm{id}$ and $\operatorname{diam}(f'[g']) < \epsilon$ for every $g' \in \mathcal{D}^1$ with $g' \subset D_g$. It can be extended via id to a homeomorphism $f'' : \partial(\operatorname{cl}(U_g)) \to \partial(\operatorname{cl}(U_g))$, which extends to a homeomorphism $f : \operatorname{cl}(U_g) \to \operatorname{cl}(U_g)$. Using Lemma 5.10 we get a homeomorphism $h' : \operatorname{cl}(U_g) \to \operatorname{cl}(U_g)$ which is the identity on the boundary and $\operatorname{diam}(h'[f'[g']]) < \epsilon$ for every $g' \in \mathcal{D}^0$ with $g' \subset \operatorname{cl}(U_g)$. The composition $h' \circ f'$ extends via id to a homeomorphism $h : M' \to M'$ which clearly satisfies the desired conditions.

Therefore the assumptions of [D, Theorem 5, p 47] are fullfilled and hence the decomposition $\mathcal{G}$ is shrinkable. Using the same arguments as in the proof of (i) we get a retraction $r_{\partial D} : M' \to \partial D$ with the desired properties.

For (ii)(b), consider a decomposition $\mathcal{G}$ of M such that $H_{\mathcal{G}} = \mathcal{D}^0 \cup (\mathcal{D}^1 \setminus \{D\})$. Using arguments similar to those above, applying Fact 3.3(ii) instead of Fact 3.3(i), we get a retraction $r_D : M \to D$ with the desired properties. We omit further details. ∎

Proof of Theorem 5.8. In both cases we will show that Θ admits a trivial associated family of extended spaces and maps, which is fine.

To see this in case (i), first note that since Θ is saturated, it is not hard to find a partition Π of the underlying tree T into finite subtrees such that every $S \in \Pi$ has a vertex t with $\partial M_t \neq \emptyset$. Therefore we can assume that every manifold from $\mathcal{M}$ has nonempty boundary. Let $e \in O_T$ be an edge and consider the manifold $M_{\omega(e)} \setminus \mathrm{int}(D_{\bar{e}})$. By Lemma 5.9(i) there is a retraction $r_e : M_{\omega(e)} \setminus \mathrm{int}(D_{\bar{e}}) \to \Sigma_{\bar{e}}$ such that $r_e[\Sigma_{e'}]$ is a point for every $e' \in N_{\omega(e)} \setminus \{\bar{e}\}$. The composition $\delta_e = \phi_{\bar{e}} \circ (r_e\lceil_{K_{\omega(e)}}) : K_{\omega(e)} \to \Sigma_e$ satisfies $\delta_e\lceil_{\Sigma_{\bar{e}}} = \phi_{\bar{e}}$, and moreover $\delta_e[\Sigma_{e'}]$ is a point for every $e' \in N_{\omega(e)} \setminus \{\bar{e}\}$. Set $\hat{K}_e = K_{\omega(e)}$ and note that $\mathcal{E} = \{\{\hat{K}_e : e \in O_T\}, \{\delta_e : e \in O_T\}\}$ is a fine, trivial associated family of extended spaces and maps for Θ.

We now show that $\dim K_t = n - 1$ for every $t \in V_T$, and thus the statement will follow from Fact 2.5. Since K_t contains the boundary of an n-cell, whose interior was removed, we have $\dim K_t \geq n-1$. To see the reverse inequality, suppose first that M_t is a closed manifold. It can be covered by a countable family $\{U_1, U_2, \ldots\}$ of open subsets, each homeomorphic to $\mathbb{R}^n$. Since $K_t = M_t \setminus \mathcal{S}(\mathcal{D}_t)$, where $\mathcal{D}_t$ is a good family of n-cells for M_t, the intersections $K_t \cap U_i$ contain no interior points and thus all have dimension $< n$ (see [AP, Theorem 20, p. 133]). Therefore $\dim K_t \leq n - 1$.

If M_t has the boundary, let $N_t = M_t \times \{0,1\}/_\sim$, where $\sim$ is the equivalence relation given by $(x,0) \sim (x,1)$ for $x \in \partial M_t$. Note that N_t is a closed n-manifold. Now $\dim(K_t \times \{0,1\}/_\sim) \leq n - 1$ by the previous argument and thus $\dim K_t \leq n - 1$. This finishes the proof of case (i).

The proof in (ii) is similar, so we omit it. ■

6. The conjecture. In this section we precisely formulate the conjecture mentioned in the Introduction, concerning the appearance of boundary trees of manifolds with boundary as ideal boundaries of certain groups. This conjecture was suggested by J. Świątkowski. To do this we need some preparation.

We start by recalling some terminology concerning simplicial complexes. Let K be a simplicial complex and let L be its subcomplex. We say that K is *flag* if every set of its vertices pairwise connected by edges spans a simplex in K. Further, L is called *full in* K if every simplex in K spanned by a set of vertices from L is a simplex in L. We say that L is 2-*geodesically convex in* K if it is full in K and every geodesic of length 2 in the 1-skeleton of K with endpoints from L is entirely contained in L (this condition was introduced in [JS], where it was called 3-*convexity*).

Before we formulate the conjecture, we introduce a new kind of regular tree systems of manifolds with boundary, the limits of which will appear in the statement of the conjecture.

Let N be a fixed n-manifold with nonempty boundary, let $\mathcal{D}^0 = \{D_i^0 : i \in \mathbb{N}\}$ be a good family of n-cells for N and let $\mathcal{D}^1 = \{D_i^1 : i \in \mathbb{N}\}$ be a good family of $(n-1)$-cells for ∂N.

Let T be a countable, locally infinite tree. Let $c_E : O_T \to \{0,1\}$ be a function such that $c_E(e) = c_E(\bar{e})$. Suppose that for every $t \in V_T$ and for $i = 0,1$ the set $\{e \in N_t : c_E(e) = i\}$ is infinite. For $i = 0,1$ set $O_T^i = \{e \in O_T : c_E(e) = i\}$ and let $c_E^i : O_T^i \to \mathbb{N}$ be a map such that $c_E^i(e) = c_E^i(\bar{e})$ and for every $t \in V_T$ the restriction $c_E^i\lceil_{N_t}$ is bijective.

For $t \in V_T$ set $M_t = N \times \{t\}$, $\mathcal{D}_t^0 = \{D_i^0 \times \{t\} : i \in \mathbb{N}\}$, $\mathcal{D}_t^1 = \{D_i^1 \times \{t\} : i \in \mathbb{N}\}$ and $K_t = M_t \setminus \mathcal{S}(\mathcal{D}_t^0)$.

For $e \in O_T^0$ set $\Sigma_e = \partial(D^0_{c_E^0(\alpha(e))} \times \{\alpha(e)\}) \subset K_{\alpha(e)}$, for $e \in O_T^1$ set $\Sigma_e = (D^1_{c_E^1(\alpha(e))} \times \{\alpha(e)\}) \subset K_{\alpha(e)}$ and for every $e \in O_T$ let $\phi_e : \Sigma_e \to \Sigma_{\bar{e}}$ be given by $\phi_e((x,\alpha(e))) = (x,\omega(e))$ (it is well defined since $c_E^i(e) = c_E^i(\bar{e})$).

Using arguments very similar to those used in the proof of Theorem 5.5 one can show that there is only one, up to isomorphism, tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ as described above. We denote it by $\Theta^{\mathrm{r}}_{\mathrm{bd}}(N)$ and call a *regular reflective boundary tree system of manifolds* N. Its limit, denoted by $\mathcal{X}^{\mathrm{r}}_{\mathrm{bd}}(N)$, is thus unique up to homeomorphism and we call it a *regular reflective boundary tree of manifolds* N. By Theorem 5.8, if $\dim N = n$, then $\dim \mathcal{X}^{\mathrm{r}}_{\mathrm{bd}}(N) = n-1$.

We now formulate the conjecture.

CONJECTURE 6.1. *Let K be a flag PL-triangulation of a manifold N with boundary and let L be its subcomplex corresponding to ∂N. Suppose that L is 2-geodesically convex in K. Let W be a Coxeter group with nerve K. Then the* CAT(0)*-boundary of W is homeomorphic to $\mathcal{X}^{\mathrm{r}}_{\mathrm{bd}}(N)$.*

Note that this hypothesis is in a natural way consistent with Fischer's result for closed PL-manifolds N. It not hard to construct examples showing that the condition of 2-geodesic convexity for L (or at least some additional condition) is necessary.

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