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SIMPLE MIXING ACTIONS WITH UNCOUNTABLY MANY PRIME FACTORS

BY

ALEXANDRE I. DANILENKO and ANTON V. SOLOMKO (Kharkiv)

Abstract. Via the (C,F)-construction we produce a 2-fold simple mixing transformation which has uncountably many non-trivial proper factors and all of them are prime.

0. Introduction. This paper is about prime factors of simple probability preserving actions. We first recall the related definitions from the theory of joinings.

Let $T=(T_g)_{g\in\Gamma}$ be an ergodic action of a locally compact second countable group Γ on a standard probability space $(X,\mathfrak{B},\mu)$. Our main interest is in $\mathbb{Z}$- and $\mathbb{R}$-actions. A measure λ on $X\times X$ is called a 2-*fold self-joining* of T if it is $(T_g\times T_g)_{g\in\Gamma}$-invariant and it projects onto μ on both coordinates. Denote by $J_2^{\mathrm{e}}(T)$ the set of all ergodic 2-fold self-joinings of T. Let $C(T)$ stand for the *centralizer* of T, i.e. the set of all μ-preserving invertible transformations of X commuting with T_g for each $g\in\Gamma$. Given a transformation $S\in C(T)$, we denote by μ_S the corresponding *off-diagonal measure* on $X\times X$ defined by $\mu_S(A\times B):=\mu(A\cap S^{-1}B)$ for all $A,B\in\mathfrak{B}$. In other words, μ_S is the image of μ under the map $x\mapsto(x,Sx)$. Of course, $\mu_S\in J_2^{\mathrm{e}}(T)$ for every $S\in C(T)$. If T is weakly mixing, $\mu\times\mu$ is also an ergodic self-joining. If $J_2^{\mathrm{e}}(T)\subset\{\mu_S\mid S\in C(T)\}\cup\{\mu\times\mu\}$ then T is called 2-*fold simple* [Ve], [dJR]. By a *factor* of T we mean a non-trivial proper T-invariant sub-σ-algebra of $\mathfrak{B}$. If T has no non-trivial proper factors then it is called *prime*. In [Ve] it was shown that if T is 2-fold simple then for each non-trivial factor $\mathfrak{F}$ of T there exists a compact (in the strong operator topology) subgroup $K_{\mathfrak{F}}\subset C(T)$ such that $\mathfrak{F}=\mathfrak{F}_{K_{\mathfrak{F}}}$, where

$$\mathfrak{F}_K=\{A\in\mathfrak{B}\mid \mu(kA\triangle A)=0 \text{ for all } k\in K\}$$

is the fixed algebra of K. In particular, $\mathfrak{F}$ (or, more precisely, the restriction of T to $\mathfrak{F}$) is prime if and only if $K_{\mathfrak{F}}$ is a maximal compact subgroup of $C(T)$.

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One of the natural questions arising after the general theory of simple actions was developed in [dJR] is: are there simple maps with non-unique prime factors? The first example of such maps was constructed by Glasner and Weiss [GlW] as an inverse limit of certain horocycle flows. For that they used some subtle facts from Ratner's theory of joinings for horocycle flows and properties of lattices in $\mathrm{SL}_2(\mathbb{R})$. Danilenko and del Junco [DdJ] later utilized a more elementary cutting-and-stacking technique to construct a weakly mixing 2-fold simple transformation which has countably many factors, all of which are prime.

Our purpose in the present paper is to use a similar cutting-and-stacking technique to produce a *mixing* transformation which has *uncountably* many factors, all of which are prime.

Via the (C,F)-construction we produce a measure preserving action T of an auxiliary group $G=\mathbb{Z}\times(\mathbb{R}\rtimes\mathbb{Z}_2)$ such that the transformation $T_{(1,0,0)}$ is mixing 2-fold simple and $C(T_{(1,0,0)})=\{T_g\mid g\in G\}$. Since all non-trivial compact subgroups of G are $G_b=\{(0,0,0),(0,b,1)\}$, $b\in\mathbb{R}$, and all of them are maximal, this gives an example of a 2-fold simple transformation with uncountably many prime factors. All these factors are 2-to-1 and pairwise isomorphic.

We also correct a gap in the proof of [DdJ, Lemma 2.3(ii)] (see Remark 2.4).

The skeleton of the proof of the main result is basically the same as in [DdJ], where the "discrete case" (i.e. when the auxiliary group is discrete) was under consideration. To work with the (C,F)-construction for actions of continuous (i.e. non-discrete) groups we use the approximation techniques from [Da2].

1. (C,F)-construction. We now briefly outline the (C,F)-construction of measure preserving actions for locally compact groups. For details see [Da1] and references therein.

Let G be a unimodular locally compact second countable (l.c.s.c.) amenable group. Fix a (σ-finite) Haar measure λ on it. Given $E,F\subset G$, we denote by EF their algebraic product, i.e. $EF=\{ef\mid e\in E,\ f\in F\}$. The set $\{e^{-1}\mid e\in E\}$ is denoted by E^{-1}. If E is a singleton, say $E=\{e\}$, then we write eF for EF. For abelian groups we use additive notation. Given a finite set A, $|A|$ will denote the cardinality of A. Given a subset $F\subset G$ of finite Haar measure, λ_F will denote the probability on F given by $\lambda_F(A):=\lambda(A)/\lambda(F)$ for each measurable $A\subset F$. If D is finite, then κ_D is the equidistributed probability on D, that is, $\kappa_D(A):=|A\cap D|/|D|$ for each A. The notation $a\neq b\in A$ will refer to two distinct elements a, b of a set A.

To define a (C,F)-action of G we need two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ of subsets in G such that the following are satisfied:

(1.1) $(F_n)_{n=0}^{\infty}$ is a Følner sequence in G,

(1.2) C_n is finite and $|C_n| > 1$,

(1.3) $F_n C_{n+1} \subset F_{n+1}$,

(1.4) $F_n c \cap F_n c' = \emptyset$ for all $c \neq c' \in C_{n+1}$.

This means that $F_n C_{n+1}$ consists of $|C_{n+1}|$ mutually disjoint 'copies' $F_n c$, $c \in C_{n+1}$, of F_n, and all these copies are contained in F_{n+1}.

First, we define a probability space (X, μ) in the following way. We equip F_n with the measure $(|C_1| \cdots |C_n|)^{-1} \lambda \restriction F_n$ and endow C_n with the equidistributed probability measure. Let $X_n := F_n \times \prod_{k>n} C_k$ stand for the product of measure spaces. Define an embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_n c_{n+1}, c_{n+2}, \ldots).$$

It is easy to see that this embedding is measure preserving. Then $X_1 \subset X_2 \subset \cdots$. Let $X := \bigcup_{n=0}^{\infty} X_n$ denote the inductive limit of the sequence of measure spaces X_n and let $\mathfrak{B}$ and μ denote the corresponding Borel σ-algebra and measure on X respectively. Then X is a standard Borel space and μ is σ-finite. It is easy to check that μ is finite if and only if

$$\lim_{n\to\infty} \frac{\lambda(F_n)}{|C_1| \cdots |C_n|} < \infty. \tag{1.5}$$

If (1.5) is satisfied then we choose (i.e., normalize) λ in such a way that $\mu(X) = 1$.

Now we define a μ-preserving action of G on X. Suppose that

(1.6) for any $g \in G$, there is $m \geq 0$ with $g F_n C_{n+1} \subset F_{n+1}$ for all $n \geq m$.

For such n, take $x \in X_n \subset X$ and write the expansion $x = (f_n, c_{n+1}, c_{n+2}, \ldots)$ with $f_n \in F_n$ and $c_i \in C_i$, $i > n$. Then we let

$$T_g x := (g f_n c_{n+1}, c_{n+2}, \ldots) \in X_{n+1} \subset X.$$

It follows from (1.6) that T_g is a well defined μ-preserving transformation of X. Moreover, $T_g T_h = T_{gh}$, i.e. $T := (T_g)_{g\in G}$ is a μ-preserving Borel action of G on X; it is called the (C, F)-*action of* G *associated with* $(C_{n+1}, F_n)_{n=0}^{\infty}$.

We now recall some basic properties of $(X, \mathfrak{B}, \mu, T)$. Given a Borel subset $A \subset F_n$, we set

$$[A]_n := \{x \in X \mid x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n \text{ and } f_n \in A\}$$

and call it an n-*cylinder*. It is clear that the σ-algebra $\mathfrak{B}$ is generated by the family of all cylinders. Given Borel subsets $A, B \subset F_n$, we have

$$[A\cap B]_n = [A]_n \cap [B]_n, \qquad [A\cup B]_n = [A]_n \cup [B]_n, \tag{1.7}$$

$$[A]_n = [AC_{n+1}]_{n+1} = \bigsqcup_{c\in C_{n+1}} [Ac]_{n+1}, \tag{1.8}$$

$$\mu([A]_n) = |C_{n+1}|\mu([Ac]_{n+1}) \qquad \text{for every } c\in C_{n+1}, \tag{1.9}$$

$$\mu([A]_n) = \mu(X_n)\lambda_{F_n}(A), \tag{1.10}$$

$$T_g[A]_n = [gA]_n \quad \text{if } gA\subset F_n, \tag{1.11}$$

$$T_g[A]_n = T_h[h^{-1}gA]_n \quad \text{if } h^{-1}gA\subset F_n. \tag{1.12}$$

Each (C,F)-action is of funny rank one (for the definition see [Fe] for the case of $\mathbb{Z}$-actions and [So] for the general case) and hence ergodic. It also follows from (1.2) that T is conservative.

2. Main result. We denote by $\mathbb{Z}_n$ a cyclic group of order n: $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0,1,\dots,n-1\}$. Let $G := \mathbb{Z}\times(\mathbb{R}\rtimes\mathbb{Z}_2)$ with multiplication law

$$(x,a,n)(y,b,m) := (x+y, a+(-1)^n b, n+m).$$

Then the center $C(G)$ of G is $\mathbb{Z}\times\{0\}\times\{0\}$. Each compact subgroup of G coincides with $G_b = \{(0,0,0),(0,b,1)\}$ for some $b\in\mathbb{R}$. Notice that G_b is a maximal compact subgroup of G for each $b\in\mathbb{R}$.

To construct the required (C,F)-action of G we will determine a sequence $(C_{n+1},F_n)_{n=0}^{\infty}$. Let $(r_n)_{n=0}^{\infty}$ be an increasing sequence of positive integers such that

$$\lim_{n\to\infty} n^4/r_n = 0. \tag{2.1}$$

Below—just after Lemma 2.1—one more restriction on the growth of $(r_n)_{n=0}^{\infty}$ will be imposed: we will assume that r_n is so large that (2.7) is satisfied. We recurrently define three other sequences $(\widetilde{a}_n)_{n=0}^{\infty}$, $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ of positive integers by setting

$$\begin{aligned}
&\widetilde{a}_0 := 1, \quad a_0 := 1,\\
&a_n := (2r_{n-1}+1)\widetilde{a}_{n-1} \qquad \text{for } n\ge 1,\\
&b_n := (2n-1)\widetilde{a}_{n-1} \qquad \text{for } n\ge 1,\\
&\widetilde{a}_n := a_n + b_n + n \qquad \text{for } n\ge 1.
\end{aligned}$$

For each $n\in\mathbb{N}$, we let

$$\begin{aligned}
I_n &:= \{-n,\dots,n\}^2 \subset \mathbb{Z}^2,\\
H_n &:= \{-r_n,\dots,r_n\}^2 \subset \mathbb{Z}^2,\\
F_n &:= (-a_n,a_n]_{\mathbb{Z}} \times (-a_n,a_n]_{\mathbb{R}} \times \mathbb{Z}_2,\\
S_n &:= (-b_n,b_n]_{\mathbb{Z}} \times (-b_n,b_n]_{\mathbb{R}} \times \mathbb{Z}_2,\\
\widetilde{F}_n &:= (-\widetilde{a}_n,\widetilde{a}_n]_{\mathbb{Z}} \times (-\widetilde{a}_n,\widetilde{a}_n]_{\mathbb{R}} \times \mathbb{Z}_2.
\end{aligned}$$

We also consider the homomorphism $\phi_n\colon \mathbb{Z}^2 \to G$ given by

$$\phi_n(i,j) := (2i\widetilde{a}_n, 2j\widetilde{a}_n, 0).$$

We then have

$$S_n \subset F_n, \quad F_nS_n = S_nF_n \subset \widetilde{F}_n \subset G, \tag{2.2}$$

$$S_{n+1} = \widetilde{F}_n\phi_n(I_n) = \bigsqcup_{h\in I_n} \widetilde{F}_n\phi_n(h) = \bigsqcup_{h\in I_n} \phi_n(h)\widetilde{F}_n, \tag{2.3}$$

$$F_{n+1} = \widetilde{F}_n\phi_n(H_n) = \bigsqcup_{h\in H_n} \widetilde{F}_n\phi_n(h) = \bigsqcup_{h\in H_n} \phi_n(h)\widetilde{F}_n, \tag{2.4}$$

We also equip F_n with a finite partition ξ_n such that:

(i) the diameter of each atom of ξ_n is less than $1/n$,
(ii) for each atom $A \in \xi_{n-1}$ and each $c \in C_n$, the subset $Ac \subset F_n$ is ξ_n-measurable, and
(iii) ξ_n is symmetric, that is, $A^{-1} \in \xi_n$ whenever $A \in \xi_n$.

For instance, we can define such partitions inductively in the following way. Each ξ_n will consist of 'rectangles' of the form $\{a\} \times \Delta \times \{m\} \subset G$, where $a \in (-a_n, a_n]_{\mathbb{Z}}$, $m \in \mathbb{Z}_2$ and $\Delta \subset (-a_n, a_n]_{\mathbb{R}}$ is a subinterval of length less than $1/n$. Let $\xi_0 := \{F_0\}$. Now suppose we have already defined ξ_{n-1}. Denote by $E_{n-1} \subset \mathbb{R}$ the finite set of endpoints of the intervals Δ for all the atoms $\{a\} \times \Delta \times \{m\} \in \xi_{n-1}$. Let $\pi_2\colon G \to \mathbb{R}$ stand for the natural projection on the second coordinate. Set

$$E'_n := E_{n-1} + \pi_2(C_n) \quad \text{and} \quad E_n := -E'_n \cup E'_n \cup \{k/n \mid k \in \mathbb{Z},\ |k| \le na_n\}.$$

The finite set $E_n \subset \mathbb{R}$ defines a partition $\xi_n^{(2)}$ of $[-a_n, a_n)_{\mathbb{R}}$ into intervals with endpoints in E_n. Denote by l_n the length of the shortest interval from $\xi_n^{(2)}$. Finally, set $\xi_n := \varepsilon_n^{(1)} \times \xi_n^{(2)} \times \varepsilon_n^{(3)}$, where $\varepsilon_n^{(1)}$ and $\varepsilon_n^{(3)}$ are partitions of $(-a_n, a_n]_{\mathbb{Z}}$ and $\mathbb{Z}_2$ respectively into single points. Properties (i)–(iii) are clearly satisfied for ξ_n.

It follows that for each measurable subset $A \subset F_n$, any $\varepsilon > 0$ and for all k large enough, there is a ξ_k-measurable subset $B \subset F_k$ such that $\mu([A]_n \triangle [B]_k) < \varepsilon$. We will denote by $\sigma(\xi_n)$ the σ-algebra on F_n generated by ξ_n.

For a finite subset D in S_n, we denote by κ_D the corresponding normalized *Dirac comb*, i.e. the measure on S_n given by $\kappa_D(A) := |A \cap D|/|D|$ for each subset $A \subset S_n$. Given two subsets $A, B \subset F_n$ define a function $f_{A,B}\colon S_n \times S_n \to \mathbb{R}$ by setting $f_{A,B}(x,y) := \lambda(Ax \cap By)/\lambda(F_n)$ for $x, y \in S_n$. We now define a finite subset D_n in S_n by

$$D_n := \{(a, k/(nl_n), m) \mid a \in (-b_n, b_n]_{\mathbb{Z}},\ k \in (-nl_nb_n, nl_nb_n]_{\mathbb{Z}},\ m \in \mathbb{Z}_2\}.$$

It is an easy exercise to check that for such ξ_n and D_n,

$$\left|\kappa_{D_n}(Ag) - \frac{1}{\lambda(S_n)}\lambda(Ag)\right| < \frac{1}{n} \tag{2.5}$$

for each ξ_n-measurable subset $A \subset F_n$ and $g \in F_n$, and

$$\left|\int_{S_n\times S_n} f_{Ag,Bh}\,d\kappa_{D_n}\,d\kappa_{D_n} - \frac{1}{\lambda(S_n)^2}\int_{S_n\times S_n} f_{Ag,Bh}\,d\lambda\,d\lambda\right| < \frac{1}{n} \tag{2.6}$$

for any ξ_n-measurable subsets $A, B \subset F_n$ and any $g, h \in F_n$ such that $AgS_n, BhS_n \subset F_n$. We also notice that $|D_n^0| = |D_n^1|$.

Given a finite (signed) measure ν on a finite set D, we let $\|\nu\|_1 := \sum_{d\in D}|\nu(d)|$. If $\pi\colon D \to E$ then clearly $\|\nu\circ\pi^{-1}\|_1 \le \|\nu\|_1$. Given a finite set Y and a mapping $s\colon Y \to D$, let $\mathrm{dist}_{y\in Y}\, s(y)$ denote the image of the equidistribution on Y under s:

$$\mathrm{dist}_{y\in Y}\, s(y) := \frac{1}{|Y|}\sum_{y\in Y}\delta_{s(y)} = \kappa_D \circ s^{-1}.$$

The following lemma easily follows from [dJ, Lemma 2.1] (cf. [Da2, Lemma 3.2]).

Lemma 2.1. *Let D be a finite set. Then given $\varepsilon, \delta > 0$, there is $R \in \mathbb{N}$ such that for each $r > R$, there exists a map $s\colon \{-r,\dots,r\}^2 \to D$ such that*

$$\left\|\mathrm{dist}_{0\le t<N}(s_n(h+(t,0)), s_n(h'+(t,0))) - \kappa_D\times\kappa_D\right\|_1 < \varepsilon$$

for each $N > \delta r$ and $h \ne h' \in \{-r,\dots,r\}^2$ with $h_1 + N < r$ and $h_1' + N < r$.

Applying this lemma with $\varepsilon = 1/n$ and $\delta = 1/n^2$ we get the following. If r_n is large enough then there is a mapping $s_n\colon H_n \to D_n$ such that for any $N > r_n/n^2$ and $h \ne h' \in H_n \cap (H_n - (N-1, 0))$ we have

$$\left\|\mathrm{dist}_{0\le t<N}(s_n(h+(t,0)), s_n(h'+(t,0))) - \kappa_{D_n}\times\kappa_{D_n}\right\|_1 < 1/n. \tag{2.7}$$

From now on we assume that r_n is so large that this condition is satisfied, and for each n we fix $s_n\colon H_n \to D_n$ satisfying (2.7).

Now we define a map $c_{n+1}\colon H_n \to G$ by setting $c_{n+1}(h) := s_n(h)\phi_n(h)$. We set $C_{n+1} := c_{n+1}(H_n)$.

The reader should have the following picture in mind. The set F_{n+1} is exactly tiled with the sets $\widetilde{F}_n\phi_n(h)$, $h \in H_n$, which may be thought of as 'windows'. Each F_n has a 'natural' translate $F_n\phi_n(h)$ in $\widetilde{F}_n\phi_n(h)$ but the translate we actually choose is the natural translate perturbed by a further translation $s_n(h)$ which is chosen in a 'random' way and does not move $F_n\phi_n(h)$ out of its window.

It is easy to derive that properties (1.1)–(1.6) are satisfied for the sequence $(F_n, C_{n+1})_{n=0}^\infty$. Hence the associated (C,F)-action $T = (T_g)_{g\in G}$ of G is well defined on a standard probability space $(X, \mathfrak{B}, \mu)$.

We now state the main result.

THEOREM 2.2. *The transformation $T_{(1,0,0)}$ is mixing and 2-fold simple. All non-trivial proper factors of $T_{(1,0,0)}$ are of the form $\mathfrak{F}_{G_b}$, $b \in \mathbb{R}$. All these factors are 2-to-1, prime and pairwise isomorphic.*

We first prove some technical lemmata. After that in Proposition 2.8 we show mixing for $T_{(1,0,0)}$, and in Proposition 2.9 we prove simplicity and describe the centralizer of $T_{(1,0,0)}$. The structure of factors then follows from Veech's theorem.

Denote by G^0 the subgroup $\mathbb{Z} \times \mathbb{R} \times \{0\}$ of index 2 in G. Given any subset A in G we set $A^0 := A \cap G^0$ and $A^1 := A \setminus A^0$. We will refer to A^0 and A^1 as *levels* of A. We will say that a subset $A \subset G$ is *ε-balanced* if

$$|\lambda(A^0) - \lambda(A^1)| < \varepsilon \lambda(A).$$

Denote by $\pi_3 \colon G \to \mathbb{Z}_2$ the natural projection on the third coordinate. Since $\kappa_{D_n} \circ \pi_3^{-1} = \kappa_{\mathbb{Z}_2}$, it follows from (2.7) that

$$\|\mathrm{dist}_{h \in H_n} \pi_3 \circ s_n(h) - \kappa_{\mathbb{Z}_2}\|_1 < 1/n. \tag{2.8}$$

In particular, for any $A^* \subset F_n$ the set $A = A^* C_{n+1}$ is $1/n$-balanced:

$$|\lambda(A^0) - \lambda(A^1)| < \frac{1}{n}\lambda(A). \tag{2.9}$$

Indeed, since

$$A^0 = \bigsqcup_{h \in s_n^{-1}(G^0)} A^{*0} c_n(h) \sqcup \bigsqcup_{h \in s_n^{-1}(G^1)} A^{*1} c_n(h),$$

we have

$$\lambda(A^0) = \lambda(A^{*0})|s_n^{-1}(G^0)| + \lambda(A^{*1})|s_n^{-1}(G^1)|,$$

and similarly

$$\lambda(A^1) = \lambda(A^{*1})|s_n^{-1}(G^0)| + \lambda(A^{*0})|s_n^{-1}(G^1)|.$$

Hence

$$\begin{aligned} |\lambda(A^0) - \lambda(A^1)| &= |\lambda(A^{*0}) - \lambda(A^{*1})| \, \big||s_n^{-1}(G^0)| - |s_n^{-1}(G^1)|\big| \\ &\le \frac{1}{|H_n|}\lambda(A)\big||s_n^{-1}(G^0)| - |s_n^{-1}(G^1)|\big|. \end{aligned}$$

It remains to notice that

$$\begin{aligned} \frac{1}{|H_n|}\big||s_n^{-1}(G^0)| - |s_n^{-1}(G^1)|\big| &\le \left|\frac{|s_n^{-1}(G^0)|}{|H_n|} - \frac{1}{2}\right| + \left|\frac{|s_n^{-1}(G^1)|}{|H_n|} - \frac{1}{2}\right| \\ &= \|\mathrm{dist}_{h \in H_n} \pi_3 \circ s_n(h) - \kappa_{\mathbb{Z}_2}\|_1 < 1/n \end{aligned}$$

by (2.9). It follows that $A = A^* C_{n+1}$ is $1/n$-balanced for each $A^* \subset F_n$.

Given $h = (h_1, h_2) \in \mathbb{Z}^2$, we let $h^* := (h_1, -h_2)$.

LEMMA 2.3. *Let $f = f'\phi_{n-1}(h)$ with $f' \in \widetilde{F}_{n-1}$ and $h \in \mathbb{Z}^2$.*

(i) *Suppose $f \in G^\alpha$ and let $\beta := 1 - \alpha$. Let*

$$L_n^- := \widetilde{F}_{n-1}^\alpha \phi_{n-1}(I_{n-2} + h) \sqcup \widetilde{F}_{n-1}^\beta \phi_{n-1}(I_{n-2} + h^*),$$
$$L_n^+ := \widetilde{F}_{n-1}^\alpha \phi_{n-1}(I_n + h) \sqcup \widetilde{F}_{n-1}^\beta \phi_{n-1}(I_n + h^*).$$

Then $L_n^- \subset fS_n \subset L_n^+$. Hence

$$\frac{\lambda(fS_n \triangle L_n^-)}{\lambda(S_n)} = \overline{o}(1).$$

(ii) *If, in addition, $fS_n \subset F_n$ then for any subset $A = A^* C_{n-1}$ with $A^* \subset F_{n-2}$ we have*

$$\frac{\lambda(AC_n \cap fS_n)}{\lambda(S_n)} = \lambda_{F_{n-1}}(A) + \overline{o}(1).$$

Here $\overline{o}(1)$ means a sequence that goes to 0 as $n \to \infty$ and does not depend on the choice of A^* in F_{n-2}.

Proof. (i) Suppose $f \in G^0$ (the case $f \in G^1$ is considered in a similar way). We have

$$\begin{aligned} fS_n &= f'\phi_{n-1}(h)\widetilde{F}_{n-1}\phi_{n-1}(I_{n-1}) \\ &= f'\widetilde{F}_{n-1}^0 \phi_{n-1}(h + I_{n-1}) \sqcup f'\widetilde{F}_{n-1}^1 \phi_{n-1}(h^* + I_{n-1}). \end{aligned}$$

Since $\widetilde{F}_{n-1}^0 \widetilde{F}_{n-1}^\alpha \subset \bigsqcup_{u \in I_1} \widetilde{F}_{n-1}^\alpha \phi_{n-1}(u)$, there exists a partition of $\widetilde{F}_{n-1}^\alpha$ into subsets A_u^α, $u \in I_1$, such that $f' A_u^\alpha \subset \widetilde{F}_{n-1}^\alpha \phi_{n-1}(u)$ for any u and $\alpha = 0, 1$. Therefore

$$fS_n = \bigsqcup_{u \in I_1} \big(f' A_u^0 \phi_{n-1}(u)^{-1} \phi_{n-1}(u+h+I_{n-1}) \sqcup f' A_u^1 \phi_{n-1}(u)^{-1} \phi_{n-1}(u+h^*+I_{n-1})\big).$$

It remains to notice that $\bigsqcup_{u \in I_1} f' A_u^\alpha \phi_{n-1}(u)^{-1} = \widetilde{F}_{n-1}^\alpha$.

(ii) Since $fS_n \subset F_n$ and $F_n = \widetilde{F}_{n-1}\phi_{n-1}(H_{n-1})$, it follows from (i) that the subsets $K := I_{n-1} + h$ and $K^* := I_{n-1} + h^*$ are contained in H_{n-1}. Therefore

$$\begin{aligned} \frac{\lambda(AC_n \cap fS_n)}{\lambda(S_n)} &= \sum_{k \in H_{n-1}} \frac{\lambda(Ac_n(k) \cap fS_n)}{\lambda(S_n)} = \sum_{k \in H_{n-1}} \frac{\lambda(Ac_n(k) \cap L_n^-)}{\lambda(S_n)} + \overline{o}(1) \\ &= \frac{1}{\lambda(S_n)} \sum_{k \in H_{n-1}} \lambda\big(As_{n-1}(k)\phi_{n-1}(k) \cap \widetilde{F}_{n-1}^\alpha \phi_{n-1}(K) \sqcup \widetilde{F}_{n-1}^\beta \phi_{n-1}(K^*)\big) + \overline{o}(1) \\ &= \frac{1}{\lambda(S_n)} \sum_{k \in K} \lambda(As_{n-1}(k) \cap \widetilde{F}_{n-1}^\alpha) + \frac{1}{\lambda(S_n)} \sum_{k \in K^*} \lambda(As_{n-1}(k) \cap \widetilde{F}_{n-1}^\beta) + \overline{o}(1). \end{aligned}$$

Notice that

$$\lambda(As_{n-1}(k)\cap \widetilde{F}^{\alpha}_{n-1}) = \begin{cases} \lambda(A^{\alpha}) & \text{if } s_{n-1}(k)\in G^0,\\ \lambda(A^{\beta}) & \text{if } s_{n-1}(k)\in G^1.\end{cases}$$

In any case, since $A = A'C_{n-1}$ is $\frac{1}{n-2}$-balanced, we conclude from (2.9) that

$$\lambda(As_{n-1}(k)\cap \widetilde{F}^{\alpha}_{n-1}) = (1/2+\overline{o}(1))\lambda(A).$$

In a similar way

$$\lambda(As_{n-1}(k)\cap \widetilde{F}^{\beta}_{n-1}) = (1/2+\overline{o}(1))\lambda(A).$$

Hence

$$\begin{aligned}\frac{\lambda(AC_n\cap fS_n)}{\lambda(S_n)} &= \frac{\lambda(A)|K|(1+\overline{o}(1))}{\lambda(S_n)} + \overline{o}(1)\\ &= \frac{\lambda(A)}{\lambda(F_{n-1})}\cdot\frac{\lambda(F_{n-1})|K|}{\lambda(S_n)}\cdot(1+\overline{o}(1))+\overline{o}(1)\\ &= \lambda_{F_{n-1}}(A)\cdot\frac{\lambda(F_{n-1})(2n-1)^2}{(2n+1)^2\lambda(\widetilde{F}_{n-1})}\cdot(1+\overline{o}(1))+\overline{o}(1)\\ &= \lambda_{F_{n-1}}(A)+\overline{o}(1). \ \blacksquare\end{aligned}$$

REMARK 2.4. We note that there is a gap in [DdJ, Lemma 2.3(ii)]. It was stated there that the claim (ii) is true for each subset $A\subset F_{n-1}$. This is not true. However—as shown in Lemma 2.3(ii) above—the claim is true if $A = A^*C_{n-1}$ for an arbitrary subset $A^*\subset F_{n-2}$. This corrected version of the claim suffices to apply it in the proof of [DdJ, Theorem 2.5] which is the only place in that paper where [DdJ, Lemma 2.3(ii)] was used.

We will also use the following simple lemma.

LEMMA 2.5. *Let A, B and S be subsets of finite Haar measure in G. Then*

$$\int_{S\times S}\lambda(Ax\cap By)\,d\lambda(x)\,d\lambda(y) = \int_{A\times B}\lambda(aS\cap bS)\,d\lambda(a)\,d\lambda(b).$$

Proof. Notice that G is unimodular. Consider two subsets in G^3:

$$\begin{aligned}\Omega_1 &:= \{(a,x,y)\mid x\in S,\ y\in S,\ a\in A\cap Byx^{-1}\}\\ &= \{(a,x,y)\mid a\in A,\ y\in S,\ x\in a^{-1}By\cap S\},\\ \Omega_2 &:= \{(a,b,y)\mid a\in A,\ b\in B,\ y\in b^{-1}aS\cap S\}\\ &= \{(a,b,y)\mid a\in A,\ y\in S,\ b\in B\cap aSy^{-1}\}.\end{aligned}$$

It is clear that the maping $\Omega_1\ni(a,x,y)\mapsto(a,axy^{-1},y)\in\Omega_2$ is 1-to-1 and

λ^3-preserving. Applying the Fubini theorem we obtain

$$\int_{S\times S} \lambda(Ax \cap By)\, d\lambda(x)\, d\lambda(y) = \lambda^3(\Omega_1) = \lambda^3(\Omega_2)$$
$$= \int_{A\times B} \lambda(aS \cap bS)\, d\lambda(a)\, d\lambda(b). \quad \blacksquare$$

The following lemma is the first step to prove mixing for $T_{(1,0,0)}$. Let $h_0 := (1,0) \in \mathbb{Z}^2$. Then $\phi_n(h_0) = (1,0,0)^{2\widetilde{a}_n}$.

Lemma 2.6. *Given a sequence of subsets $H_n^* \subset H_n$ such that $|H_n^*|/|H_n| \to \delta$ for some $\delta \geq 0$, let $C_n^* := c_n(H_{n-1}^*)$. Then*

$$\sup_{A^*, B^* \in \sigma(\xi_{n-1})} \left|\mu(T_{\phi_n(h_0)}[A^* C_n^*]_n \cap [B^*]_{n-1}) - \mu([A^* C_n^*]_n)\mu([B^*]_{n-1})\right| \to 0.$$

Proof. Let $A, B \in \sigma(\xi_n)$. We set $F_n^\circ := \{f \in F_n \mid f S_n S_n^{-1} \subset F_n\}$, $A^\circ := A \cap F_n^\circ$, $B^\circ := B \cap F_n^\circ$, $H_n' := H_n \cap (H_n - h_0)$. It is clear that $\mu(F_n \setminus F_n^\circ) \to 0$ and $|H_n'|/|H_n| \to 1$ as $n \to \infty$. Since $\phi_n(h_0) \in C(G)$ for all $n \in \mathbb{N}$, we have

$$\phi_n(h_0) A c_{n+1}(h) = A s_n(h) \phi_n(h_0 + h) = A s_n(h) s_n(h_0 + h)^{-1} c_{n+1}(h_0 + h)$$

whenever $h \in H_n'$. In particular, $\phi_n(h_0) A^\circ c_{n+1}(h) \subset F_{n+1}$ for all $h \in H_n'$. Then

$$\begin{aligned}
\mu(T_{\phi_n(h_0)}[A]_n \cap [B]_n) &= \mu(T_{\phi_n(h_0)}[A^\circ]_n \cap [B^\circ]_n) + \overline{o}(1) \\
&= \sum_{h \in H_n} \mu\big(T_{\phi_n(h_0)}[A^\circ c_{n+1}(h)]_{n+1} \cap [B^\circ]_n\big) + \overline{o}(1) \\
&= \sum_{h \in H_n'} \mu\big(T_{\phi_n(h_0)}[A^\circ c_{n+1}(h)]_{n+1} \cap [B^\circ]_n\big) + \overline{o}(1) \\
&= \sum_{h \in H_n'} \mu\big([A^\circ s_n(h) s_n(h_0+h)^{-1} c_{n+1}(h_0+h)]_{n+1} \cap [B^\circ]_n\big) + \overline{o}(1) \\
&= \sum_{h \in H_n'} \mu\big([(A^\circ s_n(h) s_n(h_0+h)^{-1} \cap B^\circ) c_{n+1}(h_0+h)]_{n+1}\big) + \overline{o}(1) \\
&= \frac{1}{|H_n|} \sum_{h \in H_n'} \mu\big([A^\circ s_n(h) s_n(h_0+h)^{-1} \cap B^\circ]_n\big) + \overline{o}(1) \\
&= \frac{1}{|H_n|} \sum_{h \in H_n'} \lambda_{F_n}\big(A^\circ s_n(h) \cap B^\circ s_n(h_0+h)\big)\mu(X_n) + \overline{o}(1) \\
&= \frac{1}{|H_n'|} \sum_{h \in H_n'} \lambda_{F_n}\big(A^\circ s_n(h) \cap B^\circ s_n(h_0+h)\big) + \overline{o}(1) \\
&= \frac{1}{|H_n'|} \sum_{h \in H_n'} \lambda_{F_n}\big(A s_n(h) \cap B s_n(h_0+h)\big) + \overline{o}(1).
\end{aligned}$$

Let $\nu_n := \mathrm{dist}_{h\in H'_n}(s_n(h), s_n(h+h_0))$. Set $f_{A,B}(x,y) := \lambda_{F_n}(Ax\cap By) = \lambda(Ax\cap By)/\lambda(F_n)$. Notice that

$$\nu_n = \frac{1}{2r_n-1}\sum_{i=-r_n}^{r_n} \mathrm{dist}_{-r_n\le t<r_n}(s_n(t,i), s_n(t+1,i)).$$

It follows from (2.7) that $\|\nu_n - \kappa_{D_n}\times\kappa_{D_n}\|_1 < 1/n$. Then by (2.6),

$$\mu(T_{\phi_n(h_0)}[A]_n\cap[B]_n) = \int_{S_n\times S_n} f_{A,B}\,d\nu_n + \overline{o}(1)$$
$$= \int_{S_n\times S_n} f_{A,B}\,d\kappa_{D_n}\,d\kappa_{D_n} + \overline{o}(1) = \frac{1}{\lambda(S_n)^2}\int_{S_n\times S_n} f_{A,B}\,d\lambda\,d\lambda + \overline{o}(1),$$

Now let $A := A^*C_n$ and $B := B^*C_n$ for some ξ_{n-1}-measurable subsets $A^*, B^*\subset F_{n-1}$. We say that elements c and c' of C_n are *partners* if $F_{n-1}cS_n\cap F_{n-1}c'S_n\neq\emptyset$. We then write $c\bowtie c'$. Since $A^*cx\cap B^*c'y=\emptyset$ for $c\not\bowtie c'$, it follows that

$$\int_{S_n\times S_n} f_{A,B}\,d\lambda\,d\lambda = \int_{S_n\times S_n}\lambda_{F_n}(A^*C_nx\cap B^*C_ny)\,d\lambda(x)\,d\lambda(y)$$
$$= \frac{1}{\lambda(F_n)}\int_{S_n\times S_n}\sum_{C_n^*\ni c\bowtie c'\in C_n}\lambda(A^*cx\cap B^*c'y)\,d\lambda(x)\,d\lambda(y).$$

Applying Lemma 2.5 we now obtain

$$\int_{S_n\times S_n} f_{A,B}\,d\lambda\,d\lambda = \frac{1}{\lambda(F_n)}\sum_{C_n^*\ni c\bowtie c'\in C_n}\int_{A^*\times B^*}\lambda(acS_n\cap bc'S_n)\,d\lambda(a)\,d\lambda(b).$$

Next, we note that

$$|\lambda(acS_n\cap bc'S_n) - \lambda(cS_n\cap c'S_n)| \le 8n\lambda(\widetilde{F}_{n-1}) = \overline{o}(1)\lambda(S_n).$$

Each $c\in C_n$ has no more than $2(4n+1)^2$ partners. Therefore

$$\mu(T_{\phi_n(h_0)}[A^*C_n^*]_n\cap[B^*]_{n-1})$$
$$= \frac{1}{\lambda(S_n)^2}\sum_{C_n^*\ni c\bowtie c'\in C_n}\int_{A^*\times B^*}\frac{\lambda(cS_n\cap c'S_n)+\lambda(S_n)\overline{o}(1)}{\lambda(F_n)}\,d\lambda(a)\,d\lambda(b) + \overline{o}(1)$$
$$= \frac{\lambda(A^*)\lambda(B^*)}{\lambda(F_{n-1})^2}\,\frac{\lambda(F_{n-1})^2}{\lambda(S_n)^2\lambda(F_n)}\sum_{C_n^*\ni c\bowtie c'\in C_n}\big(\lambda(cS_n\cap c'S_n)+\lambda(S_n)\overline{o}(1)\big) + \overline{o}(1)$$
$$= \lambda_{F_{n-1}}(A^*)\lambda_{F_{n-1}}(B^*)\theta_n \pm \frac{\lambda(F_{n-1})^2|H_n^*|2(4n+1)^2\lambda(S_n)\overline{o}(1)}{\lambda(S_n)^2\lambda(F_n)} + \overline{o}(1)$$
$$= \lambda_{F_{n-1}}(A^*)\lambda_{F_{n-1}}(B^*)\theta_n \pm \frac{\lambda(F_{n-1})^2|H_n^*|2(4n+1)^2\overline{o}(1)}{\lambda(\widetilde{F}_{n-1})^2(2n-1)^2|H_n|} + \overline{o}(1)$$
$$= \lambda_{F_{n-1}}(A^*)\lambda_{F_{n-1}}(B^*)\theta_n + \overline{o}(1),$$

where

$$\theta_n = \frac{\lambda(F_{n-1})^2}{\lambda(S_n)^2\lambda(F_n)} \sum_{C_n^* \ni c \bowtie c' \in C_n} \lambda(cS_n \cap c'S_n).$$

Substituting $A^* = B^* = F_{n-1}$ and passing to the limit we find that $\theta_n \to \delta$ as $n \to \infty$. Hence

$$\mu(T_{\phi_n(h_0)}[A^* C_n^*]_n \cap [B^*]_{n-1}) = \mu([A^* C_n^*]_n)\mu([B^*]_{n-1}) + \overline{o}(1).$$

Since $\overline{o}(1)$ does not depend on the choice of A^* and B^* inside F_{n-1}, the claim is proven. ■

Corollary 2.7. *The transformation $T_{(1,0,0)}$ is weakly mixing.*

Proof. Substituting $H_n^* := H_n$ in Lemma 2.6 we obtain

$$\sup_{A^*, B^* \in \sigma(\xi_{n-1})} \left|\mu(T_{\phi_n(h_0)}[A^*]_{n-1} \cap [B^*]_{n-1}) - \mu([A^*]_{n-1})\mu([B^*]_{n-1})\right| \to 0.$$

Since each measurable subset of X can be approximated by $[A^*]_{n-1}$ for large n and some ξ_{n-1}-measurable subset $A^* \subset F_{n-1}$, it follows that the sequence $(\phi_n(h_0))_{n=1}^\infty$ is mixing for T, that is, $\mu(T_{\phi_n(h_0)}A \cap B) \to \mu(A)\mu(B)$ for every pair of measurable subsets $A, B \subset X$. ■

Proposition 2.8. *The transformation $T_{(1,0,0)}$ is mixing.*

Proof. We have to show that

$$\lim_{n\to\infty} \mu(T_{g_n}A \cap B) = \mu(A)\mu(B)$$

for any sequence $(g_n)_{n=1}^\infty$ that goes to infinity in $C(G)$ and every pair of measurable subsets $A, B \subset X$. Let $g_n \in F_{n+1} \setminus F_n$. It suffices to show that a subsequence of $(g_n)_{n=1}^\infty$ is mixing for T. We write $g_n = f_n\phi_n(h_n)$ for some $f_n \in \widetilde{F}_n \cap C(G)$ and $h_n \in H_n$. Denote by $z\colon \mathbb{Z} \to C(G)$ the natural embedding $z(x) := (x, 0, 0)$. We may assume that $f_n \in z(\mathbb{Z}_+)$ for all n (the case $f_n \in z(\mathbb{Z}_-)$ is considered in a similar way). Let $H_n' := H_n \cap (H_n - h_n)$ and $F_n' := F_n \cap (f_n^{-1}F_n)$. Passing to a subsequence if necessary, we may also assume without loss of generality that

$$\frac{|H_n'|}{|H_n|} \to \delta_1 \quad \text{and} \quad \frac{\lambda(F_n')}{\lambda(F_n)} \to \delta_2$$

for some $\delta_1, \delta_2 \geq 0$. Partition H_n into

$$\begin{aligned} H_n^1 &:= \{h \in H_n \mid g_nF_nc_{n+1}(h) \subset F_{n+1}\phi_{n+1}(h_0)\}, \\ H_n^2 &:= \{h \in H_n \mid g_nF_nc_{n+1}(h) \subset F_{n+1}\}, \\ H_n^3 &:= H_n \setminus (H_n^1 \sqcup H_n^2). \end{aligned}$$

As before, $h_0 = (1, 0) \in \mathbb{Z}^2$. Let $C_{n+1}^i := \phi_{n+1}(H_n^i)$. It is clear that $|H_n^3| \leq 4(n+1)(2r_n+1)$ and $\left|H_n^2 \triangle H_n'\right| \leq 2r_n + 1$. Since $|H_n| = (2r_n+1)^2$, it follows that

$$\frac{|H_n^1|}{|H_n|} \to 1-\delta_1, \qquad \frac{|H_n^2|}{|H_n|} \to \delta_1, \qquad \frac{|H_n^3|}{|H_n|} \to 0.$$

Take two ξ_n-measurable subsets $A, B \subset F_n$. Since

$$\mu([AC_{n+1}^3]_{n+1}) = \frac{|C_{n+1}^3|}{|C_{n+1}|}\mu([A]_n) \le \frac{1}{2r_n+1} \to 0,$$

we have

$$\left|\mu(T_{g_n}[AC_{n+1}^3]_{n+1} \cap [B]_n) - \mu([AC_{n+1}^3]_{n+1})\mu([B]_n)\right| \to 0, \tag{2.10}$$

so $[F_nC_{n+1}^3]_{n+1}$ is negligible. It suffices to show mixing separately on each of the remaining subsets $[F_nC_{n+1}^1]_{n+1}$ and $[F_nC_{n+1}^2]_{n+1}$.

First, we note that $\phi_{n+1}(h_0)^{-1}g_nF_nC_{n+1}^1 \subset F_{n+1}$. Thus, by (1.12),

$$T_{g_n}[AC_{n+1}^1]_{n+1} = T_{\phi_{n+1}(h_0)}[\phi_{n+1}(h_0)^{-1}g_nAC_{n+1}^1]_{n+1}.$$

By Lemma 2.6 (with $C_{n+1}^* := \phi_{n+1}(h_0)^{-1}\phi_n(h_n)C_{n+1}^1$ and $A^* := f_nA$) we obtain

$$\left|\mu(T_{g_n}[AC_{n+1}^1]_{n+1} \cap [B]_n) - \mu([AC_{n+1}^1]_{n+1})\mu([B]_n)\right| \to 0. \tag{2.11}$$

It remains to consider the second case involving C_{n+1}^2. If $\delta_1 = 0$, then obviously

$$\mu([AC_{n+1}^2]_{n+1}) \to 0. \tag{2.12}$$

Suppose now that $\delta_1 > 0$. Partition A into $A_1 := A \cap f_n^{-1}F_n$, $A_2 := A \cap f_n^{-1}F_n\phi_n(h_0)$ and $A_3 := A \setminus (A_1 \sqcup A_2)$. In other words, $f_nA_1 \subset F_n$, $f_nA_2 \subset F_n\phi_n(h_0)$, $f_nA_3 \cap (F_n \sqcup F_n\phi_n(h_0)) = \emptyset$.

Note that

$$\mu([A_3C_{n+1}^2]_{n+1}) \le \mu([A_3]_n) \le \frac{2n+1}{2r_n+1} \to 0. \tag{2.13}$$

For A_1 and A_2 we argue as in the proof of Lemma 2.6. Set $F_n^\circ := \{f \in F_n \mid fS_nS_n^{-1} \subset F_n\}$, $A_1^\circ := A_1 \cap F_n^\circ$ and $B^\circ := B \cap F_n^\circ$. We have

$$\begin{aligned}
\mu(T_{g_n}[A_1C_{n+1}^2]_{n+1} \cap [B]_n) &= \sum_{h\in H_n'} \mu\big([\phi_n(h_n)f_nA_1^\circ c_{n+1}(h)]_{n+1} \cap [B^\circ]_n\big) + \overline{o}(1) \\
&= \sum_{h\in H_n'} \mu\big([(f_nA_1^\circ s_n(h)s_n(h_n+h)^{-1} \cap B^\circ)c_{n+1}(h)]_{n+1}\big) + \overline{o}(1) \\
&= \frac{1}{|H_n|}\sum_{h\in H_n'} \mu\big([f_nA_1^\circ s_n(h)s_n(h_n+h)^{-1} \cap B^\circ]_n\big) + \overline{o}(1) \\
&= \frac{\delta_1}{|H_n'|}\sum_{h\in H_n'} \lambda_{F_n}\big(f_nA_1^\circ s_n(h) \cap B^\circ s_n(h_n+h)_n\big) + \overline{o}(1) \\
&= \delta_1 \int_{S_n\times S_n} f_{A_1f_n,B}\, d\nu_n + \overline{o}(1),
\end{aligned}$$

where $\nu_n := \mathrm{dist}_{h\in H_n'}(s_n(h), s_n(h_n+h))$ and $f_{A_1 f_n,B}(x,y) = \lambda_{F_n}(A_1 f_n x \cap By)$. Write $h_n = (t_n, 0)$. Since

$$\frac{2r_n - t_n + 1}{2r_n + 1} = \frac{|H_n'|}{|H_n|} \to \delta_1 > 0$$

and

$$\nu_n = \frac{1}{2r_n - 1} \sum_{i=-r_n}^{r_n} \mathrm{dist}_{-r_n \le t \le r_n - t_n}(s_n(t,i), s_n(t+t_n, i)),$$

it follows from (2.7) and (2.6) that

$$\mu(T_{g_n}[A_1 C_{n+1}^2]_{n+1} \cap [B]_n) = \frac{\delta_1}{\lambda(S_n)^2} \int_{S_n\times S_n} f_{A_1 f_n, B}\, d\lambda\, d\lambda + \overline{o}(1).$$

Now take $A := A^* C_n^*$ and $B := B^* C_n$ for some ξ_{n-1}-measurable subsets $A^*, B^* \subset F_{n-1}$. Let $C_n' := C_n \cap F_n'$. It follows that $|C_n'|/|C_n| \to \delta_2$ and $\mu([A_1]_n \triangle [A^* C_n]_n) = \overline{o}(1)$. Hence $\mu([A_1]_n) = \delta_2 \mu([A^*]_{n-1}) + \overline{o}(1)$. Arguing as in the proof of Lemma 2.6 we obtain

$$\mu(T_{g_n}[A^* C_n' C_{n+1}^2]_{n+1} \cap [B^*]_{n-1}) = \delta_2 \mu([A^*]_{n-1})\mu([B^*]_{n-1}) + \overline{o}(1).$$

Therefore

$$\left|\mu(T_{g_n}[A_1 C_{n+1}^2]_{n+1} \cap [B]_n) - \mu([A_1 C_{n+1}^2]_{n+1})\mu([B]_n)\right| \to 0. \tag{2.14}$$

Since $T_{g_n}[A_2]_n = T_{\phi_n(h_n+h_0)}[\phi_n(h_0)^{-1} f_n A_2]$ with $\phi_n(h_0)^{-1} f_n A_2 \subset F_n$, a similar reasoning yields

$$\left|\mu(T_{g_n}[A_2 C_{n+1}^2]_{n+1} \cap [B]_n) - \mu([A_2 C_{n+1}^2]_{n+1})\mu([B]_n)\right| \to 0. \tag{2.15}$$

Since

$$[A^*]_{n-1} = [A^* C_n C_{n+1}^1]_{n+1} \sqcup \bigsqcup_{i=1}^{3} [A_i C_{n+1}^2]_{n+1} \sqcup [A^* C_n C_{n+1}^3]_{n+1},$$

it follows from (2.10)–(2.15) that

$$\lim_{n\to\infty} \sup_{A^*, B^* \in \sigma(\xi_{n-1})} \left|\mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) - \mu([A^*]_{n-1})\mu([B^*]_{n-1})\right| = 0.$$

Since ξ_n-measurable cylinders generate the entire σ-algebra $\mathfrak{B}$ as $n \to \infty$, we conclude that $(g_n)_{n=1}^\infty$ is a mixing sequence for T, as desired. ■

PROPOSITION 2.9. *The transformation $T_{(1,0,0)}$ is 2-fold simple and $C(T_{(1,0,0)}) = \{T_g \mid g \in G\}$.*

Proof. Take an ergodic joining $\nu \in J_2^{\mathrm{e}}(T_{(1,0,0)})$. Let

$$K_n := [-a_n/n^2, a_n/n^2]_{\mathbb{Z}}, \quad J_n := [-r_n/n^2, r_n/n^2]_{\mathbb{Z}}, \quad \Phi_n := K_n + 2\widetilde{a}_n J_n.$$

We claim that ν-a.e. point $(x,y)\in X\times X$ is *generic* for $T_{(1,0,0)}\times T_{(1,0,0)}$, i.e. for all cylinders $A,B\subset\bigcup_{n=1}^\infty\sigma(\xi_n)$ we have

$$\nu(A\times B)=\lim_{n\to\infty}\frac{1}{|\Phi_n|}\sum_{i\in\Phi_n}\chi_A(T_{(i,0,0)}x)\chi_B(T_{(i,0,0)}y). \tag{2.16}$$

To see this, we first note that $(\Phi_n)_{n=1}^\infty$ is a Følner sequence in $\mathbb{Z}$. Since

$$\frac{a_n}{n^2}+\frac{2\widetilde{a}_nr_n}{n^2}<\frac{\widetilde{a}_n(2r_n+1)}{n^2}<\frac{2a_{n+1}}{(n+1)^2},$$

it follows that $\Phi_n\subset K_{n+1}+K_{n+1}$ and hence $\bigcup_{m=1}^n\Phi_m\subset K_{n+1}+K_{n+1}$. This implies that $|\Phi_{n+1}+\bigcup_{m\le n}\Phi_m|\le 3|\Phi_{n+1}|$ for every $n\in\mathbb{N}$, i.e. Shulman's condition [Li] is satisfied for $(\Phi_n)_{n=1}^\infty$. By [Li], the pointwise ergodic theorem holds along $(\Phi_n)_{n=1}^\infty$ for any ergodic transformation. Since $T\times T$ is ν-ergodic, (2.16) holds for ν-a.a. $(x,y)\in X\times X$ and for every pair of cylinders $A,B\subset X$ from $\bigcup_{n=1}^\infty\sigma(\xi_n)$.

Fix a generic point $(x,y)\in X\times X$. Since $x,y\in X_n$ for all sufficiently large n, we have the expansions

$$x=(f_n,c_{n+1}(h_n),c_{n+2}(h_{n+1}),\dots),$$
$$y=(f_n',c_{n+1}(h_n'),c_{n+2}(h_{n+1}'),\dots)$$

with $f_n,f_n'\in F_n$, $h_i,h_i'\in H_i$, $i\ge n$. We let

$$H_n^-:=[-(1-1/n^2)r_n,(1-1/n^2)r_n]_{\mathbb{Z}}^2\subset H_n.$$

Since the marginals of ν are both equal to μ, we may assume without loss of generality that $h_n,h_n'\in H_n^-$. Indeed,

$$\mu\big(\{x=(f_n,c_{n+1}(h_n),c_{n+2}(h_{n+1}),\dots)\in X_n\mid h_n\notin H_i^-\}\big)<2/i^2,$$

and by the Borel–Cantelli lemma for μ-a.e. $x\in X_n$ we have $h_i\in H_i^-$ for all but finitely many i. So we may replace $x=(f_n,c_{n+1}(h_n),c_{n+2}(h_{n+1}),\dots)\in X_n$ with $x=(f_nc_{n+1}(h_n)\cdots c_m(h_{m-1}),c_{m+1}(h_m),\dots)\in X_m$ for some $m>n$ if necessary. Similarly, $h_n'\in H_n^-$.

This implies, in turn, that

$$f_{n+1}=f_nc_{n+1}(h_n)\in\widetilde{F}_n\phi_n(H_n^-)\subset[-c_n,c_n]_{\mathbb{Z}}\times[-c_n,c_n]_{\mathbb{R}}\times\mathbb{Z}_2, \tag{2.17}$$

where $c_n=\widetilde{a}_n(1+2r_n(1-1/n^2))$, and, similarly, $f_{n+1}'\in[-c_n,c_n]_{\mathbb{Z}}\times[-c_n,c_n]_{\mathbb{R}}\times\mathbb{Z}_2$.

Given $g\in\Phi_n$, there are some uniquely determined $k\in K_n$ and $j\in J_n$ such that $g=k+2\widetilde{a}_nj$, i.e. $(g,0,0)=(k,0,0)\phi_n(j,0)$. Moreover, we have $(j,0)+h_n\in H_n$ since $h_n\in H_n^-$. It also follows from (2.17) that

$$(k,0,0)f_nS_nS_n^{\pm1}\subset F_n. \tag{2.18}$$

Take $g\in\Phi_n$ and calculate $T_{(g,0,0)}x$. We have

$$x=(f_n,c_{n+1}(h_n),\dots)=(f_nc_{n+1}(h_n),\dots)=(f_ns_n(h_n)\phi_n(h_n),\dots)$$

and

$$\begin{aligned}(g,0,0)f_ns_n(h_n)\phi_n(h_n) &= (k,0,0)\phi_n(j,0)f_ns_n(h_n)\phi_n(h_n)\\ &= (k,0,0)f_ns_n(h_n)\phi_n((j,0)+h_n)\\ &= (k,0,0)f_ns_n(h_n)s_n((j,0)+h_n)^{-1}c_{n+1}((j,0)+h_n)\\ &= dc_{n+1}((j,0)+h_n),\end{aligned}$$

where $d := (k,0,0)f_ns_n(h_n)s_n((j,0)+h_n)^{-1} \in F_n$ by (2.18). This means that $T_{(g,0,0)}x = (d,\dots) \in X_n$. Similarly,

$$(g,0,0)f'_n, s_n(h'_n)\phi_n(h'_n) = d'c_{n+1}((j,0)+h'_n)$$

with $d' := (b,0,0)f'_ns_n(h'_n)s_n((t,0)+h'_n)^{-1} \in F_n$.

Now take any ξ_{n-2}-measurable subsets $A^*, B^* \subset F_{n-2}$ and set $A := A^*C_{n-1}C_n$, $B := B^*C_{n-1}C_n$. Then

$$\begin{aligned}\nu([A^*]_{n-2}\times[B^*]_{n-2}) &= \nu([A]_n\times[B]_n)\\ &= \lim_{n\to\infty}\frac{\left|\{g\in\Phi_n \mid T_{(g,0,0)}x\in[A]_n,\, T_{(g,0,0)}y\in[B]_n\}\right|}{|\Phi_n|}\\ &= \lim_{n\to\infty}\frac{\left|\{g\in\Phi_n \mid d\in A,\, d'\in B\}\right|}{|\Phi_n|}\\ &= \lim_{n\to\infty}\frac{1}{|K_n|}\sum_{k\in K_n}\frac{\left|\{j\in J_n \mid d\in A,\, d'\in B\}\right|}{|J_n|}\\ &= \lim_{n\to\infty}\frac{1}{|K_n|}\sum_{k\in K_n}\zeta_n\big(A^{-1}(k,0,0)f_ns_n(h_n)\times B^{-1}(k,0,0)f'_ns_n(h'_n)\big),\end{aligned}$$

where $\zeta_n := \operatorname{dist}_{j\in J_n}(s_n((j,0)+h_n), s_n((j,0)+h'_n))$. We distinguish two cases.

First case. Suppose first that $h_n \neq h'_n$ for infinitely many, say *bad*, n. Since $|J_n| \geq r_n/n^2$ it follows from (2.7) that $\|\zeta_n - \kappa_{D_n}\times\kappa_{D_n}\| < 1/n$. Moreover, it follows from (2.5) and the properties (ii) and (iii) of ξ_n that

$$\kappa_{D_n}(A^{-1}(k,0,0)f_ns_n(h)) = \lambda_{S_n}(A^{-1}(k,0,0)f_ns_n(h)) + \overline{o}(1).$$

Hence

$$\begin{aligned}&\frac{1}{|K_n|}\sum_{k\in K_n}\zeta_n\big(A^{-1}(k,0,0)f_ns_n(h_n)\times B^{-1}(k,0,0)f'_ns_n(h'_n)\big)\\ &\quad= \frac{1}{|K_n|}\sum_{k\in K_n}\kappa_{D_n}(A^{-1}(k,0,0)f_ns_n(h_n))\kappa_{D_n}(B^{-1}(k,0,0)f'_ns_n(h'_n)) + \overline{o}(1)\\ &\quad= \frac{1}{|K_n|}\sum_{k\in K_n}\lambda_{S_n}(A^{-1}(k,0,0)f_ns_n(h_n))\lambda_{S_n}(B^{-1}(k,0,0)f'_ns_n(h'_n)) + \overline{o}(1).\end{aligned}$$

Now we derive from Lemma 2.3(ii) that

$$\begin{aligned}\lambda_{S_n}(A^{-1}(k,0,0)f_n s_n(h_n)) &= \frac{\lambda(A^{-1}(k,0,0)f_n s_n(h_n)\cap S_n)}{\lambda(S_n)}\\ &= \frac{\lambda(A\cap(k,0,0)f_n s_n(h_n)S_n)}{\lambda(S_n)}\\ &= \lambda_{F_{n-2}}(A^*)+\overline{o}(1)\end{aligned}$$

and, in a similar way, $\lambda_{S_n}(B^{-1}(b,0,0)f'_n s_n(h'_n)) = \lambda_{F_{n-2}}(B^*)+\overline{o}(1)$. Hence

$$\begin{aligned}\nu([A^*]_{n-2}\times[B^*]_{n-2}) &= \lambda_{F_{n-2}}(A^*)\lambda_{F_{n-2}}(B^*)+\overline{o}(1)\\ &= \mu([A^*]_{n-2})\mu([B^*]_{n-2})+\overline{o}(1)\end{aligned}$$

for all bad n and all ξ_{n-2}-measurable subsets $A^*, B^* \subset F_{n-2}$. Since any measurable set can be approximated by $[A^*]_{n-2}$, it follows that in this case $\nu = \mu\times\mu$.

Second case. Now we consider the case where $h_n = h'_n$ for all n greater than some N. Then it is easy to see that $y = T_k x$, where $k = f'_N f_N^{-1} \in G$, and then it follows immediately that (x,y) is generic for the off-diagonal joining μ_{T_k}:

$$\begin{aligned}\nu([A]_n\times[B]_n) &= \lim_{n\to\infty}\frac{1}{|\Phi_n|}\sum_{i\in\Phi_n}\chi_{[A]_n}(T_{(i,0,0)}x)\chi_{[B]_n}(T_{(i,0,0)}T_k x)\\ &= \lim_{n\to\infty}\frac{1}{|\Phi_n|}\sum_{i\in\Phi_n}\chi_{[A]_n\cap T_k^{-1}[B]_n}(T_{(i,0,0)}x)\\ &= \mu([A]_n\cap T_k^{-1}[B]_n) = \mu_{T_k}([A]_n\times[B]_n)\end{aligned}$$

for all $A, B\in\sigma(\xi_n)$, since ν projects onto μ. Since each measurable set can be approximated by cylinder sets, we deduce that in this case $\nu = \mu_{T_k}$ with $k\in G$. ∎

Proof of Theorem 2.2. The conclusion follows now from Veech's theorem, Propositions 2.8, 2.9 and the fact that $\mathfrak{F}_{G_a}$ and $\mathfrak{F}_{G_b}$ are isomorphic if and only if G_a and G_b are conjugate in G [dJR, Corollary 3.3]. It is clear that $G_b = hG_a h^{-1}$ with $h = (0,(a+b)/2,1)$. ∎

3. Concluding remarks. Notice that with some additional conditions on s_n in Lemma 2.1 (cf. [Da3, Lemma 2.3]) one can show that $T_{(1,0,0)}$ is actually mixing of all orders, as well as simple of all orders (cf. [Da4, Section 6]). For the definitions of higher order simplicity we refer to [dJR].

If we replace $G = \mathbb{Z}\times(\mathbb{R}\rtimes\mathbb{Z}_2)$ with $\Gamma := \mathbb{R}\times(\mathbb{R}\rtimes\mathbb{Z}_2)$ and apply the same construction (with obvious minor changes), we obtain a probability preserving Γ-action R such that the flow $(R_{(t,0,0)})_{t\in\mathbb{R}}$ is 2-fold simple mixing and its centralizer coincides with the entire Γ-action. This gives an example of a 2-fold simple mixing *flow* with uncountably many prime factors. By [Ry], each 2-fold simple flow is simple. Moreover, since $\mathbb{Z} \subset \mathbb{R}$ is a closed co-compact subgroup, the corresponding $\mathbb{Z}$-subaction is also 2-fold simple and $C(R_{(1,0,0)}) = \{R_g \mid g \in \Gamma\}$ by [dJR, Theorem 6.1]. Thus we get examples of two non-isomorphic 2-fold simple transformations with uncountably many prime factors: $R_{(1,0,0)}$ is embeddable into a flow while $T_{(1,0,0)}$ is not.

REFERENCES

[Da1] A. I. Danilenko, *(C, F)-actions in ergodic theory*, in: Geometry and Dynamics of Groups and Spaces, Progr. Math. 265, Birkhäuser, 2008, 325–351.

[Da2] A. I. Danilenko, *Mixing actions of the Heisenberg group*, Ergodic Theory Dynam. Systems 34 (2014), 1142–1167.

[Da3] A. I. Danilenko, *Mixing rank-one actions for infinite sums of finite groups*, Israel J. Math. 156 (2006), 341–358.

[Da4] A. I. Danilenko, *On simplicity concepts for ergodic actions*, J. Anal. Math. 102 (2007), 77–117.

[DdJ] A. I. Danilenko and A. del Junco, *Cut-and-stack simple weakly mixing map with countably many prime factors*, Proc. Amer. Math. Soc. 136 (2008), 2463–2472.

[dJ] A. del Junco, *A simple map with no prime factors*, Israel J. Math. 104 (1998), 301–320.

[dJR] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynam. Systems 7 (1987), 531–557.

[Fe] S. Ferenczi, *Systèmes de rang un gauche*, Ann. Inst. H. Poincaré Probab. Statist. 21 (1985), 177–186.

[GlW] E. Glasner and B. Weiss, *A simple weakly mixing transformation with nonunique prime factors*, Amer. J. Math. 116 (1994), 361–375.

[Li] E. Lindenstrauss, *Pointwise theorems for amenable groups*, Invent. Math. 146 (2001), 259–295.

[Ry] V. V. Ryzhikov, *Around simple dynamical systems. Induced joinings and multiple mixing*, J. Dynam. Control Systems 3 (1997), 111–127.

[So] A. Sokhet, *Les actions approximativement transitives dans la théorie ergodique*, Thèse de doctorat, Univ. Paris VII, 1997.

[Ve] W. A. Veech, *A criterion for a process to be prime*, Monatsh. Math. 94 (1982), 335–341.

Alexandre I. Danilenko, Anton V. Solomko
Institute for Low Temperature Physics & Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave.
Kharkiv, 61103, Ukraine
E-mail: alexandre.danilenko@gmail.com
solomko.anton@gmail.com

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