COLLOQUIUM MATHEMATICUM

VOL. 139 2015 NO. 1

ON HEREDITARY RINGS AND THE PURE SEMISIMPLICITY CONJECTURE II: SPORADIC POTENTIAL COUNTEREXAMPLES

BY

JOSÉ L. GARCÍA (Murcia)

Abstract. It was shown in [Colloq. Math. 135 (2014), 227–262] that the pure semisimplicity conjecture (briefly, pssC) can be split into two parts: first, a weak pssC that can be seen as a purely linear algebra condition, related to an embedding of division rings and properties of matrices over those rings; the second part is the assertion that the class of left pure semisimple sporadic rings (ibid.) is empty. In the present article, we characterize the class of left pure semisimple sporadic rings having finitely many Auslander–Reiten components; the characterization is given through properties of the defining bimodules and the sequences of dimensions associated to these bimodules.

1. Introduction. A ring R is *left pure semisimple* when every left R-module is a direct sum of indecomposable submodules. The ring R is of *finite representation type* if it is left artinian and there exist only finitely many indecomposable finitely presented left R-modules, up to isomorphism. A ring is of finite representation type if and only if it is left and right pure semisimple. The *pure semisimplicity conjecture* (which we shall abbreviate as pssC) states that every left pure semisimple ring is of finite representation type. The conjecture was first discussed by Auslander [5, 6], Gruson [16] and Simson [19].

The conjecture has been proved under certain additional hypotheses [6, 21, 22, 18] but remains undecided. It is known [18] that to prove the conjecture it suffices to show that all left pure semisimple rings of the form

$$R_B = \begin{bmatrix} F & 0 \\ B & G \end{bmatrix}, \tag{1.1}$$

where F, G are division rings and B is a G-F-bimodule, are rings of finite representation type.

Suppose a ring R_B of the form (1.1) is left pure semisimple. We recall from [14, Theorem 3.8] that the indecomposable left R_B-modules (taken up to isomorphism) form a chain $\{M_\alpha \mid 0 \le \alpha \le \delta + 1\}$ (for some ordinal

2010 *Mathematics Subject Classification*: Primary 16G60.

Key words and phrases: pure semisimple ring, pure semisimplicity conjecture, finite representation type, sporadic ring, dimension sequence, Auslander–Reiten component.

DOI: 10.4064/cm139-1-4

$\delta > 0$) with $\alpha < \beta$ precisely when $\mathrm{Hom}_{R_B}(M_\alpha, M_\beta) = 0$. Let us write d_α (for $\alpha = 0, \ldots, \delta$) to denote the left dimension of $\mathrm{Hom}_{R_B}(M_{\alpha+1}, M_\alpha)$, and also $d^* = d_{-1}$ as the left dimension of $B^* = \mathrm{Hom}_G(B, G)$. Following [15], we say that the left pure semisimple ring R_B of the form (1.1) is *sporadic* in case $d_\alpha > 1$ for any α with $-1 \leq \alpha \leq \delta$. If R_B is left pure semisimple and sporadic, then it is a counterexample to the pssC. A weak form of the pssC (the weak pure semisimplicity conjecture, wpssC) has been introduced and studied in [15]. It is shown therein that the pssC holds if and only if the wpssC holds and there do not exist left pure semisimple rings R_B of the form (1.1) which are sporadic.

The present paper is devoted to the study of the class of left pure semisimple sporadic rings. Specifically, we will show that each left pure semisimple sporadic ring R_B such that R_B-ind has only finitely many Auslander–Reiten components, determines a dimension sequence in the sense of [9]. Dimension sequences were originally shown to be related to rings of finite representation type, but in view of this result, they are also connected to left pure semisimple sporadic rings. One of the main results of the paper is Theorem 5.1 which characterizes left pure semisimple sporadic rings R_B of the form (1.1) and with a finite number of connected Auslander–Reiten components, by means of their dimension sequences and the properties of the bimodule B.

We end the paper by a discussion in Section 6 of the relations between sporadic left pure semisimple rings R_B and the class of potential counterexamples constructed by Simson [26]. We show that if $G \subseteq F$ is a pair of division rings and F, viewed as a G-F-bimodule, is *left strictly sporadic* (i.e., its left G-dimension is 2, and the dimension of the successive left dual spaces is constantly 2), then the ring R_B is a counterexample to the pssC (Proposition 6.1).

This paper is a sequel to [15]. For all basic notions and terminology used (in particular, for the notations R-Mod, R-mod, R-ind or for the concepts of preinjective or preprojective modules), we therefore refer to [15] (see [1, 11] for the connections between tilting modules and pure semisimple rings). Moreover, we recall that given a hereditary left artinian ring R, the Auslander–Reiten quiver of R-mod has R-ind as its set of vertices, and an arrow $[X] \to [Y]$ exists in case there is some irreducible homomorphism $X \to Y$ (a homomorphism $f : X \to Y$ is *irreducible* if it is neither a split monomorphism nor a split epimorphism, and any factorization $f = h \circ g$ in R-mod is such that either g is a split monomorphism or h is a split epimorphism). The Auslander–Reiten components of R-ind are the connected components of the Auslander–Reiten quiver of R (see [4, 7] for more information on these topics).

Conditions for a ring R_B of the form (1.1) to have finitely many Auslander–Reiten components (indeed just two components) were already investigated in [25]; also the Auslander–Reiten components of left pure semisimple hereditary rings were studied in [2].

2. Sporadic pure semisimple rings and dimension sequences. We assume in this section that $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is a left pure semisimple sporadic ring. Under this hypothesis, all the indecomposable left R_B-modules can be ordered in a chain $\{M_\alpha \mid 0 \le \alpha \le \delta + 1\}$ so that $\alpha < \beta$ if and only if $\mathrm{Hom}_{R_B}(M_\alpha, M_\beta) = 0$ ([14, Theorem 3.8]). By [13, Proposition 3.9], the Auslander–Reiten components (from now on, AR-components) of R_B-ind are the sets $\mathcal{U}^\lambda = \{M_{\lambda+k} \mid k = 0, 1, \dots\}$ for each limit ordinal $\lambda \le \delta$. Thus $\mathcal{U}^0$ is the set of preinjective modules, and each $\mathcal{U}^\beta$ is infinite, except for the preprojective component $\mathcal{U}^\rho$ (if $\delta = \rho + n$ for a limit ordinal ρ and an integer $n \ge 0$). Moreover, it was shown in [14, Theorem 3.8(d)] that each pair of consecutive modules gives a basic tilting module $M_{\beta+k} \oplus M_{\beta+k+1}$ whose endomorphism ring is again a left pure semisimple ring of the form (1.1). We denote by d_α the left dimension of each module of homomorphisms $\mathrm{Hom}_R(M_{\alpha+1}, M_\alpha)$. Since $M_{\delta+1}$ is the simple projective, d_δ is the left dimension of B, which we denote by d (when B is understood); in addition, we set d_{-1} as $d^* = \mathrm{l.dim}(B^*)$, i.e., the left dimension of $B^* = \mathrm{Hom}_G(B, G)$.

We need the following two results from [15].

PROPOSITION 2.1 ([15, Theorem 5.1]). *Let $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be a left pure semisimple sporadic ring. Suppose that $\mathcal{U}^\lambda$ is any AR-component of R-ind which is not the preprojective component. Then there exists $n \ge 0$ such that $d_{\lambda+k} = 2$ for all $k \ge n$.*

PROPOSITION 2.2 (see [15, Proposition 2.11]). *Let $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian and $W = M \oplus N$ a rigid tilting module such that* $\mathrm{Hom}_R(N, M)$ *has the left finite dimension property. If $(\mathcal{T}, \mathcal{F})$ is the splitting torsion theory of R-*Mod *determined by W, then there is a sequence $X_0, X_1, \dots$ of finitely presented indecomposable left R-modules such that:*

(i) *$X_0 = M$ and $X_1 = N$.*
(ii) *For $k \ge 1$, if the set $\mathcal{S}_k$ of indecomposable finitely presented modules of $\mathcal{F}$ which are not isomorphic to any of the modules $X_2, \dots, X_k$ is not empty, then X_{k+1} is the only element of $\mathcal{S}_k$ (up to isomorphism) such that $A \in \mathcal{S}_k$ and $A \not\cong X_{k+1}$ imply* $\mathrm{Hom}_R(X_{k+1}, A) = 0$. *Moreover,* $\mathrm{Hom}_R(A, X_{k+1}) \ne 0$ *for any $A \in \mathcal{S}_k$.*

If there is a smallest $k \ge 0$ such that X_k is projective, then X_{k+1} is the simple projective, $\mathcal{S}_{k+1}$ is empty and the sequence is finite. Otherwise, the sequence is infinite.

Proof. This is precisely [15, Proposition 2.11], except for the added condition that $\mathrm{Hom}_R(A, X_{k+1}) \neq 0$. But this is exactly what was shown in the second paragraph of the proof of [15, Proposition 2.11] as a justification for the uniqueness of X_{k+1}. ∎

Assume that the matrix ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is left pure semisimple and sporadic, with the chain $\{M_\alpha \mid 0 \leq \alpha \leq \rho + n + 1\}$ of indecomposable left R_B-modules as above, and let (t_α, s_α) denote the d-vector of each M_α. Then for $\alpha < \beta$ we have $t_\alpha/s_\alpha > t_\beta/s_\beta$. Moreover, t_ω/s_ω is the limit of the ratios t_k/s_k for finite k; and similarly, for any limit ordinal $\lambda < \rho$ one sees that $t_{\lambda+\omega}/s_{\lambda+\omega}$ is the limit of the ratios $t_{\lambda+k}/s_{\lambda+k}$. See [15, Theorem 3.14] for details.

Let $\lambda < \rho$ be a limit ordinal. From Proposition 2.1 we infer that there is a smallest integer $k \geq 0$ such that $d_{\lambda+k} = 2 = d_{\lambda+j}$ for any $j \geq k$. We denote this value as $k(\lambda)$. If $\mu = \lambda + \omega$, then the left dimension of $\mathrm{Hom}_R(M_\mu, M_{\lambda+k(\lambda)})$ will be denoted by $r(\mu)$. By [21, Proposition 2.4(d)], $r(\mu) \geq 1$ is finite. We will write R instead of R_B when B is understood.

The following result is our first approach to the understanding of these r-values.

PROPOSITION 2.3. *Let $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be a left pure semisimple sporadic ring. Let λ be a limit ordinal such that U^λ is an AR-component which is not the preprojective component, and let $\mu = \lambda+\omega$. Let $k = k(\lambda)$. For any natural number $m \geq k$, there exists a short exact sequence $0 \to M_{\mu+1} \to M_\mu^h \to M_{\lambda+m} \to 0$ where $h = \mathrm{l.dim}(\mathrm{Hom}_R(M_\mu, M_{\lambda+m}))$.*

Proof. Since $M_\mu \oplus M_{\mu+1}$ is a rigid tilting module [14, Theorem 3.8], we know that there is a short exact sequence $0 \to M_{\mu+1}^l \to M_\mu^h \to M_{\lambda+m} \to 0$ ([15, Proposition 3.4]; that the exponent h is the dimension follows from the proof of that proposition), so all we have to show is that $l = 1$. Suppose to the contrary that $l > 1$, and let p be a prime factor of l. Let us write (t, s), (t_μ, s_μ) and (t_j, s_j) respectively for the d-vectors of $M_{\mu+1}$, M_μ and $M_{\lambda+j}$. By Proposition 2.1 and [15, Lemma 3.1], we know that for each $m \geq k$, $t_{m+1} - t_m = t_{k+1} - t_k$ and $s_{m+1} - s_m = s_{k+1} - s_k$. Then [15, Theorem 3.14(iii)] implies that

$$\frac{t_\mu}{s_\mu} = \frac{t_{m+1} - t_m}{s_{m+1} - s_m},$$

and since both fractions have coprime terms by [15, Lemma 3.8], it follows that $t_\mu = t_{m+1} - t_m$ and $s_\mu = s_{m+1} - s_m$. Thus the above short exact sequence gives us the equations

$$tl = (t_{m+1} - t_m)h - t_m, \qquad sl = (s_{m+1} - s_m)h - s_m.$$

This shows that, modulo p,

$$(t_{m+1} - t_m)h = t_m, \quad (s_{m+1} - s_m)h = s_m.$$

If $h = 0$ modulo p, then p is a divisor of both t_m and s_m, which contradicts [15, Lemma 3.8]. Moreover, p is not a divisor of s_m or of t_m, because if we had, for instance, $s_m = 0$ over $\mathbb{Z}_p$, then $s_{m+1} = s_m = 0$, and we get again a contradiction to [15, Lemma 3.8]. Then by taking h^{-1}, we obtain the equation modulo p,

$$(t_{m+1} - t_m)t_m^{-1} = (s_{m+1} - s_m)s_m^{-1},$$

and this entails $(t_{m+1} - t_m)s_m = (s_{m+1} - s_m)t_m$, always modulo p. Then modulo p we have, by simplifying the above equation,

$$t_{m+1}s_m = s_{m+1}t_m, \quad s_{m+1}t_m - t_{m+1}s_m = 0,$$

which is impossible, by [15, Lemma 3.8]. ∎

COROLLARY 2.4. *With the hypotheses of Proposition* 2.3*, let* $\mu = \lambda + \omega$*. Assume* $k = k(\lambda)$ *and* $r = r(\mu)$*. If* $m \geq k$*, then* $\mathrm{l.dim}(\mathrm{Hom}_R(M_\mu, M_{\lambda+m})) = r + (m - k)$*.*

Proof. We know $\mathrm{l.dim}(\mathrm{Hom}_R(M_\mu, M_{\lambda+k})) = r$ and this is the first step in a proof by induction of the result. So, assume $\mathrm{l.dim}(\mathrm{Hom}_R(M_\mu, M_{\lambda+m})) = r + m - k = h_m$. We shall prove the property for $M_{\lambda+m+1}$, showing that if $h' = \mathrm{l.dim}(\mathrm{Hom}_R(M_\mu, M_{\lambda+m+1}))$, then $h' = h_m + 1$.

Using the notation of the proof of Proposition 2.3, the d-vector of M_μ is $(t_{m+1} - t_m, s_{m+1} - s_m)$. Still with the same notation, and applying Proposition 2.3 twice, we get the equations

$$h_m(t_{m+1} - t_m) = t + t_m, \quad h'(t_{m+1} - t_m) = t + t_{m+1},$$

and similarly for the s-values. But then $h' = h_m + 1$, as required. ∎

Now we recall from [9] the concept of a dimension sequence, and introduce the notion of a partial dimension sequence.

DEFINITION 2.5. Let $m \geq 1$ and $d_{-1}, d_0, \ldots, d_m$ be a sequence of positive integers. This is called a *partial dimension sequence* if there is a sequence of pairs of integers $(t_{-1}, s_{-1}), (t_0, s_0), \ldots, (t_{m+2}, s_{m+2})$ with the following properties:

(i) $(t_{-1}, s_{-1}) = (0, -1)$, $(t_0, s_0) = (1, 0)$.
(ii) $t_j, s_j \geq 1$ for $j = 1, \ldots, m$.
(iii) $(t_{k+2}, s_{k+2}) = d_k(t_{k+1}, s_{k+1}) - (t_k, s_k)$ for $k = -1, 0, \ldots, m$.

The sequence $d_{-1}, d_0, \ldots, d_m$ is called a *dimension sequence* if it is a partial dimension sequence and $(t_{m+1}, s_{m+1}) = (0, 1), (t_{m+2}, s_{m+2}) = (-1, 0)$.

In [9], dimension sequences are associated to bimodules of finite representation type (see also [10] and [21]), and in [23], infinite dimension sequences are associated to hereditary artinian left pure semisimple rings R_B

in connection with generalized Artin problems for division ring extensions (see also [26]). We shall show that (finite) dimension sequences are also connected with sporadic left pure semisimple rings. To this end, we will associate a sequence to such a left pure semisimple ring.

Let $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be a left pure semisimple sporadic ring. Suppose that if we order the indecomposable left R-modules in the chain $\{M_\alpha \mid 0 \leq \alpha \leq \rho+n+1\}$ as above, we get $\rho = \omega \cdot h$ with $1 \leq h < \infty$, so that R-ind has only $h+1$ AR-components. In this situation, we may define finite sequences $s_0, \ldots, s_h$ of natural numbers as follows.

The sequence s_0 will be: $d_{-1} = d^*, d_0, \ldots, d_{k(0)} = 2$. For any $0 < j < h$, the sequence s_j will be: $1, r(\omega \cdot j) + 2, d_{\omega \cdot j}, \ldots, d_{\omega \cdot j + k(\omega \cdot j)} = 2$. Finally, the sequence s_h is: $1, r(\rho) + 2, d_\rho, \ldots, d_{\rho+n} = d$. We then have the following result.

Theorem 2.6. *If $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is a left pure semisimple sporadic ring with $h+1$ AR-components, then the sequence obtained by concatenation of the sequences $s_0, \ldots, s_h$ is a dimension sequence.*

Proof. We define the sequence of pairs (t_i, s_i) for $i = 1, \ldots, m$ (where $m = \sum_{j=0}^{h-1} k(\omega \cdot j) + 3h + n$), and check that they satisfy conditions (ii) and (iii) of Definition 2.5. These pairs (t_i, s_i) are the d-vectors of the following sequence of indecomposable modules:

$$M_1, \ldots, M_{k(0)+2}, M_\omega, \ldots, M_{\omega+k(\omega)+2}, M_{\omega \cdot 2}, \ldots, M_{\omega \cdot j}, \ldots, M_\rho, \ldots, M_{\rho+n}.$$

Condition (ii) is obviously satisfied, because the only indecomposable modules with zero in some coordinate of their d-vector are the simple modules $M_0, M_{\rho+n+1}$. In order to check (iii), the interesting equations to examine will be those of the form

$$\begin{aligned}(t_{\omega\cdot(j+1)}, s_{\omega\cdot(j+1)}) \\ = 1 \cdot (t_{\omega\cdot j+k(\omega\cdot j)+2}, s_{\omega\cdot j+k(\omega\cdot j)+2}) - (t_{\omega\cdot j+k(\omega\cdot j)+1}, s_{\omega\cdot j+k(\omega\cdot j)+1})\end{aligned}$$

and

$$\begin{aligned}(t_{\omega\cdot(j+1)+1}, s_{\omega\cdot(j+1)+1}) \\ = (r(\omega\cdot(j+1)) + 2)(t_{\omega\cdot(j+1)}, s_{\omega\cdot(j+1)}) - (t_{\omega\cdot j+k(\omega\cdot j)+2}, s_{\omega\cdot j+k(\omega\cdot j)+2}).\end{aligned}$$

Now, the first equation holds because in general, for the limit ordinal λ and $k = k(\lambda)$, one has $t_{\lambda+k+2} - t_{\lambda+k+1} = t_{\lambda+k+1} - t_{\lambda+k} = t_{\lambda+\omega}$, and similarly for s, since $d_{\lambda+k} = 2$. The second equation follows from Proposition 2.3 and Corollary 2.4. Finally, the remaining cases follow by applying the equation of [15, Lemma 3.1]. ∎

The dimension sequence determined by the left pure semisimple sporadic ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ will be referred to as the *dimension sequence of* R_B.

Note that the number of terms equal to 1 in the dimension sequence of R_B, increased by 1, is the number of AR-components of R_B-ind. On the other hand, not every dimension sequence can appear as the dimension sequence of such a ring R_B. Indeed, it follows by our construction of the sequence that:

(i) if a term of the sequence is $d_{k+1} = 1$, then $d_k = 2$ and $d_{k+2} \geq 3$;
(ii) the first two terms of the dimension sequence of R_B (i.e., d^*, d_0) are $\neq 1$, and so are the last two terms;
(iii) there are at least two terms $\neq 1$ between two terms with value 1;
(iv) the subsequence $2, 2, 1$ cannot appear in the sequence except possibly for the first three terms.

It turns out that none of the examples of dimension sequences of length ≤ 7 given in [9] satisfies these conditions. Indeed, the shortest dimension sequence that could correspond to a left pure semisimple sporadic ring has length 8. An example is $(2, 2, 1, 5, 2, 1, 3, 2)$. Likewise, we look for other necessary conditions on some of the terms of a dimension sequence of a left pure semisimple sporadic ring R_B.

We may further restrict the dimension sequences of left pure semisimple sporadic rings of the form $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ upon substitution of the endomorphism ring of a tilting module $M_\alpha \oplus M_{\alpha+1}$ for the given ring R_B. If we choose $\alpha = \lambda + k$ for some infinite limit ordinal λ and $k = k(\lambda)$, and $W = M_\alpha \oplus M_{\alpha+1}$, then $\mathrm{End}_R(W)$ is again a left pure semisimple [13, Proposition 3.4] sporadic ring such that $d = d^* = 2$ and $k(0) = 0$. Therefore its dimension sequence, in addition to the properties stated above, must be of the form $2, 2, 1, l, \ldots, 2$, where $l \geq 5$. This is because $\mathcal{U}^\omega$ is not the preprojective component by [15, Proposition 5.3], hence the modules $M_\omega, M_{\omega+1}$ (with respective d-vectors $(1, 1)$ and $(l-3, l-2)$) cannot be projective, so that $l - 3 \geq 2$.

Let $R = R_B$ be as above. Let λ be a limit ordinal and $\mathcal{U}^\lambda$ be one of the AR-components of R-ind which is not the preprojective component. We set $k = k(\lambda)$ and simplify the notation by writing (t_h, s_h) for the d-vector of $M_{\lambda+h}$ when we refer only to elements in that component. We use now the ordering of pairs (t, s) given in the proof of [15, Proposition 5.3], i.e., $(t, s) < (t', s')$ when $(t, s) \neq (t', s')$ and $t \leq t'$, $s \leq s'$. Then we say that the AR-component $\mathcal{U}^\lambda$ is *growing* when $2(t_k, s_k) < (t_{k+1}, s_{k+1})$; and $\mathcal{U}^\lambda$ is *supersporadic* when $k = 0$. Observe that if $(t_r, s_r) < (t_{r+1}, s_{r+1})$ for some $r \geq 0$, then $(t_{r+1}, s_{r+1}) < (t_{r+2}, s_{r+2})$, because $d_r \geq 2$. Note also that when $\mathcal{U}^\lambda$ is not supersporadic and $(t_{k-1}, s_{k-1}) < (t_k, s_k)$, then $\mathcal{U}^\lambda$ is growing. This is clear from the equations for p_n, q_n given before Lemma 3.2 in [15] and the fact that $d_{\lambda+k-1} > 2$.

Proposition 2.7. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be a left pure semisimple sporadic ring with only finitely many AR-components. Then there exists a non-zero limit ordinal λ such that either $r(\lambda) = 1$ or else $\mathcal{U}^\lambda$ is supersporadic with $r(\lambda) = 2$.*

Proof. Suppose to the contrary that $r(\lambda) > 1$ for all limit ordinals λ, and that $r(\lambda) > 2$ for all supersporadic components $\mathcal{U}^\lambda$. As shown in the preceding comments, we may also assume that the dimension sequence for the ring R_B is of the form $2, 2, 1, l, \dots, 2$ with $l \geq 5$ (this does not affect our hypotheses since $r(\omega) \geq 3$ with this choice). We are going to check that each component $\mathcal{U}^\lambda$ which is not the preinjective or the preprojective component is a growing component.

By the choice of the dimension sequence, the d-vectors of $M_\omega, M_{\omega+1}$ are, respectively, $(1,1)$ and $(u, u+1)$ with $u = l-3 \geq 2$. Then if $\mathcal{U}^\omega$ is supersporadic, we have $k(\omega) = 0$ and $\mathcal{U}^\omega$ is growing because $2(t_\omega, s_\omega) < (t_{\omega+1}, s_{\omega+1})$. If $\mathcal{U}^\omega$ is not supersporadic, then the inequality $(t_\omega, s_\omega) < (t_{\omega+1}, s_{\omega+1})$ and the comments just before this proposition show that, again, $\mathcal{U}^\omega$ is a growing component.

We show next that for $\lambda \geq \omega$ a limit ordinal such that $\mathcal{U}^\lambda$ is not the preprojective component, if $\mathcal{U}^\lambda$ is a growing component and $\mu = \lambda + \omega$, then $\mathcal{U}^\mu$ is growing. If $\mathcal{U}^\mu$ is not supersporadic, then it is enough to show, as in the case of $\mathcal{U}^\omega$, that $(t_\mu, s_\mu) < (t_{\mu+1}, s_{\mu+1})$. Now, let $k = k(\lambda)$ so that $(t_\mu, s_\mu) = (t_{\lambda+k+1} - t_{\lambda+k}, s_{\lambda+k+1} - s_{\lambda+k})$. Let $r = r(\mu)$ and thus $(t_{\mu+1}, s_{\mu+1}) = (rt_{\lambda+k+1} - (r+1)t_{\lambda+k}, rs_{\lambda+k+1} - (r+1)s_{\lambda+k})$. Then $t_{\mu+1} - t_\mu = (r-1)t_{\lambda+k+1} - rt_{\lambda+k}$. But $t_{\lambda+k+1} \geq t_{\lambda+k}$ and $r > 1$, and it follows that $t_{\mu+1} - t_\mu \geq 0$, and similarly for the s-value, which proves our claim in this case.

Suppose now that $\mathcal{U}^\mu$ is supersporadic. To show that it is growing, we need to prove that $2(t_\mu, s_\mu) < (t_{\mu+1}, s_{\mu+1})$. By computing as above we get

$$t_{\mu+1} - 2t_\mu = (r-2)t_{\lambda+k+1} - (r-1)t_{\lambda+k}.$$

Now since $r > 2$, again $t_{\mu+1} - 2t_\mu \geq 0$, and similarly for s, which shows that $\mathcal{U}^\mu$ is a growing component in this case, too.

If $\mathcal{U}^\rho$ is the preprojective component, then we deduce as above that $(t_\rho, s_\rho) < (t_{\rho+1}, s_{\rho+1})$; but then $(t_{\rho+m}, s_{\rho+m}) < (t_{\rho+m+1}, s_{\rho+m+1})$ for any possible finite $m \geq 0$, which gives a contradiction because the d-vector of the simple projective module is $(0,1)$. ∎

We know from [15, Proposition 5.3] that there is no left pure semisimple sporadic ring R_B such that R_B-ind has only two AR-components. We may now deduce which are the potential sporadic rings R_B with three AR-components.

Corollary 2.8. *If $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is a left pure semisimple sporadic ring with exactly three AR-components, then $r(\omega \cdot 2) = 1$.*

Proof. The proof of Proposition 2.7 shows that the component $\mathcal{U}^\omega$ is growing and that if $\mu = \omega \cdot 2$ and $r(\mu) \geq 2$, then $(t_\mu, s_\mu) < (t_{\mu+1}, s_{\mu+1})$. But $\mathcal{U}^\mu$ is the preprojective component, and this is a contradiction as in the proof of Proposition 2.7. ∎

This allows us to conclude that, upon substitution of the endomorphism ring of a tilting module for R_B, the dimension sequence determined by any of these rings with three AR-components is necessarily of the form $(2, 2, 1, 5, \ldots, 2, 1, 3, \ldots, 2)$. In particular this holds for the dimension sequence $(2, 2, 1, 5, 2, 1, 3, 2)$ mentioned above. But this is not the only case, because the sequence $(2, 2, 1, 5, 3, 4, 2, 1, 3, 2, 3, 3, 2)$ is another example of this type.

3. Acceptable bimodules. Throughout this section, G, F will be division rings, and B a G-F-bimodule with left G-dimension 2. Then the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is left artinian and hereditary. In this section we are interested in conditions on B implying that R_B is a sporadic left pure semisimple ring.

Lemma 3.1. *For G, F, B and R_B as above, the following three conditions are equivalent:*

- (i) *There is a non-zero $b \in B$ such that every element of B is of the form gbf for some $g \in G$ and $f \in F$.*
- (ii) *All the left R_B-modules which are indecomposable and have d-vector $(1, 1)$ are isomorphic.*
- (iii) *For each non-zero $b \in B$, every element of B is of the form gbf, for some elements $g \in G$ and $f \in F$.*

Proof. (iii)⇒(i) is obvious, and (i)⇒(iii) is clear because if (i) holds for a certain b, then given $b_1, b_2 \in B$ we have $b_i = g_i b f_i$, and hence $b_2 = (g_2 g_1^{-1}) b_1 (f_1^{-1} f_2)$.

Let us assume (i), and choose a left G-basis $\{b_1, b_2\}$ of B where b_2 satisfies the condition in (i). Let $\alpha : B \cong B \otimes_F F \to G$ be the G-linear map given by $\alpha(b_1) = 1$ and $\alpha(b_2) = 0$. This defines a left R_B-module M with d-vector $(1, 1)$. It is indecomposable, because the only decomposable modules with d-vector $(1, 1)$ are the direct sum of two simple modules, and the associated G-linear map is then zero.

Let L be any indecomposable left R_B-module with d-vector $(1, 1)$. Thus, L is given by a non-zero map $\beta : B \otimes F \to G$ and $\mathrm{Ker}(\beta)$ is Gb for some $b \in B$. Therefore $\mathrm{Ker}(\beta) = Gb_2 f$ for some $f \in F$. Consider the left linear map $h_1 : F \to F$ which sends 1 to f. This induces a left G-linear map $h : B \otimes_F F \to B \otimes F$ such that $h(b_2) \in Gb = \mathrm{Ker}(\beta)$. Since $Gb_2 = \mathrm{Ker}(\alpha)$, we obtain a commutative diagram of left G-vector spaces and linear maps

$$\begin{array}{ccc} B \otimes_F F & \xrightarrow{\alpha} & G \\ \downarrow & & \downarrow \\ B \otimes_F F & \xrightarrow{\beta} & G \end{array}$$

which gives an isomorphism $M \cong L$. This proves (ii).

Suppose now that (ii) holds, choose any $b_2 \in B$, fix a G-basis $\{b_1, b_2\}$ of B, and construct the module M through the linear map α as above. For any $b \neq 0 \in B$ we form the indecomposable left R_B-module L with d-vector $(1,1)$ constructed as above from the map $\beta : B \otimes_F F \to G$ with $\mathrm{Ker}(\beta) = Gb$. By (ii), there is an isomorphism $M \cong L$, so we have an F-linear map $h_1 : F \to F$ such that $(1 \otimes h_1)(b_2) \in Gb$. But then $b_2 f \in Gb$ for some non-zero $f \in F$, and $b = gb_2 f$ for some $g \in G$, as we wanted to show. ∎

When the conditions of Lemma 3.1 are satisfied and $\mathcal{B} = \{b_1, b_2\}$ is a G-basis of B, the map $\alpha : B \otimes F \to G$ given by $\alpha(b_1) = 1$ and $\alpha(b_2) = 0$ defines the left R-module M, which is uniquely determined by the bimodule B, up to isomorphism. In spite of this uniqueness, we should keep in mind that the concrete module M does depend on the basis $\mathcal{B}$, and we shall sometimes write $M(\mathcal{B})$ when we need to emphasize this fact. We will consider now certain relations which, in principle, are dependent on the choice of the G-basis of B.

Given the G-basis $\mathcal{B} = \{b_1, b_2\}$ of B, we associate to any $f \in F$ the 2×2 G-matrix $(g(f)_i^j)$ such that $b_i f = \sum_{j=1}^{2} g(f)_i^j b_j$. This association is a ring homomorphism $F \to \mathbb{M}_2(G)$ (where $\mathbb{M}_2(G)$ is the ring of 2×2 matrices over G), as is easily seen. Therefore, F is identified with a division subring of $\mathbb{M}_2(G)$.

Using this identification, we may define now the following subring $E(\mathcal{B})$ of F. An element of F (viewed as a matrix), say (g_i^j), belongs to E if and only if $g_2^1 = 0$. It is clear that $E(\mathcal{B})$ is a subring of F. Moreover, it is a division subring: if $0 \neq f \in E(\mathcal{B})$, then $b_2 f = gb_2$ for some $g \in G$, and hence $g^{-1} b_2 = b_2 f^{-1}$ so that $f^{-1} \in E(\mathcal{B})$. Note also that the mapping $E(\mathcal{B}) \to G$ which assigns to the element $f \in E(\mathcal{B}) \subseteq F$ the element $g \in G$ with the property that $b_2 f = gb_2$, is again a ring homomorphism. The image of this homomorphism is a division subring of G, which will be denoted as $G_0(\mathcal{B})$. We observe next that, even though the construction of $E(\mathcal{B})$ and its embedding into F depends on the choice of G-basis of B, the dimension of F as a left $E(\mathcal{B})$-vector space is an invariant.

LEMMA 3.2. *Let G, F, B, R_B be as in Lemma 3.1, and suppose that the equivalent conditions of Lemma 3.1 hold. Choose a G-basis $\mathcal{B} = \{b_1, b_2\}$ of B,* *and consider the left R_B-module $M(\mathcal{B}) = M$ and the subring $E(\mathcal{B})$ of F as above. Set $R := R_B$, and let E_0 be the simple injective left R-module. Then:*

(a) *The ring $E(\mathcal{B})$ is isomorphic to the endomorphism ring $\mathrm{End}_R(M)$, and F and $\mathrm{Hom}_R(M, E_0)$ are semilinearly isomorphic with respect to that isomorphism.*

(b) *The isomorphism class of $E(\mathcal{B})$ and the left $E(\mathcal{B})$-structure of F do not depend on the choice of the G-basis $\mathcal{B}$ of B.*

(c) *The dimension of F as a left $E(\mathcal{B})$-vector space is invariant under change of the G-basis of B.*

Proof. (a) For any $h \in \mathrm{End}_R(M)$, let $h_1 : F \to F$ be the corresponding left linear map and let $h_1(1) = f \in F$. Since $1 \otimes h$ must take $b_2 \otimes 1$ to an element in $\mathrm{Ker}(\alpha) = G(b_2 \otimes 1)$, we see that $b_2 f = g b_2$ for some element $g \in G$. Therefore $f \in E(\mathcal{B})$. Conversely, each $f \in E(\mathcal{B})$ defines a G-linear map $B \otimes_F F \to B \otimes_F F$ such that the image of $b_2 \otimes 1$ belongs to $\mathrm{Ker}(\alpha)$, and hence is a module homomorphism. So, the association $h \mapsto h_1(1)$ is a ring isomorphism $\mathrm{End}_R(M) \to E(\mathcal{B})$.

Now, the elements of $\mathrm{Hom}_R(M, E_0)$ are uniquely determined by the corresponding left linear map $F \to F$, hence there is a group isomorphism $\mathrm{Hom}_R(M, E_0) \cong F$. It is clear that this isomorphism is semilinear with respect to the above isomorphism $\mathrm{End}_R(M) \cong E(\mathcal{B})$.

Since the isomorphism class of M is independent of $\mathcal{B}$ by Lemma 3.1, (b) is an immediate consequence of (a), and (c) is a consequence of (b). ∎

In view of Lemma 3.2, we can write E instead of $E(\mathcal{B})$ when there is no risk of confusion. Also, the independence from the choice of basis justifies the next definition.

Definition 3.3. Let G, F be division rings and B be a G-F-bimodule with left dimension 2. We say that B is *left pre-acceptable* in case the conditions of Lemma 3.1 hold and the left dimension of F over the subring E described above is finite.

When B is left pre-acceptable, then $\mathrm{l.dim}({}_E F) > 1$, because otherwise we would have $b_2 F \subseteq G b_2$, and hence every element of B would belong to $G b_2$, which contradicts the assumption on the left dimension of B. Thus we will denote the left dimension of F over E by $\tau + 1$ with $\tau \geq 1$.

The number $\tau = \tau(B) = \mathrm{l.dim}({}_E F) - 1$ is called the *characteristic value* of the left pre-acceptable bimodule B.

For the rest of this section, we assume that B is left pre-acceptable, and we make some constructions which depend on the choice of the G-basis $\mathcal{B}$ of B.

Lemma 3.4. *Let B be a left pre-acceptable G-F-bimodule, choose a G-basis $\mathcal{B}$ of B, and consider the left R-module $M = M(\mathcal{B})$ and the ring*

$E = E(\mathcal{B})$ constructed as above, so that $\mathrm{l.dim}({}_E F) = \tau + 1$*. Let $e_0, \dots, e_\tau$ be the canonical left basis of $F^{\tau+1}$, and choose a basis $\{f_0, \dots, f_\tau\}$ of ${}_E F$. Let E_0 be the simple injective left R-module and $h : M^{\tau+1} \to E_0$ the homomorphism defined by the F-linear map $h_1 : F^{\tau+1} \to F$ where $h_1(e_i) = f_i$ for $i = 0, \dots, \tau$. Then the isomorphism class of* $\mathrm{Ker}(h)$ *is independent of the chosen E-basis of F.*

Proof. Let $h, g : M^{\tau+1} \to E_0$ be homomorphisms with associated F-linear maps $h_1, g_1 : F^{\tau+1} \to F$ with $h_1(e_i) = f_i$ and $g_1(e_i) = f'_i$ such that both $f_0, \dots, f_\tau$ and $f'_0, \dots, f'_\tau$ are left E-bases of F. We will show that there is an automorphism w of $M^{\tau+1}$ such that $wg = h$. This entails immediately that $\mathrm{Ker}(h) \cong \mathrm{Ker}(g)$, as stated.

For $i = 0, \dots, \tau$, let $f_i = \sum_{j=0}^{\tau} a_{ij} f'_j$ for $a_{ij} \in E$. Define $w_1 : F^{\tau+1} \to F^{\tau+1}$ to be the F-linear map satisfying $w_1(e_i) = \sum_{j=0}^{\tau} a_{ij} e_j$, so that $w_1 g_1 = h_1$. The fact that the elements $x \in E$ satisfy the condition $b_2 x \in G b_2$ implies easily that w_1 extends to an R-homomorphism $w : M^{\tau+1} \to M^{\tau+1}$. Moreover, since $\{f_i \mid i = 0, \dots, \tau\}$ and $\{f'_i \mid i = 0, \dots, \tau\}$ are bases, w_1 and w are isomorphisms and $wg = h$. ∎

We now introduce another left R_B-module $N(\mathcal{B})$, which depends on the G-basis $\mathcal{B}$ of B. Recall the notation $g(f)_i^j$ for elements $f \in F$ introduced before Lemma 3.2.

Lemma 3.5. *Let G, F, B, R_B be as in Lemma* 3.1, *and suppose that B is left pre-acceptable. Choose a G-basis $\mathcal{B} = \{b_1, b_2\}$ of B, and consider the ring $E = E(\mathcal{B})$ as in Lemma* 3.2*. Let $\mathcal{B}_E = \{1 = f_0, \dots, f_\tau\}$ be a basis of ${}_E F$. Suppose $\{u_1, \dots, u_\tau\}$ is the canonical basis of F^τ as a left F-vector space. Consider the set K of the elements of $B \otimes_F F^\tau$ of the form $\sum_{j=1}^{\tau} g_j (b_2 \otimes u_j)$, where $g_1, \dots, g_\tau$ are elements of G such that $\sum_{j=1}^{\tau} g_j g(f_j)_2^1 = 0$. Then:*

(a) *K is a left G-subspace of $B \otimes_F F^\tau$ with dimension $\tau - 1$.*

(b) *Let $N(\mathcal{B})$ be the left R_B-module determined by the cokernel j^c of the inclusion map $j : K \to B \otimes_F F^\tau$. Then the isomorphism class of $N(\mathcal{B})$ is invariant under changing the elements $f_1, \dots, f_\tau$ of the basis $\mathcal{B}_E$.*

Proof. (a) It is clear that K is indeed a left G-subspace of $B \otimes_F F^\tau$. Also, the vectors $b_2 \otimes u_j$ generate a subspace of left dimension τ, and the elements of K form a hyperplane of this subspace, and hence the dimension of K is $\tau - 1$.

(b) Let E_0 be the simple injective left R_B-module determined by the zero map $B \otimes_F F \to 0$. From the basis $\mathcal{B}_E$ and the canonical basis $\{e_0, \dots, e_\tau\}$ of $F^{\tau+1}$ (as a left F-vector space) we may construct the F-linear map $h_1 : F^{\tau+1} \to F$ with $h_1(e_i) = f_i$ as in Lemma 3.4, and this gives an R_B-homomorphism $h : M^{\tau+1} \to E_0$. We are going to show that $\mathrm{Ker}(h) \cong N(\mathcal{B})$.

It will then follow from Lemma 3.4 that the isomorphism class of $N(\mathcal{B})$ does not change by changing the basis $\mathcal{B}_E$.

With this aim, we define the G-linear injective map $\varphi : B \otimes_F F^\tau \to B \otimes_F F^{\tau+1}$ by setting $\varphi(b \otimes u_j) = b \otimes (f_j e_0 - e_j)$ for $j = 1, \ldots, \tau$ and any $b \in B$. This is indeed injective because the vectors $f_j e_0 - e_j$ are linearly independent in $F^{\tau+1}$. Moreover, these vectors clearly belong to the kernel of h_1, by definition. Therefore, $\mathrm{Im}(\varphi) = \mathrm{Ker}(1 \otimes h_1)$.

We next observe that φ defines an R_B-monomorphism $N(\mathcal{B}) \to M^{\tau+1}$, by showing that there is a commutative square of left G-linear maps

$$\begin{array}{ccc} B \otimes_F F^\tau & \xrightarrow{j^c} & G^{\tau+1} \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ B \otimes_F F^{\tau+1} & \xrightarrow{\alpha^{\tau+1}} & G^{\tau+1} \end{array}$$

such that the right-hand arrow is also injective. Since $K = \mathrm{Ker}(j^c)$, it will be enough to show that $\varphi(K) = \mathrm{Ker}(\alpha^{\tau+1}) \cap \mathrm{Im}(\varphi)$.

By the definition of α (see the definition of $M = M(\mathcal{B})$ after Lemma 3.1), $\mathrm{Ker}(\alpha^{\tau+1})$ is generated by the vectors $b_2 \otimes e_i$ for $i = 0, \ldots, \tau$. Now, any element of $\mathrm{Im}(\varphi)$ is

$$\varphi\Big(\sum_{j=1}^{\tau} c_j (b_1 \otimes u_j) + \sum_{j=1}^{\tau} d_j (b_2 \otimes u_j)\Big)$$
$$= \sum_{j=1}^{\tau} c_j (b_1 \otimes (f_j e_0 - e_j)) + \sum_{j=1}^{\tau} d_j (b_2 \otimes (f_j e_0 - e_j)).$$

A necessary condition for this element to belong to $\mathrm{Ker}(\alpha^{\tau+1})$ is that each $c_j = 0$ (because of the summands of the form $c_j(b_1 \otimes e_j)$). Thus the element may be written as

$$\sum_{j=1}^{\tau} d_j (b_2 f_j \otimes e_0) - \sum_{j=1}^{\tau} d_j (b_2 \otimes e_j)$$
$$= \Big(\sum_{j=1}^{\tau} d_j g(f_j)_2^1\Big)(b_1 \otimes e_0) + \Big(\sum_{j=1}^{\tau} d_j g(f_j)_2^2\Big)(b_2 \otimes e_0) - \sum_{j=1}^{\tau} d_j (b_2 \otimes e_j).$$

So, finally, the necessary and sufficient condition for this element to belong to $\mathrm{Ker}(\alpha^{\tau+1})$ is that the coefficient $\sum_{j=1}^{\tau} d_j g(f_j)_2^1$ be zero. That is, the condition is that the element belongs to $\varphi(K)$, as was to be seen. ∎

LEMMA 3.6. *Let G, F, B, R_B be as in Lemma 3.1. If B is left pre-acceptable and $M, N(\mathcal{B})$ are as above, then* $\mathrm{Hom}_{R_B}(M, N(\mathcal{B})) = 0$.

Proof. We keep the notation of the preceding proof. Suppose m is in $\mathrm{Hom}_{R_B}(M, N(\mathcal{B}))$ and $m_1 : F \to F^\tau$ is the corresponding F-linear map.

Let $m_1(1) = \sum_{j=1}^{r} x_j u_j \neq 0$ with each x_j in F. Since $1 \otimes m_1$ defines an R_B-homomorphism, $(1 \otimes m_1)(b_2 \otimes 1) \in K$. Thus

$$\sum_{j=1}^{\tau} b_2 x_j \otimes u_j = \sum_{j=1}^{\tau} g(x_j)_2^1 b_1 \otimes u_j + \sum_{j=1}^{\tau} g(x_j)_2^2 b_2 \otimes u_j \in K.$$

This entails that all $g(x_j)_2^1 = 0$, hence $x_j \in E = E(\mathcal{B})$. Moreover, we have $\sum_{j=1}^{\tau} g(x_j)_2^2 g(f_j)_2^1 = 0$. But this means that $\sum_{j=1}^{\tau} g(x_j f_j)_2^1 = g(\sum_{j=1}^{\tau} x_j f_j)_2^1 = 0$, and it follows that $\sum_{j=1}^{\tau} x_j f_j \in E$. Thus $\sum_{j=1}^{\tau} x_j f_j = x_0 \in E$ and

$$-x_0 + \sum_{j=1}^{\tau} x_j f_j = 0,$$

and thus all $x_j = 0$ because $1, f_1, \ldots, f_\tau$ is an E-basis of F. This shows that $m_1(1) = 0$ and $m = 0$. ∎

We will need a property of the subring $G_0 = G_0(\mathcal{B}) \subseteq G$ which was introduced in the paragraph before Lemma 3.2. Recall that $g \in G_0$ if and only if there exists $f \in E(\mathcal{B})$ such that $g(f)_2^2 = g$.

Lemma 3.7. *Let B be a left pre-acceptable G-F-bimodule with G-basis $\mathcal{B} = \{b_1, b_2\}$ and $\tau + 1 = \mathrm{l.dim}({}_E F)$, with $E = E(\mathcal{B})$. Assume that for each $g \in G$ there exists $f \in F$ such that $g(f)_2^1 = g$. Then the left dimension of G over $G_0 = G_0(\mathcal{B})$ equals τ.*

Proof. Given $g \in G$, let $f \in F$ with $g(f)_2^1 = g$. Choose $\{f_0 = 1, f_1, \ldots, f_\tau\}$ as E-basis of F. We have

$$f = x_0 + \sum_{j=1}^{\tau} x_j f_j$$

with $x_0, \ldots, x_\tau \in E$. By using the representation of the elements of F as G-matrices, we obtain

$$g(f)_2^1 = \sum_{j=1}^{\tau} g(x_j)_2^2 g(f_j)_2^1.$$

Since $g(x_j)_2^2 \in G_0$, the elements $g(f_j)_2^1$ form a generating system for G as a left space over G_0.

It remains to show that these are linearly independent vectors. Assume we had elements $x_1, \ldots, x_\tau \in E$ such that $\sum_{j=1}^{\tau} g(x_j)_2^2 g(f_j)_2^1 = 0$. Then $g(\sum_{j=1}^{\tau} x_j f_j)_2^1 = \sum_{j=1}^{\tau} g(x_j f_j)_2^1 = 0$, and hence $\sum_{j=1}^{\tau} x_j f_j = x_0 \in E$. Therefore

$$-x_0 f_0 + \sum_{j=1}^{\tau} x_j f_j = 0$$

so that each $x_j = 0$, because the elements $f_0, \ldots, f_\tau$ form an E-basis of F. This shows that the vectors are left G_0-linearly independent. ∎

The following lemma is a key result of this section.

LEMMA 3.8. *Let G, F, B, R_B be as in Lemma 3.1. Assume that B is left pre-acceptable, set $R := R_B$, choose a G-basis $\mathcal{B} = \{b_1, b_2\}$ of B, and construct the modules $M = M(\mathcal{B})$, $N = N(\mathcal{B})$ as above. Then the following conditions are equivalent:*

(i) $B = b_2F + Gb_2$.
(ii) *For each $g \in G$, there is $f \in F$ such that $g(f)^1_2 = g$.*
(iii) $\mathrm{Ext}^1_R(M, N) = 0$.

Proof. We start by showing (i)⇔(ii). First we assume (i). If $g \in G$, then we know that $gb_1 = b_2f + g'b_2$ for elements $f \in F$, $g' \in G$, and hence $b_2f = gb_1 - g'b_2$ so that $g = g(f)^1_2$. Conversely, asume (ii). For any $g \in G$ we have $f \in F$, $g' \in G$ with $gb_1 + g'b_2 = b_2f$ by hypothesis. Thus $gb_1 = b_2f - g'b_2$ and $Gb_1 \subseteq b_2F + Gb_2$. Since $B = Gb_1 + Gb_2$, we immediately obtain (i).

In order to prove the equivalence with (iii) we establish three consecutive claims. As in Definition 3.3, $\tau + 1$ is the left dimension of F over the subring $E = E(\mathcal{B})$; and following the notation in Lemma 3.5, we fix a left E-basis $\{1 = f_0, \dots, f_\tau\}$ of F and define $N := N(\mathcal{B})$ determined by the G-linear surjective map $j^c : B \otimes_F F^\tau \to G^{\tau+1}$, with $u_1, \dots, u_\tau$ being the canonical left basis of F^τ and with K (see Lemma 3.5) being the kernel of the map j^c defining N. In these claims we also assume that X is a left R-module which is defined by a surjective G-linear map $\gamma : B \otimes_F F^{\tau+1} \to G^{\tau+2}$. Finally, we admit the existence of an injective F-linear map $g_1 : F^\tau \to F^{\tau+1}$ and write $g_1(u_j) = v_j$ for $j = 1, \dots, \tau$.

CLAIM 1. *If $g : N \to X$ is a monomorphism of left R-modules such that $1 \otimes g_1 : B \otimes_F F^\tau \to B \otimes_F F^{\tau+1}$ is the associated injective G-linear map, then there exist $v_0 \in F^{\tau+1}$, $w_0 \in B \otimes F^\tau$, $z \in B \otimes_F F^{\tau+1}$ and $a, c \in G$ such that:* (1) *$\{v_0, \dots, v_\tau\}$ is a left basis of $F^{\tau+1}$;* (2) *a, c are not both zero;* (3) *$z = (1 \otimes g_1)(w_0) + a(b_1 \otimes v_0) + c(b_2 \otimes v_0)$; and* (4) *$\mathrm{Ker}(\gamma) = (1 \otimes g_1)(K) \oplus Gz$. Conversely, if there exist elements w_0, z, a, c as above satisfying* (1)–(4)*, then $1 \otimes g_1$ induces a monomorphism $g : N \to X$.*

Proof. First, the injectivity of g_1 implies that $v_1, \dots, v_\tau$ are linearly independent as vectors of the left F-module $F^{\tau+1}$. Hence, we may choose $v_0 \in F^{\tau+1}$ such that (1) holds. Then we know that $1 \otimes g_1$ induces a left R-monomorphism g if and only if $(1 \otimes g_1)^{-1}(\mathrm{Ker}(\gamma)) = K$. By Lemma 3.5, K has G-dimension $\tau - 1$, while $\mathrm{Ker}(\gamma)$ has dimension $2(\tau + 1) - (\tau + 2) = \tau$. Thus if $1 \otimes g_1$ induces a R-monomorphism, then $\mathrm{Ker}(\gamma)$ is the direct sum of $(1 \otimes g_1)(K)$ plus a one-dimensional subspace Gz such that $z \notin (1 \otimes g_1)(B \otimes_F F^\tau)$. Since $\mathrm{Im}(1 \otimes g_1) = B \otimes \langle v_1, \dots, v_\tau \rangle$, this shows that z has the form given in (3) and (2). The above shows also (4).

Conversely, if the four conditions hold for elements v_0, w_0, a, c, then obviously $(1 \otimes g_1)^{-1}(\mathrm{Ker}(\gamma)) = K$ and $1 \otimes g_1$ induces a left R-monomorphism. This finishes the proof of Claim 1.

CLAIM 2. *If X and g_1 are as above, then $1 \otimes g_1$ induces a monomorphism $N \to X$ if and only if there exist $v_0 \in F^{\tau+1}$ and $w_0 \in B \otimes F^\tau$ such that: (1) $\{v_0, \ldots, v_\tau\}$ is a left F-basis of $F^{\tau+1}$; (2) if we set $z = (1 \otimes g_1)(w_0) + b_2 \otimes v_0$, then* $\mathrm{Ker}(\gamma) = (1 \otimes g_1)(K) \oplus Gz$.

Proof. The sufficiency of the condition is obvious, since we obtain the properties (1)–(4) of Claim 1 by taking $a = 0$ and $c = 1$. To show the necessity, suppose that (1)–(4) of Claim 1 are fulfilled with certain elements v_0, w_0', a, c. We observe that the d-vector of X (see [15]) is $(\tau+1, \tau+2)$ by the construction of X. Since $g : N \to X$ is a monomorphism and the d-vector of N is $(\tau, \tau + 1)$, the cokernel $C = \mathrm{Coker}(g)$ has d-vector equal to $(1, 1)$. If C is not indecomposable, then one of its summands would have d-vector $(0, 1)$, and hence it would be simple projective. But in that case the short exact sequence $0 \to N \to X \to C \to 0$ would be split and X would have a simple projective direct summand. By the election of X, this does not happen, so that C has to be indecomposable. Since B is left pre-acceptable, $C \cong M$ by Lemma 3.1. Therefore, the cokernel $p : B \otimes F^{\tau+1} \to B \otimes F$ of $1 \otimes g_1$ must satisfy $p(\mathrm{Ker}(\gamma)) \subseteq \mathrm{Ker}(\alpha) = G(b_2 \otimes 1)$.

But $z' = (1 \otimes g_1)(w_0') + a(b_1 \otimes v_0) + c(b_2 \otimes v_0)$ belongs to $\mathrm{Ker}(\gamma)$ by (iv), and hence $p(z') \in G(b_2 \otimes 1)$. Since $p((1 \otimes g_1)(w_0')) = 0$ and $p(b_2 \otimes v_0) \in G(b_2 \otimes 1)$, it follows that $a(p(b_1 \otimes v_0)) \in G(b_2 \otimes 1)$. But the image of v_0 by the cokernel of g_1 is not zero, because $v_0 \notin \mathrm{Im}(g_1)$. This implies that $a = 0$, and $z' = (1 \otimes g_1)(w_0') + c(b_2 \otimes v_0)$ with $c \neq 0$. By taking $z = c^{-1}z'$, we have $z = (1 \otimes g_1)(w_0) + b_2 \otimes v_0$ for some $w_0 \in B \otimes_F F^\tau$ and $Gz = Gz'$, so that (1) and (2) hold. This finishes the proof of Claim 2.

CLAIM 3. *Given the G-basis $\mathcal{B} = \{b_1, b_2\}$ of B, the module $M = M(\mathcal{B})$ and the subring $E = E(\mathcal{B})$ of F, choose the $E(\mathcal{B})$-basis $\{1, f_1, \ldots, f_\tau\}$ of F and construct $N = N(\mathcal{B})$ (see Lemma* 3.5*). Then* $\mathrm{Ext}^1_R(M, N) = 0$ *if and only if for every $w_0 = \sum_{i=1}^2 \sum_{j=1}^\tau w_i^j b_i \otimes u_j \in B \otimes_F F^\tau$ with the $w_i^j \in G$, there exist elements $d_j \in F$ and $c_j \in G$ (for $j = 1, \ldots, \tau$) so that:*

$$\text{(3.1)} \qquad \text{(a)} \sum_{j=1}^{\tau} c_j g(f_j)_2^1 = 0, \qquad \text{(b) } g(d_j)_2^1 = w_1^j, \qquad \text{(c) } g(d_j)_2^2 = w_2^j + c_j.$$

Proof. Suppose first $\mathrm{Ext}^1_R(M, N) = 0$, and consider the element $w_0 = \sum_{i=1}^2 \sum_{j=1}^\tau w_i^j b_i \otimes u_j$. We take the canonical basis $\{v_0, \ldots, v_\tau\}$ of $F^{\tau+1}$ as a left F-vector space, define $g_1 : F^\tau \to F^{\tau+1}$ by setting $g_1(u_i) = v_i$ for $i = 1, \ldots, \tau$, write $z = (1 \otimes g_1)(w_0) + b_2 \otimes v_0$, and take γ as the cokernel of the inclusion of $(1 \otimes g_1)(K) \oplus Gz$ into $B \otimes F^{\tau+1}$. By Claim 1, $1 \otimes g_1$ induces a monomorphism $g : M \to X$, where X is the left R-module defined by the linear map γ. Since γ is surjective, X has no simple projective direct summand, and, as seen in the proof of Claim 2, there is a short exact sequence $0 \to N \xrightarrow{g} X \xrightarrow{g^c} M \to 0$. By hypothesis, g^c has to be a split epimorphism of left R-modules. In particular, $1 \otimes g_1^c : B \otimes_F F^{\tau+1} \to B \otimes_F F$ is a surjection which has a right inverse $1 \otimes h_1 : B \otimes_F F \to B \otimes F^{\tau+1}$ (here h_1 is a right inverse to g_1) that induces a homomorphism $h : M \to X$ of left R-modules.

Note now that $g_1^c(v_j) = 0$ for $j = 1, \ldots, \tau$, and hence $g_1^c(v_0) = y \neq 0$. Since we may take $v_0' = y^{-1}v_0$ without affecting the definition of g_1 and the above short exact sequence, we may assume that $g_1^c(v_0) = 1$. Thus $h_1(1) = v_0 + \sum_{j=1}^{\tau} d_j v_j$ for $d_1, \ldots, d_\tau \in F$. Since $1 \otimes h_1$ induces a homomorphism, we have the commutative diagram

$$\begin{array}{ccc} G(b_2 \otimes 1) & \to & B \otimes F \\ \downarrow & & \downarrow^{1\otimes h_1} \\ \operatorname{Ker}(\gamma) & \to & B \otimes F^{\tau+1} \end{array}$$

The left-hand map carries $b_2 \otimes 1$ to

$$x = (1 \otimes h_1)(b_2 \otimes 1) = b_2 \otimes \Big(v_0 + \sum_{j=1}^{\tau} d_j v_j\Big) = b_2 \otimes v_0 + \sum_{j=1}^{\tau} b_2 d_j \otimes v_j$$

and this element x must belong to $\operatorname{Ker}(\gamma)$. Set $w = (1 \otimes g_1)(w_0)$ so that $z = w + b_2 \otimes v_0$. Then by the construction of $\operatorname{Ker}(\gamma)$, x must be the sum of $g'z$ for some $g' \in G$, and an element in $(1 \otimes g_1)(K)$. By Lemma 3.5 and the definition of g_1, this second summand will be

$$\sum_{j=1}^{\tau} c_j (b_2 \otimes v_j)$$

where the coefficients $c_1, \ldots, c_\tau \in G$ are subject to the condition $\sum c_j g(f_j)_2^1 = 0$. Therefore

$$x = \sum_{i=1}^{2} \sum_{j=1}^{\tau} g' w_i^j b_i \otimes v_j + \sum_{j=1}^{\tau} c_j (b_2 \otimes v_j) + g' b_2 \otimes v_0.$$

By equating the coefficients we conclude that $g' = 1$, and consequently

$$\sum_{j=1}^{\tau} g(d_j)_2^1 b_1 \otimes v_j + \sum_{j=1}^{\tau} g(d_j)_2^2 b_2 \otimes v_j = w + \sum_{j=1}^{\tau} c_j (b_2 \otimes v_j).$$

This proves that there exist elements $d_j \in F$ and $c_j \in G$ (these last elements giving a row which is left orthogonal to $(g(f_j)_2^1)$ so that (a) of (3.1) holds) satisfying

$$g(d_j)_2^1 = w_1^j \quad \text{and} \quad g(d_j)_2^2 = w_2^j + c_j.$$

This shows that these elements satisfy (b) and (c) of equation (3.1).

To complete the proof of the claim, we must show the converse. Accordingly, we consider an extension $0 \to N \xrightarrow{g} X \to M \to 0$ and prove that, under the hypotheses (3.1), it splits. First, the d-vector of X is $(\tau+1, \tau+2)$ and X has no simple projective direct summand because M has no simple projective direct summand. We may assume that the injective map $g_1 : F^\tau \to F^{\tau+1}$ corresponding to the monomorphism g is defined as in the beginning of this proof, and hence by Claim 2, we know that there exists $w_0 \in B \otimes F^\tau$ such that conditions (1) and (2) of that claim are fulfilled. Let $w_0 = \sum w_i^j b_i \otimes u_j$ and $w = (1 \otimes g_1)(w_0)$. By our assumption, there exist $d_j \in F$ and $u = \sum_{j=1}^{\tau} c_j b_2 \otimes u_j \in K \subseteq B \otimes_F F^\tau$ such that the conditions (3.1) are satisfied. These conditions show that, if we set $v = \sum_{j=1}^{\tau} d_j v_j \in F^{\tau+1}$, then

$$b_2 \otimes (v_0 + v) = b_2 \otimes v_0 + w + (1 \otimes g_1)(u).$$

Define the F-linear map $m_1 : F \to F^{\tau+1}$ that carries 1 to $v_0 + v$. The G-linear map $1 \otimes m_1 : B \otimes_F F \to B \otimes_F F^{\tau+1}$ carries $\mathrm{Ker}(\alpha)$ to $\mathrm{Ker}(\gamma)$ by (2) of Claim 2. Therefore $1 \otimes m_1$ induces a module homomorphism $m : M \to X$; moreover, the composed map $B \otimes_F F \xrightarrow{1\otimes m_1} B \otimes_F F^{\tau+1} \xrightarrow{1\otimes g_1^c} B \otimes_F F$ is non-zero, since the image of $v + v_0$ by g_1^c is $0 + g_1^c(v_0) \neq 0$, because $v_0 \notin \mathrm{Im}(g_1)$. Thus the composed homomorphism $M \to X \to M$ is a non-zero homomorphism, hence an automorphism of M, because the endomorphism ring of M is a division ring (Lemma 3.2). This proves that the given short exact sequence splits, and finishes the proof of Claim 3.

To end the proof of Lemma 3.8, we show the equivalence of (ii) and (iii).

(iii)⇒(ii). Suppose that $\mathrm{Ext}_R^1(M, N) = 0$ and take any $g \in G$. Then we may choose $w_0 = \sum w_i^j b_i \otimes u_j$ so that $w_1^1 = g$. By Claim 3, there exists $d_1 \in F$ such that $g(d_1)_2^1 = w_1^1 = g$. Thus (ii) holds.

(ii)⇒(iii). We show that, given $w_0 = \sum w_i^j b_i \otimes u_j$, there exist elements $d_1, \ldots, d_\tau, c_1, \ldots, c_\tau$ satisfying the equations (a)–(c) of (3.1). From Claim 3, (iii) will follow. By hypothesis, there exist $d_j \in F$ such that $g(d_j)_2^1 = w_1^j$ for each $j = 1, \ldots, \tau$, and this proves (b).

We note that these elements d_j are not unique, because if d satisfies the condition $g(d)_2^1 = g \in G$, then so does $d + u$ whenever $u \in E$.

Recall that $G_0 = G_0(\mathcal{B})$ was defined before Lemma 3.2 and that the left dimension of G over G_0 is, under our current hypothesis, equal to τ by Lemma 3.7. Consider two left G_0-subspaces of G^τ. First, G_0^τ is a G_0-subspace of G^τ with dimension τ. Then consider the subset S of G^τ consisting of elements (c_j) such that $\sum_{j=1}^{\tau} c_j g(f_j)_2^1 = 0$. Since this is a left G-subspace which is, as such, a hyperplane (given by a linear equation), its left dimension over G_0 will be $(\tau-1)\tau$. Finally, the left dimension of G^τ itself is τ^2. On the other hand, the intersection of the subspaces G_0^τ and S is zero, because the elements $g(f_j)_2^1$ form a G_0-basis of G (see the proof of Lemma 3.7). Since $\tau^2 = (\tau-1)\tau + \tau$, G^τ is the direct sum of G_0^τ and S.

To complete our proof, we need to show that there exist $c_1, \dots, c_\tau \in G$ such that $(c_1, \dots, c_\tau) \in S$ while $c_j = g(d_j)_2^2 - w_2^j$ for each $j = 1, \dots, \tau$. Let $\widehat{w} = (w_2^j)$ and $y = (g(d_j)_2^2)$. As shown above, d_j may be replaced by $d_j' = d_j + u_j$ with $u_j \in E$, and condition (b) still holds. This substitution would replace $g(d_j)_2^2$ with $g(d_j)_2^2 + t_j$ for some $t_j \in G_0$, and would hence replace y with $y+t$ for some $t \in G_0^\tau$. For the given $\widehat{w}, y \in G^\tau$, we know that $y - \widehat{w} \in S + G_0^\tau$, and hence $x = (y+t) - \widehat{w} \in S$ for some $t \in G_0^\tau$. If we take $(c_1, \dots, c_\tau) = x$ and $(d_1', \dots, d_\tau')$, then (a)–(c) of (3.1) are fulfilled, so that (iii) holds by Claim 3. ■

Let G, F, B, R_B be as in Lemma 3.1, and assume that B is left pre-acceptable. Choose a G-basis $\mathcal{B} = \{b_1, b_2\}$ of B. We say that the basis $\mathcal{B}$ is *acceptable* if the equivalent conditions of Lemma 3.8 hold.

PROPOSITION 3.9. *Let B be a left pre-acceptable bimodule with an acceptable G-basis $\mathcal{B} = \{b_1, b_2\}$. If $M, N = N(\mathcal{B})$ are as in Lemma* 3.8, *then $M \oplus N$ is a tilting module.*

Proof. Set $R := R_B$ and let E_0 be the simple injective left R-module. By Lemmas 3.6 and 3.8, $\mathrm{Hom}_R(M,N) = 0$ and $\mathrm{Ext}_R^1(M,N) = 0$. Moreover, by Lemma 3.5 we have a short exact sequence

$$0 \to N \to M^{\tau+1} \to E_0 \to 0,$$

so for any module X we get an exact sequence $\mathrm{Ext}_R^1(X,N) \to \mathrm{Ext}_R^1(X,M)^{\tau+1} \to 0$. Thus, if $\mathrm{Ext}_R^1(X,N) = 0$, then we may infer that $\mathrm{Ext}_R^1(X,M) = 0$. This implies immediately that $\mathrm{Ext}_R^1(M,M) = 0$.

Consider now $h_i : F^\tau \to F$ as the left F-linear map taking the basis vector u_i to 1 and the other u_j to 0. Let $K \subseteq B \otimes_F F^\tau$ as in Lemma 3.5, so that a basis of K is given by vectors of the form $\sum g_j b_2 \otimes u_j$. Thus $(1 \otimes h_i)(K) \subseteq G(b_2 \otimes 1)$, and hence h_i induces a homomorphism $N \to M$ which is an epimorphism. Then the family $h_1, \dots, h_\tau$ determines an epimorphism $\rho : N \to M^\tau$. The d-vector of $\mathrm{Ker}(\rho)$ is $(0,1)$, that is, $\mathrm{Ker}(\rho)$ is isomorphic to the simple projective P_0. We thus have a short exact sequence

$$0 \to P_0 \to N \to M^\tau \to 0.$$

Now, if X is any module, we get an exact sequence $\mathrm{Ext}_R^1(M,X)^\tau \to \mathrm{Ext}_R^1(N,X) \to 0$, and this implies that if $\mathrm{Ext}_R^1(M,X) = 0$, then $\mathrm{Ext}_R^1(N,X) = 0$. Since $\mathrm{Ext}_R^1(M,N) = \mathrm{Ext}_R^1(M,M) = 0$, we deduce that $\mathrm{Ext}_R^1(N,N) = \mathrm{Ext}_R^1(N,M) = 0$, and consequently $\mathrm{Ext}_R^1(M \oplus N, M \oplus N) = 0$.

The previous sequence gives a coresolution for P_0 by means of modules in $\mathrm{add}(M \oplus N)$. If $\tau = 1$, then N has d-vector $(1,2)$ and $K = 0$ (see Lemma 3.5), so that the G-linear map defining N is an isomorphism. Hence N is projective and isomorphic to the non-simple projective P_1. It follows that also P_1 has a coresolution by modules in $\mathrm{add}(M \oplus N)$.

Suppose now that $\tau > 1$ so that N is not projective, and let us use the notation of Lemma 3.5 for N. We consider the homomorphism $\mu : N \to M^{\tau-1}$ given through the F-linear maps $h_1, \ldots, h_{\tau-1}$ as above. Then μ is an epimorphism whose kernel is defined by a map $\lambda : B \otimes F = B \otimes \langle u_\tau \rangle \to G^2$. Now, $\mathrm{Ker}(\lambda)$ consists of elements $g_1 b_1 \otimes u_\tau + g_2 b_2 \otimes u_\tau$ whose image in $B \otimes F^\tau$ lies inside K. But this condition entails that $g_1 = 0$ and $(0, 0, \ldots, g_2)$ is left orthogonal to $(g(f_1)_2^1, \ldots, g(f_\tau)_2^1)$, by Lemma 3.5. Hence $g_2 g(f_\tau)_2^1 = 0$, which implies $g_2 = 0$. Therefore λ is an isomorphism, and so $\mathrm{Ker}(\mu)$ is isomorphic to the projective module P_1. This gives a short exact sequence $0 \to P_1 \to N \to M^{\tau-1} \to 0$, so we have a coresolution for P_1 by modules in $\mathrm{add}(M \oplus N)$. One shows that $M \oplus N$ is tilting, by applying the definition of tilting module (see, for instance, [8]). ∎

THEOREM 3.10. *Let B be a left pre-acceptable G-F-bimodule. If there exists an acceptable G-basis of B, then every G-basis $\mathcal{B}$ is acceptable, $N(\mathcal{B})$ is an indecomposable module, and the isomorphism class of $N(\mathcal{B})$ is independent of the choice of the G-basis $\mathcal{B}$ of B.*

Proof. Let $\mathcal{B} = \{b_1, b_2\}$ be an acceptable G-basis of B, and consider the module $N = N(\mathcal{B})$ constructed in Lemma 3.5. By Proposition 3.9, $M \oplus N$ is a tilting module, and by [27, Theorem 1.5] it is a direct sum of indecomposable modules belonging to only two isomorphism classes. Now, M cannot be isomorphic to a direct summand of N by Lemma 3.6, and therefore N has to be a direct sum of isomorphic indecomposable direct summands. But having d-vector $(\tau, \tau + 1)$ with these two values coprime, it cannot be a direct sum of isomorphic different modules, so it is indecomposable. Since $\varphi : N \to M^{\tau+1}$ is a monomorphism (see the proof of Lemma 3.5), N is isomorphic to an indecomposable maximal submodule of $M^{\tau+1}$.

Let L be any indecomposable maximal submodule of $M^{\tau+1}$, so that L will be the kernel of an epimorphism $h : M^{\tau+1} \to E_0$, where E_0 is the simple injective left R-module. Let $h_1 : F^{\tau+1} \to F$ be the F-linear map determined by the homomorphism h, and consider the elements $h_1(e_0), \ldots, h_1(e_\tau)$ of F, where $e_0, \ldots, e_\tau$ is the canonical basis of $F^{\tau+1}$. Recall from Lemma 3.2 that $E = E(\mathcal{B})$ is isomorphic to $\mathrm{End}_R(M)$. We claim that $h_1(e_0), \ldots, h_1(e_\tau)$ are left E-linearly independent. Indeed, suppose that they are linearly dependent. Then there is a non-zero homomorphism $g : M \to M^{\tau+1}$ such that $gh = 0$. Bearing in mind that $\mathrm{End}_R(M)$ is a division ring, we deduce that $M^{\tau+1} = X \oplus Y$ such that $Y \cong M$ and $h(Y) = 0$. Then Y is a proper direct summand of $\mathrm{Ker}(h) = L$ and L is not indecomposable. This contradiction proves that $h_1(e_0), \ldots, h_1(e_\tau)$ are left E-linearly independent elements of F, and consequently they form an E-basis of F.

By Lemma 3.4, we infer that $\mathrm{Ker}(h) = L \cong N$. Thus, we have shown that, up to isomorphism, $M = M(\mathcal{B})$ is such that all indecomposable maximal submodules of $M^{\tau+1}$ are isomorphic. Since the isomorphism class of M does not depend on the choice of the G-basis of B, this property holds for every G-basis. Also, the module $N(\mathcal{B})$ does not depend, up to isomorphism, on the choice of the G-basis $\mathcal{B}$.

Therefore, the isomorphism class of $M(\mathcal{B}) \oplus N(\mathcal{B})$ is independent of the choice of G-basis of B, so that $\mathrm{Ext}_1^R(M, N) = 0$ for any choice of G-basis of B. By Lemma 3.8, any G-basis of B is acceptable, as required. ∎

Definition 3.11. Let G, F, B, R_B be as in Lemma 3.1, M as in Lemma 3.2 and N as in Lemma 3.5. Then B is called *left weakly acceptable* if B admits an acceptable G-basis. The bimodule B is called *left acceptable* when B is left weakly acceptable and $\mathrm{Hom}_R(N, M)$ is left finite-dimensional over $\mathrm{End}_R(N)$.

By Theorem 3.10, if the bimodule B is left weakly acceptable, then every G-basis of B is acceptable. If B is left (weakly) acceptable, we say that $M \oplus N$ is the *tilting module determined by* B. Recall from [15, Definition 2.4] the concept of a rigid tilting module.

Lemma 3.12. *Assume that B is a left weakly acceptable G-F-bimodule, and let $M \oplus N$ be the tilting module determined by B. Then $M \oplus N$ is a rigid tilting module if and only if B is left acceptable.*

Proof. Note that $\mathrm{End}_R(N)$ (as well as $\mathrm{End}_R(M)$) is a division ring as follows from the proof of [17, Lemma 4.1] (see also [14, Theorem 3.1(g)]). Moreover, $\mathrm{Hom}_R(M, N) = 0$ by Lemma 3.6 and $\mathrm{Hom}_R(N, M) \neq 0$ as shown in the proof of Lemma 3.5. Since B is left weakly acceptable, $M \oplus N$ is a basic tilting module. According to [15, Definition 2.4], it is rigid if and only if $\mathrm{Hom}_R(N, M)$ is left finite-dimensional. ∎

When B is left acceptable, the endomorphism ring of $M \oplus N$ is again of the form (1.1), with the bimodule $B' = \mathrm{Hom}_R(N, M)$ which is an H-E-bimodule, where $H = \mathrm{End}_R(N)$ and $E = E(\mathcal{B}) \cong \mathrm{End}_R(M)$ are division rings. Though the proper construction of B' and H depends on the choice of the basis $\mathcal{B}$ of B, we have seen above that their isomorphism classes are independent of that choice, as well as of the left H-dimension of B', which is finite by our hypothesis that B is left acceptable. In this situation, we say that B' is the *acceptance bimodule* of the left acceptable bimodule B. Moreover, as shown before, the characteristic value $\tau(B) = \mathrm{l.dim}({}_E F) - 1$ is also independent of the choice of basis.

We have considered in Lemma 3.7 the subring $G_0(\mathcal{B})$ of G, and we have seen that the left dimension of G over this subring is τ when the G-F-bimodule B is left acceptable (see Lemma 3.8). Hence this dimension is invariant under the change of the G-basis $\mathcal{B}$ of B. We end this section by introducing another division subring of G isomorphic to $G_0(\mathcal{B})$.

Let the G-basis $\mathcal{B} = \{b_1, b_2\}$ of B be chosen, and consider the division subring $E = E(\mathcal{B}) \subseteq F$ formed with the elements $f \in F$ such that $g(f)_2^1 = 0$, as constructed before Lemma 3.2. Consider the map $E \ni f \mapsto g(f)_1^1 \in G$. This is a ring homomorphism $E \to G$ whose image is a division subring of G isomorphic to E. We shall denote this subring as $G_1 = G_1(\mathcal{B})$.

Proposition 3.13. *Let B be a left acceptable bimodule, choose a G-basis $\{b_1, b_2\}$ of B, and let E, G_1, τ be as above. Let $B^* = \mathrm{Hom}_G(B, G)$ viewed as an F-G-bimodule, and let $d^* = \mathrm{l.dim}(B^*)$. If $\widehat{d}$ is the left dimension of G over G_1, then*

$$\tau + \widehat{d} = (\tau + 1)d^*.$$

In particular, $d^* = 2$ if and only if $\widehat{d} = \tau + 2$.

Proof. The F-G-bimodule $B^* = \mathrm{Hom}_G(B, G)$ has a left F-module structure coming from the right F-structure of B. There is a canonical isomorphism of right G-vector spaces $B^* \cong G^2$. On the other hand, we mentioned after Lemma 3.1 that F is isomorphic to a subring $\hat{F}$ of $\mathbb{M}_2(G)$, the ring of 2×2 matrices over G. Then it is easy to see that there is a semilinear isomorphism ${}_F B^* \cong {}_{\hat{F}} G^2$. Thus, $\mathrm{l.dim}(B^*)$ is the left dimension of G^2 over the ring of matrices $\hat{F}$ corresponding to F. Let $\hat{E}$ be the image of $E \subseteq F$ under the isomorphism $F \cong \hat{F}$. Thus $\hat{E}$ consists of those matrices belonging to $\hat{F}$ whose $(2, 1)$ entry is zero.

Let d^* be the left dimension of B^* (finite or infinite), and call m the left dimension of G^2 over $\hat{E}$. Since the left dimension of F over E equals $\tau + 1$, we have $m = (\tau + 1)d^*$.

Observe that $L = G \times 0 \subseteq G^2$ is an $\hat{E}$-subspace of G^2. Moreover, the isomorphism $E \cong G_1$ described above results in an isomorphism $\hat{E} \cong G_1$, given by the map $(x_{ij}) \mapsto x_{11}$. It follows that there is a semilinear isomorphism ${}_{\hat{E}} L \cong {}_{G_1} G$, and hence the left dimension of L equals $\widehat{d}$.

Let now $g_1, \ldots, g_\tau$ be a basis of G viewed as a left space over G_0 (see Lemma 3.7). Then $(0, g_1), \ldots, (0, g_\tau)$ are elements of G^2 which are linearly independent over $\hat{E}$. Indeed, any dependence relation $\sum (x_{ij}^k)(0, g_k) = 0$ gives $\sum x_{22}^k g_k = 0$, whence each $x_{22}^k \in G_0$ is zero, and this implies that $(x_{ij}^k) = 0$. The $\hat{E}$-subspace generated by these elements has (as we have essentially just shown) zero intersection with L, and their direct sum obviously equals G^2. Thus $m = \tau + \widehat{d}$, hence $\tau + \widehat{d} = (\tau + 1)d^*$ as required. ∎

Remark 3.14. By Proposition 3.13, the left dimension of G over its subring G_1 is independent of the G-basis $\mathcal{B}$ of the bimodule B. This is a consequence also of the following fact, which can be shown in a standard way: if $E(\mathcal{B})$ is the subring of F in Lemma 3.2 and E_1 is the non-simple indecomposable injective left R-module, then $G_1 \cong E(\mathcal{B})$, and $\mathrm{Hom}_R(M, E_1)$ is semilinearly isomorphic (with respect to the isomorphism $E \cong G_1$) to the left G_1-module G. Consequently, $\mathrm{l.dim}(\mathrm{Hom}_R(M, E_1)) = \mathrm{l.dim}({}_{G_1}G)$.

4. Sporadic bimodules. Recall from [15] that if F and G are division rings, then a G-F-bimodule B has the *left finite dimension property* when B and all the successive left dual bimodules $B^* = \mathrm{Hom}_G(B, G)$, $B^{*2} = \mathrm{Hom}_F(B^*, F)$, $\ldots$ are left finite-dimensional.

DEFINITION 4.1. Let F, G be division rings and B a G-F-bimodule that has the left finite dimension property. We say that B is *left strictly sporadic* when $\mathrm{l.dim}(B^{*n}) = 2$ for each $n = 0, 1, \ldots$. Here, $B^{*0} = B$, $B^{*1} = B^*$ and B^{*n} denotes the nth left dual bimodule of B for $n > 1$.

THEOREM 4.2. *Let F, G be division rings, and let B be a G-F-bimodule which is left strictly sporadic and left acceptable. Let $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$. There is a well-ordered sequence*

$$X_0, \ldots, X_{\omega+1}$$

of elements of R-ind *with the following properties:*

(i) *For $\alpha \le \omega + 1$, let $\mathcal{S}_\alpha = R\text{-ind} \setminus \{X_\beta \mid \beta < \alpha\}$. Then X_α is the only element of $\mathcal{S}_\alpha$ satisfying $\mathrm{Hom}_R(X_\alpha, Y) = 0$ for all $Y \in \mathcal{S}_\alpha$ such that $Y \not\cong X_\alpha$. Moreover, $\mathrm{Hom}_R(Y, X_\alpha) \neq 0$ for each $Y \in \mathcal{S}_\alpha$.*

(ii) *The rigid tilting module determined by the left acceptable bimodule B is $W = X_\omega \oplus X_{\omega+1}$. If $(\mathcal{T}, \mathcal{F})$ is the torsion theory of* R-Mod *defined by W, then a finitely presented indecomposable module Y belongs to $\mathcal{T}$ if and only if $Y \cong X_\alpha$ for some $0 \le \alpha \le \omega + 1$.*

(iii) *If τ is the characteristic value of the left acceptable bimodule B, set $d'_{-1} = d^* = \mathrm{l.dim}(B^*)$ and $d'_0 = 1$, $d'_1 = \tau + 2$. Then the sequence*

$$d'_{-1}, d'_0, d'_1$$

is a partial dimension sequence.

Proof. We develop the proof of (i) and (ii) under a slightly weaker hypothesis: we only assume that B is left acceptable and has the left finite dimension property and R_B is not of finite representation type. By Proposition 2.2, there is an infinite sequence of finitely presented indecomposable left R-modules

$$X_0, X_1, \ldots$$

such that each X_k on the list is unique satisfying $\mathrm{Hom}_R(X_k, Y) = 0$ for any finitely presented indecomposable left R-module Y with $Y \not\cong X_i$ for $i \le k$ (and moreover $\mathrm{Hom}_R(Y, X_k) \neq 0$ for such Y). This means that (i) of this theorem holds for all $\alpha < \omega$.

Let us consider now the dimensions $d_{-1} = d^*, d_0, d_1, \ldots$ (where $d_k =$ l.dim$(\mathrm{Hom}_R(X_{k+1}, X_k))$ for $k \geq 0$) and construct the corresponding continued fractions $[d^*, d_0, \ldots, d_k]$ as in [15, Lemma 3.2]. We shall make the additional assumption that the limit (as $k \to \infty$) of the sequence of the convergents $[d^*, d_0, \ldots, d_k]$ equals 1. Note that this condition holds under the hypothesis of the theorem, because the sequence of fractions $[2, 2, 2, \ldots, 2]$ has limit 1. By [15, Proposition 2.11], the infinite sequence of modules

$$X_0, X_1, \ldots$$

consists precisely of all the preinjective finitely presented indecomposable left R-modules. On the other hand, by [15, Lemmas 3.2 and 3.3], the d-vector (t_k, s_k) of each of the preinjective modules X_k satisfies the inequality $t_k > s_k$, because of the assumption that 1 is the limit of the convergents. Also, [15, Proposition 3.11] shows that the preinjective modules are the only finitely presented indecomposable left R-modules with a d-vector (t, s) such that $t > s$.

Let $M \oplus N$ be the rigid tilting module determined by B (see Proposition 3.9). Then $M \oplus N$ defines a torsion theory $(\mathcal{T}, \mathcal{F})$ of R-Mod where $X \in \mathcal{T}$ if and only if $\mathrm{Ext}^1_R(M \oplus N, X) = 0$ (see [8]). The endomorphism ring S of $M \oplus N$ is left artinian and hereditary [15, Proposition 2.5], and thus $(\mathcal{T}, \mathcal{F})$ is splitting [3, Lemma 4.5]. We also have a pair of equivalences H, H' from $\mathcal{T}, \mathcal{F}$ to the corresponding categories of S-Mod.

We observe that every preinjective module X_k belongs to the class $\mathcal{T}$. Indeed, by the construction of the sequence, we have $\mathrm{Hom}_R(X_k, M \oplus N) = 0$. Consider now any short exact sequence $0 \to X_k \to A \to M \to 0$ for some left R-module A. Since the d-vector (t, s) of X_k has $t > s$, the d-vector (t', s') of A has $t' > s'$ as well. Therefore, some of the indecomposable direct summands of A, say A_1, has a d-vector (t'', s'') such that $t'' > s''$. Thus A_1 is preinjective, and hence $\mathrm{Hom}_R(A_1, M) = 0$. It follows that A_1 is isomorphic to a direct summand of X_k, but this entails that $X_k \cong A_1$ and the sequence splits. This shows that $\mathrm{Ext}^1_R(M, X_k) = 0$. Finally, we have seen in the proof of Proposition 3.9 that this implies $\mathrm{Ext}^1_R(M \oplus N, X_k) = 0$, so that X_k is torsion.

Next we show that if Y is a finitely presented indecomposable torsion module and $Y \not\cong N$, then Y is M-generated. To see this, suppose that $Y \in \mathcal{T}$. We know that the equivalence H carries the torsion module N to the simple projective left S-module $H(N)$, while $H(M)$ is the non-simple projective. Since $H(Y) \not\cong H(N)$, there exists an epimorphism of left S-modules $H(M)^k \to H(Y)$, and the equivalence justifies the existence of the corresponding non-zero homomorphism $f : M^k \to Y$. If this is not an epimorphism, then there exists a non-zero epimorphism $g : Y \to C$ such that the composition $fg = 0$. Since C is torsion and non-zero, $H(C)$

is non-zero, and there exists a non-zero homomorphism $H(Y) \to H(C)$ which annihilates the epimorphism $H(M)^k \to H(Y)$, which is impossible. Consequently, $M^k \to Y$ is an epimorphism.

Now we claim that every finitely presented indecomposable non-preinjective torsion module Z is either isomorphic to M or to N. If $Z \ncong N$, there is an epimorphism $M^k \to Z$, as we have just seen. This gives a short exact sequence $0 \to K \to M^k \to Z \to 0$. Let (t, s) be the d-vector of Z, so that $t \leq s$. Let us first assume that $t < s$. Then K has an indecomposable direct summand with a d-vector (t', s') such that $t' > s'$, and this means that K has a preinjective direct summand. But K is isomorphic to a submodule of M^k and preinjective modules have no non-zero homomorphisms to M, which gives a contradiction.

It remains to consider the case when the d-vector of Z is (t, t) for some integer $t \geq 1$. Observe first that the non-simple indecomposable projective left R-module has d-vector $(1, 2)$ because $2 = \mathrm{l.dim}(B)$ and hence M is not projective. According to Lemma 3.5 and [15, Proposition 2.5], the d-vector of the indecomposable module $D(\mathrm{Tr}(M))$ equals $(\delta\tau - 1, \delta\tau + \delta - 1)$, where $\tau \geq 1$ and δ is the left dimension of $B' = \mathrm{Hom}_R(N, M)$, which is left finite-dimensional by the hypothesis that B is left acceptable. By [15, Proposition 3.9], the d-vector of the left S-module $H(Z)$ is of the form

$$\big(t(\tau + 1) - t\tau, t\delta\tau + t\delta - t - t\delta\tau + t\big) = (t, t\delta).$$

Since δ is the left dimension of B' and $S \cong R_{B'}$, the linear map defining $H(Z)$ would have to be an isomorphism and then the non-simple projective left S-module would be isomorphic to a direct summand of $H(Z)$, hence to $H(Z)$, so that $t = 1$ and $Z \cong M$, because B is left pre-acceptable. This completes the proof of the claim.

Let $\mathcal{S}_\omega$ be the set of all modules in R-ind which are not preinjective, and let $Y \in \mathcal{S}_\omega$. By the claim, $\mathrm{Hom}_R(M, Y) = 0$ whenever $Y \ncong M$ and $Hom_R(N, Y) = 0$ whenever $Y \ncong M$ and $Y \ncong N$. It then follows that in the set of all modules of R-ind that are not preinjective and not isomorphic to M, a module N has $\mathrm{Hom}_R(N, Y) = 0$ whenever $Y \ncong N$. Also, if Y is torsionfree, then $\mathrm{Ext}^1_S(H'(Y), H(N)) \neq 0$, because $H(N)$ is the simple projective left S-module and $H'(Y)$ is not projective. By [15, Lemma 2.2], we have $\mathrm{Hom}_R(Y, N) \neq 0$ and the monomorphism $N \to M^{\tau+1}$ of the proof of Proposition 3.9 shows also that $\mathrm{Hom}_R(Y, M) \neq 0$. This means that by taking $X_\omega = M$ and $X_{\omega+1} = N$, we may extend the sequence above to a sequence whose existence is claimed in item (i) of the statement. This proves (i) and (ii).

Finally, under the hypothesis that B is left strictly sporadic, $d'_{-1} = 2$. Let us consider the d-vectors (t'_1, s'_1) to (t'_3, s'_3) of the sequence of indecomposable left R-modules $X_1, X_\omega, X_{\omega+1}$. These are, respectively, $(2, 1), (1, 1), (\tau, \tau + 1)$.

Then one checks directly that the formulas (iii) of Definition 2.5 hold for these values, and hence the sequence of the d'_i is a partial dimension sequence. ∎

LEMMA 4.3. *Let B be a left strictly sporadic and left acceptable bimodule. Then B^* is left strictly sporadic and left acceptable, the acceptance bimodules of B, B^* are (semilinearly) isomorphic, and if τ is the characteristic value of B, then $\tau + 1$ is the characteristic value of B^*.*

Proof. It is obvious that B^* is left strictly sporadic. To see that it is left acceptable, let $w_1, w_2 \in B^*$. Each of these maps can be viewed in a natural way as a G-linear map $B \otimes_F F \to G$. Thus each defines a left R_B-module M_1, resp. M_2 whose d-vector is $(1, 1)$. Since we may assume that both elements are non-zero, M_1, M_2 are indecomposable, and by Lemma 3.1 there is an isomorphism $M_1 \to M_2$ because B is left pre-acceptable. The isomorphism yields the commutative diagram

$$\begin{array}{ccc} B \otimes_F F & \xrightarrow{w_1} & G \\ {\scriptstyle 1\otimes m_2}\downarrow & & \downarrow{\scriptstyle m_1} \\ B \otimes_F F & \xrightarrow{w_2} & G \end{array}$$

where m_1 is given by right multiplication by some $g \in G$, while m_2 is right multiplication by some element $f \in F$. Referring to the structure of B^* as an F-G-bimodule, the commutative diagram above shows that $w_1 g = f w_2$, and hence $w_1 = f w_2 g^{-1}$. This shows that $w_2 \in B^*$ satisfies condition (i) in Lemma 3.1, whence B^* is left pre-acceptable.

Choose now a left G-basis $\mathcal{B} = \{b_1, b_2\}$ of B, and let $b^* \in B^*$ be such that $b^*(b_1) = 1$ and $b^*(b_2) = 0$. Using this basis, we consider the inclusion $F \subseteq \mathbb{M}_2(G)$ as in the second paragraph after Lemma 3.1. Then B^* admits a representation $B^* = G^2$, where the right G-structure of B^* is natural and the left F-structure is the restriction of the matrix product of elements of $\mathbb{M}_2(G)$ and elements of G^2.

Like this, b^* is viewed as an element in G^2, the element $(1, 0)$. Then Fb^* consists of all elements of G^2 which are left columns of some element of $F \subseteq \mathbb{M}_2(G)$. Likewise b^*G contains all elements of G^2 of the form $(g, 0)$ for $g \in G$. Let (g_1, g_2) be any element of G^2. Since B is left acceptable, there exists an element of $F \subseteq \mathbb{M}_2(G)$ whose first column is (x, g_2) by Theorem 3.10 and Lemma 3.8. Thus $(x, g_2) \in Fb^*$. Hence, $(g_1, g_2) = (x, g_2) - (g_1 - x, 0) \in Fb^* + b^*G$, and this shows that B^* is left weakly acceptable.

Let now $M \oplus N$ be the rigid tilting left R-module determined by the left acceptable bimodule B, and let $R' = R_{B^*} = \left[\begin{smallmatrix} G & 0 \\ B^* & F \end{smallmatrix}\right]$. The reflection functor (see [23, Lemma 3.1]) H' yields an equivalence between the category of finitely presented left R_B-modules which have no simple injective direct summand and the category of finitely presented left R'-modules which have no simple projective direct summand. If X is a finitely presented indecomposable left R_B-module with d-vector $(t, s) \neq (1, 0)$ and (t', s') is the d-vector of $H'(X)$, then we know from [26, Lemma 2.3] that

$$t' = s, \qquad s' = 2s - t.$$

Thus $H'(M)$ has d-vector $(1, 1)$ and $H'(N)$ has d-vector $(\tau + 1, \tau + 2)$. It follows that $H'(M)$ is the only indecomposable left R'-module (up to isomorphism) with d-vector $(1, 1)$.

On the other hand, if E_1 is the non-simple injective left R-module, then $H'(E_1)$ is the simple injective left R'-module, and hence the characteristic value of B^* is $\mathrm{l.dim}(\mathrm{Hom}_{R'}(H'(M), H'(E_1))) - 1$. By the equivalence H', this equals $\mathrm{l.dim}(\mathrm{Hom}_R(M, E_1)) - 1$. By Proposition 3.13 and Remark 3.14, the characteristic value of B^* is $\tau + 1$. Moreover, there is an epimorphism $M^{\tau+2} \to E_1$ whose kernel is isomorphic to N, in view of [15, Proposition 3.4]. Since H' preserves cokernels, there is an exact sequence of R'-Mod, $H'(N) \to H'(M)^{\tau+2} \to H'(E_1) \to 0$. The values of the d-vectors of these three modules show that the first homomorphism $H'(N) \to H'(M)^{\tau+2}$ is a monomorphism. Thus $H'(N)$ is isomorphic to an indecomposable maximal submodule of $H'(M)^{\tau+2}$, and it follows by the characterization in Theorem 3.10 that $H'(M) \oplus H'(N)$ is the tilting module determined by the left weakly acceptable bimodule B^*. This proves that B^* is left acceptable and its acceptance bimodule $\mathrm{Hom}_{R'}(H'(N), H'(M))$ is (semilinearly) isomorphic to the acceptance bimodule of B. ■

Theorem 4.2 may be extended to a more general setting. To this end, we need the following definitions.

Definition 4.4. Let F, G be division rings and B a G-F-bimodule that has the left finite dimension property. We say that B is a (*left*) *sporadic bimodule* if:

(i) There is $n \geq 0$ such that the bimodule B^{*n} is left strictly sporadic.
(ii) $\mathrm{l.dim}(B^{*i}) > 1$ for $i = 0, \dots, n - 1$.

When B is left sporadic, then there is a smallest $m \geq 0$ such that B^{*m} is left strictly sporadic. The bimodule B^{*m} will be called the *initial bimodule* of the left sporadic bimodule B.

Let B be a left sporadic bimodule and B^{*m} its initial bimodule. The sequence of the dimensions $\mathrm{l.dim}(B), \mathrm{l.dim}(B^*), \dots, \mathrm{l.dim}(B^{*m}) = 2$ will be called the *fundamental sequence* of the sporadic bimodule B. When B is left strictly sporadic, its fundamental sequence has only one term. Otherwise, $\mathrm{l.dim}(B^{*(m-1)}) > 2$.

We now combine the concepts about left sporadic bimodules and the left acceptable bimodules of Section 3.

Definition 4.5. Let F, G be division rings and let B be a G-F-bimodule. We say that B is a *left acceptably sporadic bimodule* when it is left sporadic and its initial bimodule B^{*m} is left acceptable. In this case, the acceptance bimodule B' of B^{*m} is called the *left derivated bimodule* of B.

When the left derivated bimodule B' of a left acceptably sporadic bimodule B is again left acceptably sporadic, its left derivated bimodule will be called *left 2nd-derivated bimodule* $B^{(2)}$ of B, and so successively.

Theorem 4.6. *Let F, G be division rings, and let B be a G-F-bimodule which is left acceptably sporadic such that $B_0 = B^{*m}$ is its initial bimodule and $m > 0$. Let $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$. There is a well-ordered sequence*

$$X_0, \dots, X_{\omega+1}$$

*of elements of R-*ind *with the following properties:*

- (i) *For $\alpha \leq \omega + 1$, let $\mathcal{S}_\alpha = R\text{-ind} \setminus \{X_\beta \mid \beta < \alpha\}$. Then X_α is the only element of $\mathcal{S}_\alpha$ having the property that if $Y \ncong X_\alpha$ and $Y \in \mathcal{S}_\alpha$, then $\mathrm{Hom}_R(X_\alpha, Y) = 0$. Moreover, $\mathrm{Hom}_R(Y, X_\alpha) \neq 0$ for each $Y \in \mathcal{S}_\alpha$.*
- (ii) *$X_\omega \oplus X_{\omega+1}$ is a rigid tilting left R-module and the left derivated bimodule of B is isomorphic to the bimodule $\mathrm{Hom}_R(X_{\omega+1}, X_\omega)$.*
- (iii) *Set $d_{-1} = d^* = \mathrm{l.dim}(B^*)$ and $d_i = \mathrm{l.dim}(B^{*(i+2)})$ for $i = 0, 1, \dots$. If τ is the characteristic value of the left acceptable bimodule B_0, set $d'_i = d_i$ for $i = -1, 0, \dots, m-2$, and $d'_{m-1} = 1$, $d'_m = \tau + 2$. Then the sequence $d'_{-1}, d'_0, \dots, d'_m$ is a partial dimension sequence.*

Proof. Since B has the left finite dimension property and is not of finite representation type, the preinjective left R-modules can be ordered in an infinite chain

$$X_0, X_1, \dots$$

which satisfies the equations of (i) for $\alpha < \omega$, by Proposition 2.2.

By hypothesis, $B_0 = B^{*m}$ is left acceptable and left strictly sporadic, and hence $B_1 = B^{*(m+1)}$ is again left acceptable and left strictly sporadic by Lemma 4.3. By iterated application of [15, Lemma 2.7], $B_1 \cong \mathrm{Hom}_R(X_m, X_{m-1})$, and $W = X_{m-1} \oplus X_m$ is a rigid tilting left R-module.

Let $R_1 = R_{B_1}$, which is a ring of the form (1.1) isomorphic to $\mathrm{End}_R(W)$. Since B_1 is left acceptable and left strictly sporadic, Theorem 4.2 shows that there is a sequence

$$Y_0, \dots, Y_{\omega+1}$$

of elements of R-ind satisfying conditions (i)–(iii) of Theorem 4.2. Since W is a rigid tilting left R-module, there is a splitting torsion theory $(\mathcal{T}, \mathcal{F})$ in R-Mod induced by W, and an equivalence H' between $\mathcal{F}$ and a full subcategory $\mathcal{X}$ of R_1-Mod, where $(\mathcal{X}, \mathcal{Y})$ is a splitting torsion theory of R_1-Mod. Since $\mathrm{Hom}_R(W, X_j) = 0$ when $j \geq m + 1$ while $\mathrm{Hom}_R(W, X_j) \neq 0$ when

$j \leq m$, the left R-modules $X_0, \ldots, X_m$ are in $\mathcal{T}$, and all the remaining (up to isomorphism) finitely presented indecomposable left R-modules belong to $\mathcal{F}$. By the defining properties of the sequence $Y_0, Y_1, \ldots$, it follows that $k \geq 0 \Rightarrow Y_k \cong H'(X_{m+k+1})$, and hence $Y_\omega \cong H'(X_\omega)$ and $Y_{\omega+1} \cong H'(X_{\omega+1})$ for certain finitely presented indecomposable left R-modules X_ω, $X_{\omega+1}$ satisfying condition (i) of the statement.

Notice that $X_\omega, X_{\omega+1}$ have non-zero homomorphisms to any of the modules X_k $(k < \omega)$. Now, each X_k generates X_i if $i \leq k$, by [15, Proposition 2.5], and hence $X_\omega, X_{\omega+1}$ generate each of the X_k $(k < \omega)$. Also, $\mathrm{Ext}^1_R(X_\omega \oplus X_{\omega+1}, X_k) = 0$ for $k < \omega$, by [15, Lemmas 2.1, 2.2]. Hence $X_\omega \oplus X_{\omega+1}$ is a tilting left R-module. Moreover, $\mathrm{Hom}_R(X_{\omega+1}, X_\omega)$ is isomorphic to $\mathrm{Hom}_{R_1}(Y_{\omega+1}, Y_\omega)$, and thus $X_\omega \oplus X_{\omega+1}$ is a rigid tilting module and $\mathrm{Hom}_R(X_{\omega+1}, X_\omega)$ is the acceptance bimodule of B_1, by Theorem 4.2. By Lemma 4.3 it is also the acceptance bimodule of B_0, hence it is the left derivated bimodule of B, up to isomorphism. This shows (ii).

To prove (iii), we show that the d-vectors of the sequence of modules

$$X_1, \ldots, X_m, X_\omega, X_{\omega+1}$$

are the values (t'_k, s'_k) which make $d'_{-1}, d'_0, \ldots, d'_n$ a partial dimension sequence, according to Definition 2.5. We check that the equations (iii) of that definition are fulfilled with these values for the d-vectors (t'_i, s'_i) and the d'_i.

Note first that the required equations do hold for $k = -1, 0, \ldots, m-2$. This is because $d'_k = \mathrm{l.dim}(B^{*(k+2)}) = \mathrm{l.dim}(\mathrm{Hom}_{R_B}(X_{k+1}, X_k))$ for these k by [15, Lemma 2.7], and the equations follow from [15, Lemma 3.1]. So, we have only to check that

$$(t_\omega, s_\omega) = (t'_{m+1}, s'_{m+1}) = (t_{m+1}, s_{m+1}) - (t_m, s_m),$$
$$(t_{\omega+1}, s_{\omega+1}) = (t'_{m+2}, s'_{m+2}) = (\tau + 2)(t_\omega, s_\omega) - (t_{m+1}, s_{m+1}),$$

where we give in the form (t_k, s_k) the d-vector of the left R-module X_k. By considering the left acceptable left strictly sporadic bimodule B_1 and applying Theorem 4.2, we find that the d-vectors of the left R_1-modules $Y_0, Y_\omega, Y_{\omega+1}$ are, respectively, $(1, 0), (1, 1), (\tau_1, \tau_1 + 1)$ where τ_1 is the characteristic value of B_1. By Lemma 4.3, $\tau_1 = \tau + 1$. Now, we use the equations of [15, Proposition 3.9] and compare the d-vectors of X_ω and $Y_\omega = H'(X_\omega)$:

$$1 = s_\omega t_m - t_\omega s_m, \qquad 1 = s_\omega t_{m+1} - t_\omega s_{m+1},$$

and for the d-vectors of $X_{\omega+1}$ and $Y_{\omega+1} = H'(X_{\omega+1})$:

$$\tau + 1 = s_{\omega+1} t_m - t_{\omega+1} s_m, \qquad \tau + 2 = s_{\omega+1} t_{m+1} - t_{\omega+1} s_{m+1}.$$

The first pair of equations is a linear system of two equations for the unknowns t_ω, s_ω and the determinant of the system is $s_{m+1} t_m - s_m t_{m+1} = 1$

by [15, Lemma 3.8]. Thus it has a unique solution, and we just check that $(t_{m+1} - t_m, s_{m+1} - s_m)$ solves the system, which is straightforward.

The second pair is again a system with the same determinant, and thus it is enough to check that $((\tau+2)t_\omega - t_{m+1}, (\tau+2)s_\omega - s_{m+1})$ solves the system. ∎

PROPOSITION 4.7. *Let F, G be division rings and B a G-F-bimodule. If the ring $R = R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is a left pure semisimple sporadic ring, then B is left acceptably sporadic. Moreover, if B' is its derivated bimodule, then the ring $R_{B'} = \left[\begin{smallmatrix} E & 0 \\ B' & H \end{smallmatrix}\right]$ is left pure semisimple sporadic, and so B' is also left acceptably sporadic.*

Proof. By [15, Proposition 2.10], the bimodule B has the left finite dimension property. Moreover, B is left sporadic by [15, Theorem 5.1 and Lemma 2.7]. Now, let $B_0 = B^{*s}$ be the initial bimodule of the left sporadic bimodule B, and, without loss of generality, suppose it is a G-F-bimodule.

It follows from [15, Lemma 2.7] and [14, Theorem 3.2] that the ring $R_0 = R_{B_0} = \left[\begin{smallmatrix} F & 0 \\ B_0 & G \end{smallmatrix}\right]$ is left pure semisimple and left sporadic. Hence, there is only one indecomposable left R_0-module, up to isomorphism, with d-vector $(1,1)$. This entails that B_0 is left pre-acceptable, in view of Lemma 3.2.

Order the finitely presented indecomposable left R_0-modules $\{X_\alpha \mid -1 \le \alpha \le \sigma\}$ as in [14, Theorem 3.8]. Since B_0 is left strictly sporadic, each of the modules X_n (for $n < \omega$) has d-vector $(n+1, n)$ and, by [15, Theorem 3.14], the indecomposable left R_0-module M with d-vector $(1,1)$ is precisely X_ω. If $r = r(\omega) = \tau + 1$, we conclude from Proposition 2.3 that $N = X_{\omega+1}$ has d-vector $(\tau, \tau+1)$. By [14, Theorem 3.8], $M \oplus N$ is a rigid tilting module, and thus B_0 is left acceptable, by Lemma 3.8. According to Definition 4.5, B is left acceptably sporadic, as asserted.

If $W = M \oplus N$, then $B' = \mathrm{Hom}_{R_0}(N, M)$ is the left derivated bimodule of B. To prove the final sentence, note that $R_{B'} \cong \mathrm{End}_{R_0}(W)$ and this ring is again left pure semisimple (because W is tilting [14, Theorem 3.2]), and left sporadic (because of the equivalences induced by the tilting module W, and [15, Lemmas 2.1 and 2.2]). Then B' is left acceptably sporadic, by the first part of the proof. ∎

5. Sporadic rings with finitely many AR-components. Our aim in this section is to characterize the G-F-bimodules B such that R_B is left pure semisimple sporadic with a finite number of AR-components. By Proposition 4.7, if B is such a bimodule, then B is left sporadic and its initial bimodule $B_0 = B^{*m}$ is left strictly sporadic with a corresponding ring R_{B_0} which is again left pure semisimple sporadic. In view of the close relationship between the rings R_B and R_{B_0} in this case, we study this problem for left strictly sporadic bimodules, but the translation to the general situation is straightforward. Since there are no left pure semisimple sporadic rings with just two AR-components [15, Proposition 5.3], we consider from the start that the number of AR-components is $h+1$ with $h>1$.

THEOREM 5.1. *Let F, G be division rings and B a left strictly sporadic G-F-bimodule. Let $h>1$ be an integer. Then the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is a left pure semisimple sporadic ring with $h+1$ AR-components if and only if the following conditions are satisfied:*

(i) *B is left acceptably sporadic and has a sequence of left derivated bimodules $B', B^{(2)}, \dots, B^{(h)}$ which are left acceptably sporadic.*
(ii) *For each $j=0,\dots,h$, let τ_j be the characteristic value of the initial bimodule of the left sporadic bimodule $B^{(j)}$, and for $j\geq 1$, let $d_{j,1},\dots,d_{j,n_j+1}$ with $n_j \geq 0$ be the fundamental sequence of the left sporadic bimodule $B^{(j)}$. Also, let $d_{i+1,0} = \tau_i + 2$ for $i = 0,\dots,h-1$. Then*

$$2, 1, d_{1,0}, \dots, d_{1,n_1+1}, 1, d_{2,0}, \dots, d_{2,n_2+1}, 1, \dots, 1, d_{h,0}, \dots, d_{h,n_h+l}$$

is a dimension sequence (for some $l\geq 1$ and with each $d_{h,n_h+i}=2$ if $i\geq 1$).

Proof. The proof is in several steps.

STEP 1. If R_B is left pure semisimple sporadic with $h+1$ AR-components, then (i) holds. By Proposition 4.7, B is left acceptably sporadic and, since it is its own initial bimodule, it is left acceptable. Proposition 4.7 shows also that the derivated bimodule B' is left acceptably sporadic and $R_1 = R_{B'}$ is left pure semisimple sporadic. By iterating the application of Proposition 4.7, we see that all the successive left derivated bimodules $B^{(j)}$ do exist and are left acceptably sporadic, hence (i) holds.

STEP 2. Assume (i) and, moreover, assume that the ring $R := R_B$ satisfies the following property (for a certain integer $j\geq 1$): There is a sequence

$$X_0, \dots, X_\omega, \dots, X_{\omega\cdot j}, X_{\omega\cdot j+1} \tag{5.1}$$

of modules of R-ind such that the following conditions (a)–(d) hold:

(a) For $\alpha \leq \omega\cdot j+1$, let $\mathcal{S}_\alpha = R\text{-ind} \setminus \{X_\beta \mid \beta<\alpha\}$. Then X_α is the only element of $\mathcal{S}_\alpha$ having the property that if $Y\in\mathcal{S}_\alpha$ and $Y\not\cong X_\alpha$, then $\mathrm{Hom}_R(X_\alpha, Y) = 0$ (and, moreover, $\mathrm{Hom}_R(Y, X_\alpha)\neq 0$).
(b) The sequence

$$2, 1, d_{1,0}, \dots, d_{1,n_1+1}, 1, d_{2,0}, \dots, d_{2,n_2+1}, 1, \dots, 1, d_{j,0} \tag{5.2}$$

is a partial dimension sequence; and the pairs (t_i, s_i) (for $i\geq 1$) of integers that correspond to this sequence in the sense of Definition 2.5

are the d-vectors of the modules

$$X_1, X_\omega, X_{\omega+1}, \dots, X_{\omega+n_1+2}, X_{\omega\cdot 2}, \dots, X_{\omega\cdot j}, X_{\omega\cdot j+1}.$$

(c) $W = X_{\omega\cdot j} \oplus X_{\omega\cdot j+1}$ is a rigid tilting module with endomorphism ring $R_j \cong R_{B(j)}$.

(d) Let $(\mathcal{T}, \mathcal{F})$ be the splitting torsion theory determined by the tilting module W. For any $Z \in R$-ind, Z belongs to $\mathcal{T}$ if and only if $Z \cong X_\alpha$ for some $\alpha \leq \omega \cdot j + 1$.

Then: (1) if the set of isomorphism classes of indecomposable finitely presented left R-modules that belong to $\mathcal{F}$ is empty or finite, then condition (ii) of the statement holds for $h = j$; and (2) if the set of modules of R-ind that belong to $\mathcal{F}$ is infinite, then the sequence of modules (5.1) can be extended to a sequence

$$X_0, \dots, X_\omega, \dots, X_{\omega\cdot j+1}, X_{\omega\cdot j+2}, \dots, X_{\omega\cdot(j+1)}, X_{\omega\cdot(j+1)+1}$$

which again satisfies the properties (a)–(d) with $j+1$ substituted for j.

We first prove part (2). By Proposition 2.2 and assumptions (c) and (d), we obtain an extended sequence of modules of R-ind, by adjoining to the sequence (5.1) an infinite subsequence of modules of $\mathcal{F}$, which we denote thus

$$X_{\omega\cdot j+2}, \dots, X_{\omega\cdot j+i}, \dots,$$

such that the whole sequence still satisfies condition (a).

Since $B^{(j)}$ is left acceptably sporadic by (i), we may apply Theorem 4.6 (or Theorem 4.2 in case $B^{(j)}$ is left strictly sporadic) and obtain a sequence of indecomposable finitely presented left R_j-modules

$$Y_0, \dots, Y_{\omega+1} \tag{5.3}$$

satisfying the statement in that theorem. Let $(\mathcal{X}, \mathcal{Y})$ be the splitting torsion theory of R_j-Mod provided by the tilting module W, and let H, H' be the equivalence functors $H : \mathcal{T} \to \mathcal{Y}$ and $H' : \mathcal{F} \to \mathcal{X}$. Since all $X_{\omega\cdot j+i+2}$ belong to $\mathcal{F}$, we see that the $H'(X_{\omega\cdot j+i+2})$ are torsion left R_j-modules, and therefore they satisfy the defining properties for the sequence of the Y_i, hence $Y_i \cong H'(X_{\omega\cdot j+i+2})$ for $i < \omega$. Since the indecomposable projective left R-modules belong to $\mathcal{F}$ and none of the Y_i (for $i < \omega$) is isomorphic to $H'(P)$ for a projective P, it follows that $Y_\omega, Y_{\omega+1} \in \mathcal{X}$, by condition (i) of Theorem 4.6 for the chain of the Y_α. Hence, there exist indecomposable left R-modules in $\mathcal{F}$, which we call $X_{\omega\cdot(j+1)}$, $X_{\omega\cdot(j+1)+1}$, such that $H'(X_{\omega\cdot(j+1)+i}) \cong Y_{\omega+i}$ for $i = 0, 1$. Thus if we consider the sequence of indecomposable finitely presented left R-modules

$$X_{\omega\cdot j+2}, \dots, X_{\omega\cdot j+i}, \dots, X_{\omega\cdot(j+1)}, X_{\omega\cdot(j+1)+1},$$

then the sequence obtained by adjoining this one to the previous sequence (5.1), being the translation by H' of the above sequence (5.3) of R_j-modules, still satisfies condition (a).

To prove (b), we have to add the subsequence

$$d_{j,1}, \dots, d_{j,n_j+1}, 1, d_{j+1,0}$$

to the sequence (5.2), and show that this gives a partial dimension sequence for the d-vectors of the modules in assumption (b) followed by the modules

$$X_{\omega\cdot j+2}, \dots, X_{\omega\cdot j+n_j+2}, X_{\omega\cdot(j+1)}, X_{\omega\cdot(j+1)+1}.$$

Note first that, as seen in the first paragraph of the proof of [15, Proposition 2.11], $X_{\omega\cdot j+2}$ is isomorphic to $D(\mathrm{Tr}(X_{\omega\cdot j}))$. On the other hand, $d_{j,1}$ is the left dimension of $\mathrm{Hom}_R(X_{\omega\cdot j+1}, X_{\omega\cdot j})$ by assumption (c). Then [15, Proposition 2.5] shows that the d-vector of $X_{\omega\cdot j+2}$ is obtained from $d_{j,1}$ and the d-vectors of the two modules preceding it in the sequence above.

Then Theorem 4.6 implies that

$$d_{j,2}, \dots, d_{j,n_j+1}, 1, d_{j+1,0}$$

is also a partial dimension sequence corresponding through the equations of Definition 2.5 to the d-vectors of the left R_j-modules

$$Y_1, \dots, Y_{n_j}, Y_\omega, Y_{\omega+1},$$

i.e.,

$$H'(X_{\omega\cdot j+3}), \dots, H'(X_{\omega\cdot j+n_j+2}), H'(X_{\omega\cdot(j+1)}), H'(X_{\omega\cdot(j+1)+1}).$$

The linear relationship between the d-vectors of $X_{\omega\cdot j+\alpha}$ and the corresponding Y_β, given in [15, Proposition 3.9], entails that the equations of Definition 2.5 which are satisfied for the d's with the d-vectors of the Y_β's, still hold for the same d's and the d-vectors of the $X_{\omega\cdot j+\alpha}$. This proves the inductive step for (b).

To show (c), let $W' = X_{\omega\cdot(j+1)} \oplus X_{\omega\cdot(j+1)+1}$. By the construction of the sequence, we know that $\mathrm{Hom}_R(W', X_\alpha) \neq 0$ for each $\alpha \leq \omega\cdot(j+1)+1$. For the same ordinals α, $\mathrm{Ext}^1_R(W', X_\alpha) = 0$, again by the above construction. Indeed, if $\alpha > \omega\cdot j+1$, then $X_\alpha \in \mathcal{F}$ by assumption (d), and hence $\mathrm{Ext}^1_R(W', X_\alpha) \cong \mathrm{Ext}^1_{R_j}(Y_\omega \oplus Y_{\omega+1}, Y_k) = 0$. Moreover, if $\alpha \leq \omega\cdot j+1$, then $X_\alpha \in \mathcal{T}$ and $W' \in \mathcal{F}$, hence $\mathrm{Ext}^1_R(W', X_\alpha) = 0$. On the other hand, if Z is a finitely presented indecomposable left R-module which is not isomorphic to any of the X_α (for $\alpha \leq \omega\cdot(j+1)+1$), then $\mathrm{Hom}_R(W', Z) = 0$. A standard argument shows now that W' generates all the modules X_α for $\alpha \leq \omega\cdot(j+1)+1$, and this is precisely the subset of R-ind generated by W'. Finally, since $\mathrm{Ext}^1_{R_j}(Y_\omega \oplus Y_{\omega+1}, Z) \neq 0$ if $Z \in R$-ind is not isomorphic to any of the modules $Y_0, Y_1, \dots,$ we see that $\mathrm{Ext}^1_R(W', Z) \neq 0$ if Z is not any of the modules X_α (for $\alpha \leq \omega\cdot(j+1)+1$). It follows that W' is also a tilting module; and it is a rigid tilting module, since $Y_\omega \oplus Y_{\omega+1}$ is a rigid tilting left R_j-module. This proves (c).

Finally, the splitting torsion theory of R-Mod determined by W' has the property that a module $X \in R$-ind belongs to the torsion class if and only if it is W'-generated, hence if and only if it is isomorphic to X_α for $\alpha \leq \omega \cdot (j+1) + 1$. This shows (d) and completes part (2) of Step 2.

Consider now part (1). First, if $\mathcal{F}$ has no finitely presented left R-modules, then $X_{\omega\cdot j}, X_{\omega\cdot j+1}$ are the indecomposable projective left R-modules. Thus $B^{(j)} \cong B$ and $n_j = 0$. By assumptions (a), (b), the sequence obtained by adding $d_{j,1} = 2$ and the pair $(-1, 0)$ to the series of dimensions and d-vectors of (b) is now a dimension sequence, and (ii) holds. In case $\mathcal{F}$ has only one element $X_{\omega\cdot j+2}$ in R-ind, assumption (a) holds for the extended sequence and $X_{\omega\cdot j+1}, X_{\omega\cdot j+2}$ are the indecomposable projective left R-modules. Thus $(B^{(j)})^* \cong B$ and $n_j \leq 1$. By (a), (b), the sequence obtained by adding $d_{j,1}, d_{j,2} = 2$ and the pairs $(0, 1), (-1, 0)$ to the series of dimensions and d-vectors of (b) is now a dimension sequence, and (ii) holds.

Suppose now that $\mathcal{F}$ contains at least two non-isomorphic finitely presented indecomposable left R-modules. We follow the arguments of part (2). By Proposition 2.2 and assumptions (c) and (d), we obtain an extended sequence of modules of R-ind, by adjoining to the sequence (5.1) the sequence

$$X_{\omega\cdot j+2}, \ldots, X_{\omega\cdot j+r}$$

for some $r > 2$, and $X_{\omega\cdot j+r-1}, X_{\omega\cdot j+r}$ are the projective indecomposable modules. The whole sequence satisfies condition (a).

On the other hand, we apply Theorem 4.6 for $B^{(j)}$ and obtain a sequence of indecomposable finitely presented left R_j-modules

$$Y_0, \ldots, Y_{\omega+1}$$

that satisfies the statement of that theorem. Let H, H' be the equivalence functors $H : \mathcal{T} \to \mathcal{Y}$ and $H' : \mathcal{F} \to \mathcal{X}$ provided by the tilting module W. Again by comparing both sequences we deduce that $Y_i \cong H'(X_{\omega\cdot j+i+2})$ for $i < r-1$. In particular, $\mathrm{Hom}_{R_j}(Y_{r-2}, Y_{r-3}) \cong \mathrm{Hom}_R(X_{\omega\cdot j+r}, X_{\omega\cdot j+r-1})$, which in turn is isomorphic to B. Thus $B \cong (B^{(j)})^{*(r-1)}$. Since B is left strictly sporadic, we see that the initial bimodule of $B^{(j)}$ is $(B^{(j)})^{*s}$ with $s < r$, that is, $n_j < r$ and we may set $r = n_j + l$ with $l \geq 1$. Also, $d_{j,n_j+1} = \cdots = d_{j,n_j+l} = 2$.

As in the first part of the proof of Step 2, the sequence

$$2, 1, d_{1,0}, \ldots, d_{1,n_1+1}, 1, d_{2,0}, \ldots, d_{2,n_2+1}, 1, \ldots, 1, d_{j,0}, \ldots, d_{j,n_j+l}$$

is a partial dimension sequence; and the pairs (t, s) of integers that correspond to this sequence in the sense of Definition 2.5 are the d-vectors of the modules

$$X_1, X_\omega, X_{\omega+1}, \dots, X_{\omega+n_1+2}, X_{\omega\cdot 2}, \dots, X_{\omega\cdot j}, \dots, X_{\omega\cdot j+n_j+l}$$

plus the pair $(-1,0)$. Since the two final pairs are $(0,1)$ and $(-1,0)$, this is a dimension sequence, and hence (ii) holds.

STEP 3. If property (i) of the statement holds, then the ring $R = R_B$ satisfies the property of Step 2 for $j = 1$. This follows from Theorem 4.2.

STEP 4. If property (i) of the statement holds, then one and only one of the following possibilities occurs: either conditions (a)–(d) hold for any positive integer j, or else property (ii) holds for some index h, and then R is left pure semisimple. Indeed, by Steps 2 and 3, just one of the following two cases occurs: either for each $j \geq 1$ there is a sequence of finitely presented indecomposable left R-modules as in (a), or else there is $j = h \geq 2$ such that condition (ii) holds for this value of j. The resulting sequence of finitely presented indecomposable left R-modules has elements $X_{\omega\cdot j+r-1}, X_{\omega\cdot j+r}$ which are projective, hence the sequence contains all finitely presented indecomposable left R-modules. In the first case, conditions (a)–(d) hold for any positive integer j; in the second case, all the finitely presented indecomposable left R-modules can be ordered by ordinals as X_α and $\alpha < \beta$ implies $\operatorname{Hom}_R(X_\alpha, X_\beta) = 0$. This entails that there are no infinite sequences of non-isomorphic non-zero homomorphisms between finitely presented indecomposable left R-modules

$$M_1 \to M_2 \to \cdots ,$$

and then R is left pure semisimple by [20, Theorem 1.3]. In view of the uniqueness of the sequence (5.1), these two possibilities are incompatible.

STEP 5. Suppose that B is left strictly sporadic and $R := R_B$ is left pure semisimple with $h+1$ AR-components. By Step 1, (i) holds. Moreover, we know from [14, Theorem 3.8] that the finitely presented indecomposable left R-modules may be well-ordered as

$$M_0, \dots, M_\omega, \dots, M_{\omega\cdot h}, M_{\omega\cdot h+1}, \dots, M_{\omega\cdot h+r}$$

with the property of this ordering as in condition (a) of Step 2. This forces the ordering of the M_α to coincide with the ordering of the X_α of condition (a). Consequently, the second possibility in Step 4 occurs and (ii) holds.

Conversely, assume that (i) and (ii) hold. By (i), Steps 2 and 3 show that there exist partial dimension sequences and sequences of finitely presented indecomposable left R-modules as in (b), for successive values of the index j. For $j = h$, the sequence obtained is a dimension sequence, and thus the module $X_{\omega\cdot h+n_h+l}$ is the simple projective. As above, this shows that the first possibility of Step 4 does not occur, and consequently R is left pure semisimple with $h+1$ AR-components. ∎

6. A relation with Simson's first potential counterexample. We discuss in this section some similarities and differences between the first potential counterexample to the pure semisimplicity conjecture given by Simson [24] and sporadic rings. First we remark that Simson's potential counterexample is not sporadic. However, it is related to sporadic rings; in fact, it belongs to a class of rings that was called almost sporadic in [15]. It turns out that we can specialize our results to give another presentation of this potential counterexample.

PROPOSITION 6.1. *Let $G \subsetneq F$ be a pair of division rings such that F, viewed as a G-F-bimodule, is left strictly sporadic. Then the ring*

$$R = R_F = \begin{bmatrix} F & 0 \\ F & G \end{bmatrix}$$

is a counterexample to the pssC.

Proof. Set $B := F$ and view it as a G-F-bimodule. It is clear that B satisfies the condition in Lemma 3.1. Following the notation of Lemma 3.2 and taking a G-basis $\mathcal{B} = \{f_0, 1\}$ of B, the subring E of F is the division ring G and, by hypothesis, F has left dimension 2 over $E = G$. It follows that F is left pre-acceptable (see Definition 3.3). Since condition (i) of Lemma 3.8 is also obviously satisfied, F is left weakly acceptable.

Let now $M \oplus N$ be the tilting module obtained in Proposition 3.9. Since the left dimension of F over $E = G$ is 2, the d-vector of N is $(1, 2)$. But the only finitely presented indecomposable left R-module with d-vector $(1, 2)$ is the non-simple projective P_1. This way, it is easy to see that $\mathrm{End}_R(N) \cong F$ and, with the natural left F-structure, $\mathrm{Hom}_R(N, M) \cong F$. Therefore $\mathrm{Hom}_R(N, M)$ has left dimension 1, and hence B is left acceptable.

We now apply Theorem 4.2 to get a sequence of finitely presented indecomposable left R-modules

$$X_0, \ldots, X_\omega = M, X_{\omega+1} = N$$

containing all the finitely presented indecomposable modules, except the simple projective P_0. Therefore, all the finitely presented indecomposable left R-modules are well-ordered in that form. Hence R is left pure semisimple. Since there are an infinite number of isomorphism classes of finitely presented indecomposable modules, R is not of finite representation type. ∎

With the notation of Section 3, the bimodule $B' = \mathrm{Hom}_R(N, M)$ is isomorphic to F and so is the endomorphism ring $H = \mathrm{End}_R(N)$. Hence the bimodule B' is an F-E-bimodule, with $E = G$, as we saw in the proof of Proposition 6.1. We obtain the following noteworthy consequence.

COROLLARY 6.2. *Let $G \subsetneq F$ be a pair of division rings such that F is, as a G-F-bimodule, left strictly sporadic. Then the right dimension of F over G is infinite.*

Proof. Following the notation above, the rigid tilting module $M \oplus N$ is the key module for the left pure semisimple ring R (see [12]). By [12, Theorem 3.6], the module $M \oplus N$ is not endofinite and, in view of [14, Proposition 3.7(b)], M is not endofinite. Therefore $B' = F$ is right infinite-dimensional over the endomorphism ring E of M. Since $E = G$, the corollary follows. ■

Simson's potential counterexample belongs to a wider class of possible counterexamples to the conjecture. In fact, it is easy to see that Proposition 6.1 may be generalized as follows.

PROPOSITION 6.3. *Let $G \subseteq F$ be a pair of division rings such that the left G-dimension of F is* 2. *Let R be the ring*

$$R = R_F = \begin{bmatrix} F & 0 \\ F & G \end{bmatrix},$$

and suppose that as a G-F-bimodule, $B := F$ has the left finite dimension property. Let $d, d^*, d_0, d_1, \ldots$ be the sequence of the left dimensions of $B, B^*, B^{**}, \ldots$, and consider the sequence of the convergents $[d^*, d_0, \ldots, d_k]$ as defined in [15, Section 3]. *If the sequence is defined for any k, and* 1 *is its limit, then the ring R is left pure semisimple and is a counterexample to the pure semisimplicity conjecture.*

Proof. As in the proof of Proposition 6.1, one shows that F, as a G-F-bimodule, is left acceptable with characteristic value $\tau = 1$. Then the proof of Theorem 4.2 shows that the preinjective left R-modules form a chain

$$X_0, X_1, \ldots$$

such that $\mathrm{Hom}_R(X_n, X_m) = 0$ if $n < m$. Also from the proof of Proposition 6.1 it follows that if $M \oplus N$ is the rigid tilting module determined by the left acceptable bimodule F, then the torsion class $\mathcal{T}$ defined by this tilting module includes all preinjective modules X_n along with M, N, while the remaining finitely presented indecomposable left R-modules are torsionfree.

If the ring R is of finite representation type, then the sequence of the dimensions $d^*, d_0, d_1, \ldots$ repeats cyclically [9, Proposition 1] and the sequence of the convergents $[d^*, d_0, \ldots, d_k]$ is not defined for all k, which contradicts the hypothesis that its limit is 1. Thus R is not of finite representation type. Then, according to [15, Proposition 3.11], the finitely presented indecomposable left R-modules with d-vector (t, s) such that $t > s$ are precisely the preinjective modules; and again as in the proof of Proposition 6.1 we construct the well-ordered sequence

$$X_0, \ldots, X_{\omega+1}$$

where $X_\omega = M$, $X_{\omega+1} = N$, and $\mathrm{Hom}_R(X_\omega, Y) = 0 = \mathrm{Hom}_R(X_{\omega+1}, Y)$ for any $Y \not\cong X_\alpha$ ($\alpha \leq \omega + 1$). Since the d-vector of X_ω is $(1,1)$ by construction, and since $\tau = 1$, the d-vector of $X_{\omega+1}$ is $(1,2)$, which means that $X_{\omega+1}$ is the non-simple projective indecomposable left R-module. By the foregoing observations, each of the remaining non-preinjective finitely presented indecomposable modules is isomorphic to the simple projective P_0, and hence the above list completed by the module P_0 contains all the elements of R-ind. Since R-ind has a well-ordering that satisfies the same property of the sequence in Theorem 4.2, the ring R is left pure semisimple. ∎

REMARK 6.4. The class of potential counterexamples presented in [26, Theorem 4.16] which correspond to infinite dimension sequences v of the first kind (in the sense of [26, Definition 4.4]) are essentially included in the class of rings of Proposition 6.3 as long as the operations ξ_m for constructing the sequence v from the principal infinite dimension sequence $\boldsymbol{w} = (\ldots, 2, 2, \ldots, 2, 1, \infty)$ do not change the last two terms $2, 1$ in $\boldsymbol{w}$. In that case, the limit of the convergents is equal to 1 and Proposition 6.3 applies.

Acknowledgements. The author acknowledges support from the Spanish Ministry of Education (MTM-2010-20940-C02-02). He is also grateful to Prof. D. Simson and to the referee for many helpful comments and suggestions on a previous version of this paper.

REFERENCES

[1] L. Angeleri Hügel, *A key module over pure-semisimple hereditary rings*, J. Algebra 307 (2007), 361–376.

[2] L. Angeleri Hügel and D. Herbera, *Auslander–Reiten components over pure-semisimple hereditary rings*, J. Algebra 331 (2011), 285–303.

[3] L. Angeleri Hügel and H. Valenta, *A duality result for almost split sequences*, Colloq. Math. 80 (1999), 267–292.

[4] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, Vol. 1, London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge, 2006.

[5] M. Auslander, *Representation theory of artin algebras II*, Comm. Algebra 1 (1974), 269–310.

[6] M. Auslander, *Large modules over artin algebras*, in: Algebra, Topology and Category Theory, Academic Press, New York, 1976, 3–17.

[7] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Univ. Press, Cambridge, 1995.

[8] R. R. Colby and K. R. Fuller, *Tilting, cotilting and serially tilted rings*, Comm. Algebra 18 (1990), 1585–1615.

[9] P. Dowbor, C. M. Ringel and D. Simson, *Hereditary artinian rings of finite representation type*, in: Representation Theory II, Lecture Notes in Math. 832, Springer, Berlin, 1980, 232–241.

[10] P. Dowbor and D. Simson, *A characterization of hereditary rings of finite representation type*, Bull. Amer. Math. Soc. 2 (1980), 300–302.

[11] N. V. Dung and J. L. García, *Preinjective modules over pure semisimple rings*, J. Pure Appl. Algebra 212 (2008), 1207–1221.

[12] N. V. Dung and J. L. García, *On the endofiniteness of a key module over left pure semisimple rings*, Proc. Amer. Math. Soc. 138 (2010), 2269–2278.

[13] N. V. Dung and J. L. García, *Indecomposable modules over pure semisimple hereditary rings*, J. Algebra 371 (2012), 577–595.

[14] N. V. Dung and J. L. García, *Splitting torsion pairs over pure semisimple rings*, J. Pure Appl. Algebra (online).

[15] J. L. García, *On hereditary artinian rings and the pure semisimplicity conjecture: rigid tilting modules and a weak conjecture*, Colloq. Math. 135 (2014), 227–262.

[16] L. Gruson, *Simple coherent functors*, in: Representations of Algebras, Lecture Notes in Math. 488, Springer, Berlin, 1975, 156–159.

[17] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 272 (1982), 399–443.

[18] I. Herzog, *A test for finite representation type*, J. Pure Appl. Algebra 95 (1994), 151–182.

[19] D. Simson, *Pure semisimple categories and rings of finite representation type*, J. Algebra 48 (1977), 290–296; Corrigendum, ibid. 67 (1980), 254–256.

[20] D. Simson, *On pure semisimple Grothendieck categories II*, Fund. Math. 110 (1980), 107–116.

[21] D. Simson, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra 71 (1981), 195–218.

[22] D. Simson, *Indecomposable modules over one-sided serial local rings and right pure semisimple rings*, Tsukuba J. Math. 7 (1983), 87–103.

[23] D. Simson, *On right pure semisimple hereditary rings and an Artin problem*, J. Pure Appl. Algebra 104 (1995), 313–332.

[24] D. Simson, *An Artin problem for division ring extensions and the pure semisimplicity conjecture I*, Arch. Math. (Basel) 66 (1996), 114–122.

[25] D. Simson, *On the representation theory of artinian rings and Artin's problems on division rings extensions*, Bull. Greek Math. Soc. 42 (1999), 97–112.

[26] D. Simson, *An Artin problem for division ring extensions and the pure semisimplicity conjecture, II*, J. Algebra 227 (2000), 670–705.

[27] H. Valenta, *Existence criteria and construction methods for certain classes of tilting modules*, Comm. Algebra 22 (1994), 6047–6071.

José L. García
Department of Mathematics
University of Murcia
30100 Murcia, Spain
E-mail: jlgarcia@um.es

Received 1 July 2014;
revised 17 October 2014 (6302)