COLLOQUIUM MATHEMATICUM

VOL. 139 2015 NO. 1

A HYPERSURFACE DEFECT RELATION FOR A FAMILY OF MEROMORPHIC MAPS ON A GENERALIZED p-PARABOLIC MANIFOLD

BY

QI HAN (Worcester, MA)

Abstract. This paper establishes a hypersurface defect relation, that is, $\sum_{j=1}^{q}\delta(D_j,f) \le (n+1)/d$, for a family of meromorphic maps from a generalized p-parabolic manifold M to the projective space $\mathbb{P}^n$, under some weak non-degeneracy assumptions.

1. Introduction. Let $f:\mathbb{C}^m\to\mathbb{P}^n$ be a non-constant meromorphic map, and let $D_1,\dots,D_q$ be q ($\ge n+1$) hypersurfaces of degree d (≥ 1) in $\mathbb{P}^n$ such that $f(\mathbb{C}^m)\nsubseteq D_j$ for each $j=1,\dots,q$. In 1972, Carlson and Griffiths [4] showed that

$$\sum_{j=1}^{q}\delta(D_j,f)\le\frac{n+1}{d}\tag{1.1}$$

when $m\ge n=\operatorname{rank} f$ and $D_1,\dots,D_q$ have normal crossings. This result was extended by Griffiths and King [8] in 1973, and then by Shiffman [13] in 1975. When f is assumed to be algebraically non-degenerate, it is a conjecture that (1.1) is still true without the restriction $m\ge n$ (see Griffiths [7] and Shiffman [14]).

When $d=1$, (1.1) is classical, first studied by R. Nevanlinna and then furthered by Cartan, Ahlfors, the Weyls', Stoll, Vitter, Wong and Ru, etc. However, when $d>1$, it is extremely difficult to prove this inequality, which still remains open. We refer the reader to [9, 11, 12, 17, 18, 20, 21] and the references therein for more details.

When we use the weaker condition of being "in general position" on the hypersurfaces and assume f is non-constant, then $2n$ is the best possible upper bound for (1.1) by Shiffman [15] or Eremenko and Sodin [6]. When f is algebraically non-degenerate, then $n+1$ is a nice upper bound to (1.1), independent of the degree, by Ru [11, 12].

Employing a concept of weak non-degeneracy of degree d, Biancofiore [2, 3] proved that (1.1) holds for a class of meromorphic maps that are

---

2010 *Mathematics Subject Classification*: Primary 32H30; Secondary 32Q15, 32H25.

Key words and phrases: meromorphic maps, defect relation, generalized p-parabolic manifolds, non-degeneracy.

DOI: 10.4064/cm139-1-5

"projections of maximal linear deficiency". Moreover, he provided examples to show that his results are sharp.

The purpose of this paper is two-fold: we further weaken the assumptions in [2, 3] and then extend all these results to certain generalized p-parabolic manifolds. The assumptions and notation will be detailed later as appropriate.

We remark that the essential ideas used here are due to Biancofiore [2, 3].

Write $y = (y_0, y_1, \dots, y_n) \in \mathbb{C}^{n+1}$, and let $\mathsf{V}_d^n := \operatorname{span}\{y_{i_1} y_{i_2}^{d-1}\}_{i_1, i_2 = 0}^n$ be a linear subspace of $\mathbb{C}_{(d)}^{n+1}$, the space of all homogeneous polynomials of degree d (≥ 1) in $\mathbb{C}[y_0, y_1, \dots, y_n]$. Denote by $\mathcal{D}$ the collection of all hypersurfaces generated by elements in V_d^n in $\mathbb{P}^n$. In addition, let $\{\epsilon_0, \epsilon_1, \dots, \epsilon_n\}$ be the standard basis of $\mathbb{C}^{n+1}$.

Our main result may be simply formulated as follows.

Theorem 1.1. *Let M be either an affine algebraic variety, or an algebraic vector bundle over an affine algebraic variety, or its projectivization, let $f : M \to \mathbb{P}^n$ be a linearly non-degenerate, transcendental meromorphic map such that $f(M) \not\subseteq D$ for every hypersurface D in $\mathcal{D}$, and let $D_1, \dots, D_q$ be q $(\geq n+1)$ hypersurfaces of degree d (≥ 1) in $\mathbb{P}^n$ having normal crossings at $\epsilon_0, \epsilon_1, \dots, \epsilon_n$. Then* (1.1) *holds provided the $n+1$ coordinate hyperplanes $H_i := \mathbf{P}(y_i^{-1}(0))$ in $\mathbb{P}^n$ are such that*

$$\sum_{i=0}^{n} N_f^{(1)}(H_i; r, s) = o(T_f(r, s)).$$

Note that the assumption on non-degeneracy of f is weaker than non-degeneracy of degree d, and the one on linear deficiency of H_i is weaker than $\sum_{i=0}^{n} N_f(H_i; r, s) = o(T_f(r, s))$ (see [2, 3]); as a matter of fact, the latter condition is satisfied by the example provided in [3] to show that the weak non-degeneracy condition of degree d is sharp.

The interested reader may also consult Aihara and Mori [1], or Hu and Yang [10].

1.1. Generalized manifolds.

Originally, the notion of parabolic manifold (see Stoll [17, 18]) has an affine algebraic variety as a prototype, and the concept of *parabolicity* is based on the existence of some non-negative plurisubharmonic exhaustion τ (≥ 0), defined on a Kähler manifold (M, ω), such that $\phi := \log \tau$ satisfies the complex Monge–Ampère equation

$$(dd^c\phi)^m \equiv 0 \tag{1.2}$$

on $M \setminus \{\tau = 0\}$, with $m := \dim M$, subject to $(dd^c\phi)^{m-1} \not\equiv 0$.

Instead, the concept of *p-parabolicity* depends on the existence of a non-negative plurisubharmonic exhaustion τ defined again on (M, ω) such that ϕ satisfies a generalized complex Monge–Ampère equation, i.e., for some integer $p \in (1, m]$,

$$(dd^c\phi)^p \wedge \omega^{m-p} \equiv 0 \tag{1.3}$$

on $M \setminus \{\tau = 0\}$, yet $(dd^c\phi)^{p-1} \wedge \omega^{m-p} \not\equiv 0$. Note that m-parabolicity is just parabolicity. One thing of interest is that (see [21, Theorem 2.10]), for a parabolic Stein manifold M of dimension m having a strictly positive plurisubharmonic exhaustion τ_M (> 0), any holomorphic vector bundle E of rank $r \geq 2$ over M, its dual vector bundle E^*, as well as the associated projectivizations $\mathbf{P}(E)$ and $\mathbf{P}(E^*)$ over M are not parabolic but they do satisfy identities analogous to (1.3); for example, for $\mathbf{P}(E)$, we have

$$(dd^c\phi)^{m-1} \wedge \omega^{r-1} \not\equiv 0 \quad \text{and} \quad (dd^c\phi)^m \wedge \omega^{r-1} \equiv 0, \tag{1.4}$$

where ϕ is the pull-back of $\phi_M := \log \tau_M$ on M and ω is some Kähler metric on $\mathbf{P}(E)$ (see [21, Lemma 2.9]). We refer the interested reader to Chandler and Wong [5, 19] and the references therein for more details on this subject.

In view of this, we follow the Wongs' [21] in giving the following definition.

Definition 1.2. Given $p \in (1, m]$, a Kähler (complex) manifold (M, ω) of dimension m is said to be a *generalized p-parabolic manifold* when there exists a plurisubharmonic function ϕ such that

(A1) $\{\phi = -\infty\}$ is a closed subset of M of strictly lower dimension;
(A2) ϕ is smooth on the open dense set $M \setminus \{\phi = -\infty\}$, with $dd^c\phi \geq 0$, such that

$$(dd^c\phi)^{p-1} \wedge \omega^{m-p} \not\equiv 0 \quad \text{and} \quad (dd^c\phi)^p \wedge \omega^{m-p} \equiv 0. \tag{1.5}$$

We remark here that when M is the projectivization of an algebraic vector bundle E over an affine algebraic variety or its dual bundle E^*, we shall assume $\operatorname{rank}(E) \geq 2$ to guarantee the existence of a non-trivial Kähler metric on M.

1.2. Nevanlinna theory. We write, for $d^c := \frac{i}{4\pi}(\bar\partial - \partial)$,

$$\tau := e^{\phi} \quad \text{and} \quad \sigma := d^c\phi \wedge (dd^c\phi)^{p-1} \wedge \omega^{m-p}, \tag{1.6}$$

with τ (≥ 0) called a *p-parabolic exhaustion* on M, and we have

$$(dd^c\tau)^j = \tau^j\{(dd^c\phi)^j + j d\phi \wedge d^c\phi \wedge (dd^c\phi)^{j-1}\} \tag{1.7}$$

for $j = 1, \ldots, p$, such that

$$(dd^c\tau)^p \wedge \omega^{m-p} \not\equiv 0 \quad \text{and} \quad d\sigma = (dd^c\phi)^p \wedge \omega^{m-p} \equiv 0. \tag{1.8}$$

Naturally, set $\Omega := (dd^c\tau)^p \wedge \omega^{m-p}$ to be the volume form on M. Also, for any $r > 0$, write $M[r] := \{x \in M : \tau(x) \leq r^2\}$ and $M\langle r\rangle := \{x \in M : \tau(x) = r^2\}$.

Let $\nu : M \to \mathbb{Z}^+$ be a smooth divisor, with $D_\nu := \operatorname{supp}\nu$. Non-trivially, $\dim(D_\nu) = m-1$ and the singular set Σ_{D_ν} of D_ν is analytic with $\dim(\Sigma_{D_\nu}) \leq m-2$.

Given any integer $k \geq 1$, denote by

$$\nu^{(k)} := \min\{k, \nu\} : M \to [0, k] \tag{1.9}$$

the kth *truncated divisor* associated with ν. Then, for $1 \leq k_1 \leq k_2$, one has

$$\nu^{(k_1)} \leq \nu^{(k_2)} \leq \frac{k_2}{k_1}\nu^{(k_1)}. \tag{1.10}$$

Fix $s > 0$. When $r > s$, the *counting functions* of ν and ν_k are defined as

$$N(\nu; r, s) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]\cap D_\nu} \nu(dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{1.11}$$

$$N^{(k)}(\nu; r, s) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]\cap D_\nu} \nu^{(k)}(dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{1.12}$$

respectively. Clearly, $0 \leq N^{(1)}(\nu; r, s) \leq N^{(k)}(\nu; r, s) \leq N(\nu; r, s)$.

Next, let $f : M \to \mathbb{P}^n$ be a meromorphic map defined on M, and let $\mathfrak{f} : M \to \mathbb{C}^{n+1}$ be a reduced representation associated with f. That is, $\mathfrak{f}$ is a holomorphic vector function on $M \setminus \mathfrak{f}^{-1}(0)$, with $\dim(\mathfrak{f}^{-1}(0)) \leq m-2$, such that $\mathbf{P} \circ \mathfrak{f} = f$ on $M \setminus \mathfrak{f}^{-1}(0)$. The map f is said to be *non-degenerate of degree* d provided that, for any hypersurface D of degree d in $\mathbb{P}^n$, $f(M) \not\subseteq D$, and f is said to be *linearly non-degenerate* if $d = 1$. When f is non-degenerate of degree d for all $d \geq 1$, then f is said to be *algebraically non-degenerate.*

Finally, let $\omega_{\rm FS}$ be the Fubini–Study metric on $\mathbb{P}^n$, and let D be a hypersurface of degree d in $\mathbb{P}^n$. The *characteristic function* of f and the *counting* and the kth *truncated counting functions* of f with respect to D are defined as, respectively, for all $r > s$,

$$T_f(r, s) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]} f^*(\omega_{\rm FS}) \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{1.13}$$

$$N_f(D; r, s) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]\cap D_{\nu_{f,D}}} \nu_{f,D}(dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{1.14}$$

$$N_f^{(k)}(D; r, s) := \int_s^r \frac{dt}{t^{2p-1}} \int_{M[t]\cap D_{\nu_{f,D}}} \nu_{f,D}^{(k)}(dd^c\tau)^{p-1} \wedge \omega^{m-p}. \tag{1.15}$$

Here, on any local holomorphic coordinate chart (z, U_z) of M, one has $f^*(\omega_{\rm FS})|_{U_z} := dd^c \log[\sum_{i=0}^n \mathtt{f}_i^2]$, with $\mathfrak{f} = (\mathtt{f}_0, \mathtt{f}_1, \ldots, \mathtt{f}_n)$ being a reduced representation of f, $\nu_{f,D}|_{U_z} := dd^c \log[\alpha \circ \mathfrak{f}]|_{U_z}$ the divisor generated by

$D \circ f$ through the *Poincaré–Lelong formula*, with $D := \mathbf{P}(\alpha^{-1}(0))$ being generated by $\alpha \in \mathbb{C}^{n+1}_{(d)}$, and $\nu^{(k)}_{f,D} := \min\{k, \nu_{f,D}\}$.

One has the following *first main theorem* on M.

First Main Theorem. *Let $f : M \to \mathbb{P}^n$ be a non-constant meromorphic map on a generalized p-parabolic manifold M, and let D be a hypersurface of degree d in $\mathbb{P}^n$ with $f(M) \not\subseteq D$. Then, for $r > s > 0$,*

$$d\,T_f(r,s) \geq N_f(D; r, s) + O(1). \tag{1.16}$$

Accordingly, the *defect* of f with respect to D is defined to be

$$\delta(D,f) := 1 - \limsup_{r\to\infty} \frac{N_f(D;r,s)}{d\,T_f(r,s)}. \tag{1.17}$$

Remark. Henceforth, we shall assume that M is either an affine algebraic variety, or an algebraic vector bundle over an affine algebraic variety, or its dual bundle or their projectivizations, and keep in mind the remark after (1.5).

We have the following *second main theorem* on M.

Second Main Theorem. *Let $f : M \to \mathbb{P}^n$ be a linearly non-degenerate meromorphic map on M, and let $H_1, \dots, H_q$ be q $(\geq n+1)$ hyperplanes in $\mathbb{P}^n$ in general position. Then, for $r > s > 0$,*

$$(q-n-1)T_f(r,s) \leq \sum_{j=1}^{q} N^{(n)}_f(H_j; r, s) + O(\log(rT_f(r,s))). \tag{1.18}$$

Proof. The first proof for this result of high dimensional value distribution theory was given by Smiley [16] in the curve case, and her method can be extended to meromorphic maps on Stein manifolds, as detailed in Stoll [18, Section 13]. ∎

For more details on the preceding subject, see the Wongs' [21] or Han [9].

2. Normal crossings. Recall we write $y = (y_0, y_1, \dots, y_n) \in \mathbb{C}^{n+1}$. Let $\mathbb{C}^{n+1}_{(d)}$ be the space of all homogeneous polynomials of degree d (≥ 1) in $\mathbb{C}[y_0, y_1, \dots, y_n]$. Then any member in $\mathbb{C}^{n+1}_{(d)}$ is of the form $\sum_{I \in \mathsf{K}^n_d} a_I y^I$. Here, $I = (i_0, i_1, \dots, i_n) \in (\mathbb{Z}^+)^{n+1}$, K^n_d is the family of all I's satisfying $|I| := i_0 + i_1 + \cdots + i_n = d$, and $y^I = y_0^{i_0} y_1^{i_1} \cdots y_n^{i_n}$. Note that the *Veronese embedding theorem* implies that $\dim(\mathbb{C}^{n+1}_{(d)}) = n_d + 1 := \binom{n+d}{d}$.

Let $D_1, \dots, D_q$ be q hypersurfaces of degree d (≥ 1) in $\mathbb{P}^n$, and let $\alpha_1, \dots, \alpha_q$ be elements in $\mathbb{C}^{n+1}_{(d)}$ that generate them. For $\mathbf{y} = \mathbf{P}(y) \in \mathbb{P}^n$, write

$$\iota = \iota_{\mathbf{y}} := \#\{j \in [1,q] : \alpha_j(y) = 0\}. \tag{2.1}$$

Then $0 \le \iota \le q$. Call $\iota = \iota_{\mathbf{y}}$ the *crossings number* of $D_1, \ldots, D_q$ at $\mathbf{y}$. When $\iota \ge 1$, there is a unique injective map $\kappa = \kappa_{\mathbf{y}} : [1, \iota] \to [1, q]$ with $\alpha_{\kappa(j)}(y) = 0$ for each $j \in [1, \iota]$. Call $\kappa = \kappa_{\mathbf{y}}$ the *crossings selector* of $D_1, \ldots, D_q$ at $\mathbf{y}$, and

$$\mathfrak{J}(\mathbf{y}) = \mathfrak{J}[D_1, \ldots, D_q; \mathbf{y}] := [d\alpha_{\kappa(1)} \wedge \cdots \wedge d\alpha_{\kappa(\iota)}](y) \tag{2.2}$$

the *crossings Jacobian* of $D_1, \ldots, D_q$ at $\mathbf{y}$.

One says that $D_1, \ldots, D_q$ have *normal crossings at* $\mathbf{y} \in \mathbb{P}^n$ if $\iota_{\mathbf{y}} \ge 1$ and $\mathfrak{J}(\mathbf{y}) \ne 0$, and $D_1, \ldots, D_q$ have *normal crossings* if they have normal crossings at each $\mathbf{y} \in \bigcup_{j=1}^{q} (\operatorname{supp} D_j) \setminus \{0\}$. If $D_1, \ldots, D_q$ have normal crossings at $\mathbf{y}$, then $\iota_{\mathbf{y}} \le n$; if $D_1, \ldots, D_q$ have normal crossings, then they are all smooth. We say that $D_1, \ldots, D_q$ are *in general position* whenever $\bigcap_{i=0}^{n} (\operatorname{supp} D_{j_i}) = \{0\}$ for each subset $\{j_0, j_1, \ldots, j_n\}$ of distinct elements of $\{1, \ldots, q\}$. Hence, having normal crossings is stronger than being in general position. In particular, when $d = 1$, these notions coincide.

We say a hypersurface D in $\mathbb{P}^n$ is *generated* by an $\alpha \in \mathbb{C}^{n+1}_{(d)}$ if D is associated with $\mathbf{P}(\alpha^{-1}(0))$. By convention, we set $\operatorname{supp} D := \alpha^{-1}(0) \subseteq \mathbb{C}^{n+1}$. Thus, $f(M) \not\subseteq D$ if and only if $f(M) \cap \mathbf{P}(\mathbb{C}^{n+1} \setminus \operatorname{supp} D) \ne \emptyset$, i.e., $\alpha \circ f \not\equiv 0$.

Fix $N \ge n$ and a surjective linear map $\varphi : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$. Let D be a hypersurface of degree d in $\mathbb{P}^n$, generated by $\alpha \in \mathbb{C}^{n+1}_{(d)}$ (denoted by $D[\alpha]$), and let $\tilde{D}$ be the hypersurface of degree d in $\mathbb{P}^N$, generated by $\beta := \alpha \circ \varphi \in \mathbb{C}^{N+1}_{(d)}$ (denoted by $\tilde{D}[\beta]$).

For $w = (w_0, w_1, \ldots, w_N) \in \mathbb{C}^{N+1}$, write the members in $\mathbb{C}^{N+1}_{(d)}$ as $\sum_{L \in \mathsf{K}^N_d} b_L w^L$, with K^N_d the set of all $L = (l_0, l_1, \ldots, l_N) \in (\mathbb{Z}^+)^{N+1}$ satisfying $|L| = d$, and $w^L = w_0^{l_0} w_1^{l_1} \cdots w_N^{l_N}$. Write $\mathsf{B} := [0, N] \times [0, N]$. When $d \ge 3$, denote by B_d the subset of $L \in \mathsf{K}^N_d$ with $l_t \ge d - 1$ for some $t \in [0, N]$. Then define $\gamma_d : \mathsf{B} \to \mathsf{B}_d$ by $\gamma_d(h, k)(t) = l_t = 0$ if $t \ne h \ne k$, $\gamma_d(t, k)(t) = l_t = 1$ if $t \ne k$, $\gamma_d(h, t)(t) = l_t = d - 1$ if $t \ne h$, and $\gamma_d(t, t)(t) = l_t = d$ for all $t \in [0, N]$. If $d = 2$, then $\mathsf{B}_2 = \mathsf{K}^N_2$ and we define γ_2 to be the identity map.

Take $\beta = \sum_{L \in \mathsf{K}^N_d} b_L w^L \in \mathbb{C}^{N+1}_{(d)}$. Write $b_{hk} := b_{\gamma_d(h,\,k)}$ and set

$$\mathfrak{y}(\beta) := \sum_{L \in \mathsf{B}_d} b_L w^L = \sum_{h,\,k=0}^{N} b_{hk} w_h w_k^{d-1}. \tag{2.3}$$

When $d = 2$, we have $\mathfrak{y}(\beta) \equiv \beta$ and $b_{hk} = b_L$; moreover, we assume all the homogeneous polynomials of degree 2 are symmetric. For each $k = 0, 1, \ldots, N$, set

$$\mathfrak{y}^k(\beta) := \sum_{h=0}^{N} b_{hk} w_h \quad \text{so that} \quad \mathfrak{y}(\beta) = \sum_{k=0}^{N} \mathfrak{y}^k(\beta) w_k^{d-1}. \tag{2.4}$$

Next, let $\mathscr{L}_\varphi$ be the class of all injective maps $\mu : [0, n] \to [0, N]$ such that $\{\varphi(e_{\mu(0)}), \varphi(e_{\mu(1)}), \ldots, \varphi(e_{\mu(n)})\}$ is a basis of $\mathbb{C}^{n+1}$. Since φ is surjective,

$\mathscr{L}_\varphi \neq \emptyset$. Here, $\{e_0, e_1, \dots, e_N\}$ denotes the standard basis of $\mathbb{C}^{N+1}$. Fix $s \in [0, N]$ and define $\mathscr{L}_\varphi^s$ to be the set of all maps $\lambda : [0, s] \times [0, n] \to [0, N]$ such that

(B1) $\lambda_\imath := \lambda(\imath, \Box) \in \mathscr{L}_\varphi$ for each $\imath = 0, 1, \dots, s$;
(B2) $\lambda^o := \lambda(\Box, 0) : [0, s] \to [0, N]$ is injective.

Clearly, given $\imath \in [0, s]$, $\lambda_\imath(\jmath) = \lambda^o(\imath)$ if and only if $\jmath = 0$, i.e., $\lambda_\imath(0) = \lambda^o(\imath)$.

Define $\tilde{\lambda} : [0, s] \times [0, n] \to \mathsf{B}$ by $\tilde{\lambda}(\imath, \jmath) := (\lambda_\imath(\jmath), \lambda^o(\imath)) = (\lambda_\imath(\jmath), \lambda_\imath(0))$, and let T_λ be the image of $\tilde{\lambda}$ in B for each $\lambda \in \mathscr{L}_\varphi^s$. Then, for all $\beta \in \mathbb{C}_{(d)}^{N+1}$, write

$$\mathfrak{y}_\lambda(\beta) := \sum_{(h,k) \in T_\lambda} b_{hk} w_h w_k^{d-1} = \sum_{\imath=0}^{s} \sum_{\jmath=0}^{n} b_{\lambda_\imath(\jmath)\lambda_\imath(0)} w_{\lambda_\imath(\jmath)} w_{\lambda_\imath(0)}^{d-1}, \tag{2.5}$$

associated with $\mathfrak{y}(\beta)$ and λ, and, for each $\imath = 0, 1, \dots, s$, write

$$\mathfrak{y}_\lambda^{\lambda_\imath(0)}(\beta) := \sum_{\jmath=0}^{n} b_{\lambda_\imath(\jmath)\lambda_\imath(0)} w_{\lambda_\imath(\jmath)} \quad \text{so that} \quad \mathfrak{y}_\lambda(\beta) = \sum_{\imath=0}^{s} \mathfrak{y}_\lambda^{\lambda_\imath(0)}(\beta) w_{\lambda_\imath(0)}^{d-1}. \tag{2.6}$$

Given $\imath \in [0, s]$, one has $\beta(e_{\lambda_\imath(0)}) = \mathfrak{y}(\beta)(e_{\lambda_\imath(0)}) = \mathfrak{y}^{\lambda_\imath(0)}(\beta)(e_{\lambda_\imath(0)}) = \mathfrak{y}_\lambda(\beta)(e_{\lambda_\imath(0)}) = \mathfrak{y}_\lambda^{\lambda_\imath(0)}(\beta)(e_{\lambda_\imath(0)}) = b_{\lambda_\imath(0)\lambda_\imath(0)}$. Also, when $b_{\lambda_\imath(0)\lambda_\imath(0)} = 0$, it follows that

$$d\mathfrak{y}_\lambda(\beta)(e_{\lambda_\imath(0)}) = \begin{cases} 2d\mathfrak{y}_\lambda^{\lambda_\imath(0)}(\beta)(e_{\lambda_\imath(0)}), & d = 2, \\ d\mathfrak{y}_\lambda^{\lambda_\imath(0)}(\beta)(e_{\lambda_\imath(0)}), & d > 2. \end{cases} \tag{2.7}$$

By condition (B1), $\{\varphi(e_{\lambda_\imath(0)}), \varphi(e_{\lambda_\imath(1)}), \dots, \varphi(e_{\lambda_\imath(n)})\}$ is a basis of $\mathbb{C}^{n+1}$. Let

$$\begin{bmatrix} \varphi_{00} & \varphi_{01} & \cdots & \varphi_{0N} \\ \varphi_{10} & \varphi_{11} & \cdots & \varphi_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n0} & \varphi_{n1} & \cdots & \varphi_{nN} \end{bmatrix}_{(n+1)\times(N+1)}$$

be the matrix representation of φ in terms of the standard bases of $\mathbb{C}^{N+1}$ and $\mathbb{C}^{n+1}$. Then the linear map $\varphi_{\lambda_\imath} : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$ with matrix representation

$$\begin{bmatrix} 0 & \cdots & \varphi_{0\lambda_\imath(0)} & \cdots & 0 & \cdots & \varphi_{0\lambda_\imath(1)} & \cdots & 0 & \cdots & \varphi_{0\lambda_\imath(n)} & \cdots & 0 \\ 0 & \cdots & \varphi_{1\lambda_\imath(0)} & \cdots & 0 & \cdots & \varphi_{1\lambda_\imath(1)} & \cdots & 0 & \cdots & \varphi_{1\lambda_\imath(n)} & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \varphi_{n\lambda_\imath(0)} & \cdots & 0 & \cdots & \varphi_{n\lambda_\imath(1)} & \cdots & 0 & \cdots & \varphi_{n\lambda_\imath(n)} & \cdots & 0 \end{bmatrix}_{(n+1)\times(N+1)},$$

associated with φ and λ, is also surjective. Notice that $\varphi_{\lambda_\imath}$ is the composition of an elementary map $\mathfrak{e}_{\lambda_\imath}$ with φ, i.e., $\varphi_{\lambda_\imath} = \varphi \circ \mathfrak{e}_{\lambda_\imath}$. When $\beta = \alpha \circ \varphi \in \mathbb{C}_{(d)}^{N+1}$

for an $\alpha \in \mathbb{C}^{n+1}_{(d)}$, set $\beta^{\lambda_\imath} := \alpha \circ \varphi_{\lambda_\imath} = \beta \circ \mathfrak{e}_{\lambda_\imath}$. Then, for all $\imath \in [0, s]$,

$$\mathfrak{y}_\lambda^{\lambda_\imath(0)}(\beta) \equiv \mathfrak{y}^{\lambda_\imath(0)}(\beta^{\lambda_\imath}). \tag{2.8}$$

As $e_{\lambda_\imath(0)} \in \mathbb{C}^{N+1} \setminus \ker\varphi$ for all $\imath \in [0, s]$, we can prove the following result.

LEMMA 2.1. *$D_1, \ldots, D_q$ have normal crossings at $\mathbf{y}_\imath = \mathbf{P}(\varphi(e_{\lambda_\imath(0)})) \in \mathbb{P}^n$ if and only if $\tilde{D}_1^{\lambda_\imath}, \ldots, \tilde{D}_q^{\lambda_\imath}$ have normal crossings at $\mathbf{w}_\imath = \mathbf{P}(e_{\lambda_\imath(0)}) \in \mathbb{P}^N$. Moreover, they have the same crossings number ι and the same crossings selector κ.*

Proof. Since $D_\jmath = D[\alpha_\jmath]$, $\tilde{D}_\jmath^{\lambda_\imath} = \tilde{D}[\beta_\jmath^{\lambda_\imath}]$ for $\beta_\jmath^{\lambda_\imath} := \alpha_\jmath \circ \varphi_{\lambda_\imath}$, and $\varphi(e_{\lambda_\imath(0)}) = \varphi_{\lambda_\imath}(e_{\lambda_\imath(0)})$, the crossings number ι and the crossings selector κ are the same. In addition,

$$\mathfrak{J}[\tilde{D}_1^{\lambda_\imath}, \ldots, \tilde{D}_q^{\lambda_\imath}; \mathbf{w}] = \varphi_{\lambda_\imath}^*\{\mathfrak{J}[D_1, \ldots, D_q; \mathbf{y}]\} \tag{2.9}$$

follows from [2, Lemma 3.2]. As $\varphi_{\lambda_\imath}$ is surjective, $\varphi_{\lambda_\imath}^*$ is injective. So, $\mathfrak{J}[\tilde{D}_1^{\lambda_\imath}, \ldots, \tilde{D}_q^{\lambda_\imath}; \mathbf{w}] \neq 0$ if and only if $\mathfrak{J}[D_1, \ldots, D_q; \mathbf{y}] \neq 0$. ■

Thus, (2.7), (2.8) and [2, Lemma 3.3]—applied to $\beta^{\lambda_\imath}$, $\mathfrak{y}(\beta^{\lambda_\imath})$ and $\mathfrak{y}^{\lambda_\imath(0)}(\beta^{\lambda_\imath})$—imply that, for $\tilde{D}^\lambda = \tilde{D}[\mathfrak{y}_\lambda(\beta)]$, $\tilde{R}^{\lambda_\imath} = \tilde{D}[\mathfrak{y}(\beta^{\lambda_\imath})]$ and $\tilde{R}^{\lambda_\imath(0)} = \tilde{D}[\mathfrak{y}^{\lambda_\imath(0)}(\beta^{\lambda_\imath})]$, one has

COROLLARY 2.2. *If $\{D_\jmath\}_{\jmath=1}^q$ have normal crossings at $\mathbf{y}_\imath = \mathbf{P}(\varphi(e_{\lambda_\imath(0)})) \in \mathbb{P}^n$, then so do $\{\tilde{D}_\jmath^{\lambda_\imath}\}_{\jmath=1}^q$, $\{\tilde{D}_\jmath^{\lambda}\}_{\jmath=1}^q$, $\{\tilde{R}_\jmath^{\lambda_\imath}\}_{\jmath=1}^q$ and $\{\tilde{R}_\jmath^{\lambda_\imath(0)}\}_{\jmath=1}^q$ at $\mathbf{w}_\imath = \mathbf{P}(e_{\lambda_\imath(0)}) \in \mathbb{P}^N$. Moreover, they have the same crossings number ι and the same crossings selector κ.*

Proof. This can be proved using the same discussion as in [2, Corollary 3.4]. ■

Henceforth, we assume, without loss of generality, that $D_1, \ldots, D_q$ have normal crossings at each $\mathbf{y}_\imath = \mathbf{P}(\varphi(e_{\lambda_\imath(0)})) \in \mathbb{P}^n$ for $\imath = 0, 1, \ldots, s$.

$\lambda \in \mathscr{L}_\varphi^s$ is said to be *effective* for φ provided there is a $\mu \in \mathscr{L}_\varphi$ such that, for every $\imath \in [0, s]$, there is a permutation $\mathfrak{p}_\imath$ of $\{0, 1, \ldots, n\}$ such that $\lambda(\imath, \jmath) = \mu(\mathfrak{p}_\imath(\jmath))$ for all $\jmath \in [0, n]$. Then λ is said to be *generated* by μ, or μ is a *generator* of λ.

Here, $s \leq n$ follows from condition (B2) and the fact that $\lambda([0, s] \times \{0\}) \subseteq \mu([0, n])$. Therefore, $\{\varphi(e_{\lambda^o(\imath)})\}_{\imath=0}^s$ is a linearly independent subset of $\{\varphi(e_{\mu(\jmath)})\}_{\jmath=0}^n$. This in turn determines which $\lambda \in \mathscr{L}_\varphi^s$ can be effective for φ. Replace condition (B2) by

(B3) For some $s \leq n$, $\lambda^o = \lambda(\square, 0) : [0, s] \to [0, N]$ can be extended to an element in $\mathscr{L}_\varphi$.

One finds that, in a sense, this is an optimal requirement for effectiveness.

Lemma 2.3. *Let* $\varphi : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$ *be a surjective linear map, and let* $\lambda^o : [0, s] \to [0, N]$ *be injective, with* $\{\varphi(e_{\lambda^o(\imath)})\}_{\imath=0}^{s} (s \leq n)$ *linearly independent in* $\mathbb{C}^{n+1}$. *Then there exists a* $\lambda \in \mathscr{L}_{\varphi}^{s}$ *effective for* φ *such that* $\lambda^o(\imath) = \lambda(\imath, 0)$ *for each* $\imath = 0, 1, \ldots, s$.

Proof. This is similar to [3, Lemma 3.2]. Yet, it seems to me that Professor Biancofiore might not have realized the above observation on the necessary condition for effectiveness, and thereby, his original proof was not quite compatible with what he really needed later. Therefore, we shall detail a different (and somewhat easier) proof below.

First, *a priori*, in order to derive $\lambda^o(\imath) = \lambda(\imath, 0)$, by definition, $\lambda^o(\imath) = \mu(\mathfrak{p}_\imath(0))$ must hold for every $\imath \in [0, s]$. To choose $s + 1$ permutations $\mathfrak{p}_\imath$ of $\{0, 1, \ldots, n\}$ for $\imath \in [0, s]$, $\{\mathfrak{p}_0(0), \mathfrak{p}_1(0), \ldots, \mathfrak{p}_s(0)\} \subseteq \mu([0, n])$ should be pairwise distinct by assumption.

Without loss of generality, suppose $s = n$. (Otherwise, we can always extend λ^o to an element in $\mathscr{L}_\varphi$.) By hypothesis, $\mu := \lambda^o$ is in $\mathscr{L}_\varphi$. Given $\imath \in [0, n]$, let $\mathfrak{p}_\imath$ be the permutation of $\{0, 1, \ldots, n\}$ that switches $\{0, \imath\}$ and fixes the other elements in $\{0, 1, \ldots, n\} \setminus \{0, \imath\}$ when $\imath \neq 0$, while set $\mathfrak{p}_0$ to be the identity map for $\{0, 1, \ldots, n\}$. Define

$$\lambda(\imath, \jmath) := \mu(\mathfrak{p}_\imath(\jmath)) \in \mathscr{L}_{\varphi}^{s} \tag{2.10}$$

for every $(\imath, \jmath) \in [0, n] \times [0, n]$. In matrix form,

$$[\lambda(\imath, \jmath)]_{(n+1)\times(n+1)}$$
$$= \begin{bmatrix} \lambda^o(0) & \lambda^o(1) & \lambda^o(2) & \cdots & \lambda^o(n-1) & \lambda^o(n) \\ \lambda^o(1) & \lambda^o(0) & \lambda^o(2) & \cdots & \lambda^o(n-1) & \lambda^o(n) \\ \lambda^o(2) & \lambda^o(1) & \lambda^o(0) & \cdots & \lambda^o(n-1) & \lambda^o(n) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda^o(n-1) & \lambda^o(1) & \lambda^o(2) & \cdots & \lambda^o(0) & \lambda^o(n) \\ \lambda^o(n) & \lambda^o(1) & \lambda^o(2) & \cdots & \lambda^o(n-1) & \lambda^o(0) \end{bmatrix}_{(n+1)\times(n+1)}.$$

It can be easily verified that λ is generated by μ, i.e., λ^o, and is effective for φ, so that $\lambda^o(\imath) = \lambda(\imath, 0)$ for each $\imath = 0, 1, \ldots, n$, as claimed. ∎

We remark here that, in the notation of [3, Lemma 3.2], $\omega_\imath$ should be a permutation of $\{\tau(0), \tau(1), \ldots, \tau(n)\}$, not of $\{0, 1, \ldots, N\}$, so that $\omega_\imath \circ \tau$ is again τ.

Henceforth, without loss of generality, let us take $s \leq n$ and denote by $\mathscr{L}_{\varphi}^{s,n}$ the subclass of $\mathscr{L}_{\varphi}^{s}$ of all maps such that conditions (B1) and (B3) are satisfied.

Finally, define $\eta_0 : \mathbb{C} \to \{0, 1\}$ by $\eta_0(b) = 1$ if $b \neq 0$ while $\eta_0(0) = 0$. When $\beta = \sum_{L \in \mathsf{K}_d^N} b_L w^L \in \mathbb{C}_{(d)}^{N+1}$, write $\eta_d(\beta) := \sum_{L \in \mathsf{K}_d^N} \eta_0(b_L) \left|w^L\right|$. Then we have

Proposition 2.4. *Assume that* $\varphi : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$ *is linear and surjective, and* $\lambda \in \mathscr{L}_{\varphi}^{s,n}$ *is effective for* φ *with a generator* $\mu \in \mathscr{L}_{\varphi}$. *If* $D_1, \dots, D_q$ *have normal crossings at* $\{\mathbf{y}_\imath = \mathbf{P}(\varphi(e_{\lambda_\imath(0)}))\}_{\imath=0}^{s}$, *then there exists a constant* $C_\lambda > 0$ *such that*

$$\prod_{j=1}^{q} \eta_d(\mathfrak{y}_\lambda(\beta_j)) \geq C_\lambda \|w_{\lambda_\imath}\|^{qd-n-1} \prod_{\jmath=0}^{n} |w_{\mu(\jmath)}|. \tag{2.11}$$

Here, $w_{\lambda_\imath} := (w_{\lambda_0(0)}, w_{\lambda_1(0)}, \dots, w_{\lambda_s(0)})$ *and* $\|w_{\lambda_\imath}\| = \sqrt{\sum_{\imath=0}^{s} |w_{\lambda_\imath(0)}|^2}$.

Proof. This can be proved by using the same discussions as in [2, Proposition 3.5]. For completeness, we sketch a proof below.

Similar to estimate (3.3) in [2], from (2.6) and (2.8), one finds that

$$\begin{aligned} \prod_{j=1}^{q} \eta_d(\mathfrak{y}_\lambda(\beta_j)) &= \prod_{j=1}^{q} \sum_{\imath=0}^{s} \eta_1(\mathfrak{y}^{\lambda_\imath(0)}(\beta_j^{\lambda_\imath})) |w_{\lambda_\imath(0)}|^{d-1} \\ &\geq \sum_{\imath=0}^{s} \Big\{ \prod_{j=1}^{q} \eta_1(\mathfrak{y}^{\lambda_\imath(0)}(\beta_j^{\lambda_\imath})) \Big\} |w_{\lambda_\imath(0)}|^{qd-q}. \end{aligned} \tag{2.12}$$

Given $\imath \in [0, s]$, considering the surjective linear map $\varphi_{\lambda_\imath} : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$ and taking $\mathbf{y}_\imath = \mathbf{P}(\varphi(e_{\lambda_\imath(0)}))$, just like estimate (3.4) in [2], Claim 1 in [2] says that

$$\prod_{j=1}^{q} \eta_1(\mathfrak{y}^{\lambda_\imath(0)}(\beta_j^{\lambda_\imath})) \geq |w_{\varepsilon_\imath(1)}|\,|w_{\varepsilon_\imath(2)}| \cdots |w_{\varepsilon_\imath(\iota_{\mathbf{y}_\imath})}|\,|w_{\lambda_\imath(0)}|^{q-\iota_{\mathbf{y}_\imath}}. \tag{2.13}$$

Here, $\iota_{\mathbf{y}_\imath}$ $(\leq n)$ is the crossings number of $\tilde{D}_1^\lambda, \dots, \tilde{D}_q^\lambda$ at $\mathbf{y}_\imath$ in view of Corollary 2.2, and $\varepsilon_\imath : [1, \iota_{\mathbf{y}_\imath}] \to \{\lambda_\imath(1), \dots, \lambda_\imath(n)\}$ is an injective map.

Finally, in view of the assumption that $\lambda \in \mathscr{L}_{\varphi}^{s,n}$ is generated by $\mu \in \mathscr{L}_{\varphi}$, together with the conclusion of Lemma 2.3, (2.12) and (2.13) may be combined to yield

$$\prod_{j=1}^{q} \eta_d\big(\mathfrak{y}_\lambda(\beta_j)\big) \geq \sum_{\imath=0}^{s} |w_{\mu(0)}|\,|w_{\mu(1)}| \cdots |w_{\mu(n)}|\,|w_{\lambda_\imath(0)}|^{qd-n-1}, \tag{2.14}$$

which along with Claim 2 in [2] gives (2.11). ■

3. Defect relation. In this section, we follow Section 4 of [2] and Sections 2, 4 and 5 of [3] to obtain a defect relation, under a slightly weaker hypothesis.

Definition 3.1. Suppose that $f : M \to \mathbb{P}^n$ is a transcendental meromorphic map. Then we write $f \in \mathscr{D}$ provided that there is a meromorphic map $g : M \to \mathbb{P}^N$ and a surjective linear map $\varphi : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}$ such that

(C1) $\mathsf{u}\mathfrak{f} = \varphi \circ \mathfrak{g}$, with $\mathfrak{f}, \mathfrak{g}$ reduced representations of f, g, respectively, and u a function holomorphic on $M \backslash (\mathfrak{f}^{-1}(0) \cup \mathfrak{g}^{-1}(0))$ with $N_{\mathsf{u}}(0; r, s) = o(T_g(r, s))$;

(C2) $\sum_{l=0}^{N} N_g^{(1)}(\tilde{H}_l; r, s) = o(T_g(r, s))$ for the $N + 1$ hyperplanes $\tilde{H}_l := \mathbf{P}(w_l^{-1}(0))$ in $\mathbb{P}^N$.

If conditions (C1) and (C2) are satisfied, then (g, φ) is called a *decomposition* of $f \in \mathscr{D}$. In addition, this decomposition (g, φ) is said to be *reduced* when g is linearly non-degenerate and $\varphi(e_l) \neq 0$ for each $l = 0, 1, \dots, N$.

Similar to [2, Proposition 4.3], we can prove the following result.

Proposition 3.2. *Let (g, φ) be a decomposition of $f \in \mathscr{D}$. Then*

$$T_f(r, s) \leq T_g(r, s) + O(1). \tag{3.1}$$

In addition, when this decomposition (g, φ) is reduced, then

$$T_g(r, s) \leq T_f(r, s) + o(T_f(r, s)). \tag{3.2}$$

Proof. The proof of (3.1) depends entirely on the hypothesis that u is holomorphic and φ is linear, as then $N_{\mathsf{u}}(\infty; r, s) = O(1)$ and $T_{\mathbf{P}(\varphi \circ \mathfrak{g})}(r, s) \leq T_g(r, s) + O(1)$. In particular, as f is transcendental, this further implies that g is also transcendental.

In fact, as $\mathsf{u}\mathfrak{f} = \varphi \circ \mathfrak{g}$, noticing $f^*(\omega_{\mathrm{FS}}) + \nu_{\mathsf{u},0} - \nu_{\mathsf{u},\infty} = (\varphi \circ g)^*(\omega_{\mathrm{FS}})$, we immediately get

$$T_f(r, s) + N_{\mathsf{u}}(0; r, s) - N_{\mathsf{u}}(\infty; r, s) = T_{\mathbf{P}(\varphi \circ \mathfrak{g})}(r, s), \tag{3.3}$$

which then gives (3.1) in view of the above arguments.

On the other hand, as shown in [2, Proposition 4.3], there exists a linear function $\omega : \mathbb{C}^{n+1} \to \mathbb{C}$ such that $\omega(\varphi(e_l)) \neq 0$, as $\varphi(e_l) \neq 0$, for each $l \in [0, N]$. Set $\chi := \omega \circ \varphi : \mathbb{C}^{N+1} \to \mathbb{C}$, and write $\tilde{H} := \mathbf{P}(\chi^{-1}(0))$, the hyperplane generated by χ in $\mathbb{P}^N$. Then $\tilde{H}, \tilde{H}_0, \dots, \tilde{H}_N$ are in general position and $\mathsf{u}(\omega \circ \mathfrak{f}) = \chi \circ \mathfrak{g}$. Thus, we have

$$\begin{aligned} T_g(r, s) &\leq N_g^{(N)}(\tilde{H}; r, s) + O(\log(rT_g(r, s))) \\ &\leq N_f^{(N)}(H; r, s) + N_{\mathsf{u}}^{(N)}(0; r, s) + o(T_g(r, s)) \\ &\leq T_f(r, s) + N_{\mathsf{u}}(0; r, s) + o(T_g(r, s)). \end{aligned} \tag{3.4}$$

Here, the estimates (1.10), (1.16) and (1.18) were applied, and $H := \mathbf{P}(\omega^{-1}(0))$ is the hyperplane generated by ω in $\mathbb{P}^n$. So, (3.2) follows from (3.1) and (3.4). ∎

Let $g : M \to \mathbb{P}^N$ be a meromorphic map, with $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_N)$ being a reduced representation, and set $\widehat{\mathsf{U}}_0 := \{\beta \in \mathbb{C}_{(d)}^{N+1} : \beta \circ \mathfrak{g} \equiv 0\}$, a linear subspace of $\mathbb{C}_{(d)}^{N+1}$. Take $\lambda \in \mathscr{L}_{\varphi}^{s,n}$ effective for φ, with a generator μ, and write $\widehat{\mathsf{W}}_\lambda := \operatorname{span}\{\mathfrak{y}_\lambda(\beta) : \beta \in \mathbb{C}^{N+1}_{(d)}\}$. Here, as in Section 1, "span" means linear span.

Definition 3.3. Given λ, μ, we call g *weakly non-degenerate of degree d for φ* provided $\widehat{\mathsf{U}}_0 \cap \widehat{\mathsf{W}}_\lambda = \{0\}$, and, for $g_\mu := \mathbf{P} \circ \mathfrak{g}_\mu$ with $\mathfrak{g}_\mu := (\mathfrak{g}_{\mu(0)}, \mathfrak{g}_{\mu(1)}, \ldots, \mathfrak{g}_{\mu(s)})$,

$$T_g(r,s) = T_{g_\mu}(r,s) + o(T_g(r,s)). \tag{3.5}$$

Any $\lambda \in \mathscr{L}^{s,\,n}_\varphi$ as above is said to be *compatible for* (g, φ).

Definition 3.4. Suppose that $f : M \to \mathbb{P}^n$ is a transcendental meromorphic map. Then we write $f \in \mathscr{W}$ provided $f \in \mathscr{D}$ admits a reduced decomposition (g, φ) such that g is weakly non-degenerate of degree d for φ.

When $s = n = N$, Proposition 4.2 of [3] and the example there show that Definition 4.3 does provide a weaker assumption than non-degeneracy of degree d.

Finally, we shall require the following general assumptions.

(D1) $f \in \mathscr{W}$, with (g, φ) being a reduced decomposition;
(D2) $\lambda \in \mathscr{L}^{s,\,n}_\varphi$ is compatible for the decomposition (g, φ) with $s \le n$;
(D3) $D_1, \ldots, D_q$ have normal crossings at $\{\mathbf{y}_\imath = \mathbf{P}(\varphi(e_{\mu(\imath)}))\}_{\imath=0}^s$ with $q \ge n+1$.

Like [2, Lemma 4.5] and [3, Section 4], we can prove the following result.

Lemma 3.5. *Suppose* (D1) *and* (D2) *hold, and $D = D[\alpha]$ is a hypersurface of degree d associated with $\alpha \in \mathbb{C}^{n+1}_{(d)}$ in $\mathbb{P}^n$. Then*

$$\int_{M\langle r\rangle} [\log(\eta_d(\mathfrak{y}_\lambda(\alpha \circ \varphi)) \circ \mathfrak{g})]\sigma \le N_f(D; r, s) + o(T_f(r,s)). \tag{3.6}$$

Proof. For $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_N)$, let

$$\tilde{\mathfrak{g}} := \psi_d \circ \mathfrak{g} = (\mathfrak{g}_0^{l_0}\mathfrak{g}_1^{l_1} \cdots \mathfrak{g}_N^{l_N} \mid L \in \mathtt{K}^N_d) : M \to \mathbb{C}^{N_d+1}.$$

Here, $\psi_d : \mathbb{C}^{N+1} \to \mathbb{C}^{N_d+1}$ denotes the dth *Veronese map*, with $N_d := \binom{N+d}{d} - 1$.

Identify $\mathbb{C}^{N+1}_{(d)}$ and $\mathbb{C}^{N_d+1}_{(1)}$ to see that $\beta \circ \mathfrak{g} = \beta \circ \tilde{\mathfrak{g}}$ for each $\beta \in \mathbb{C}^{N+1}_{(d)}$.

Let $\mathsf{U}_0 := \{z \in \mathbb{C}^{N_d+1} : \beta(z) = 0 \text{ for all } \beta \in \widehat{\mathsf{U}}_0\}$ be the adjoint subspace of $\widehat{\mathsf{U}}_0$. Choose $\gamma_d(T_\lambda) \subseteq \mathtt{K} \subseteq \mathtt{K}^N_d$ such that $\widehat{\mathsf{U}}_0 \cap \widehat{\mathsf{U}}_1 = \{0\}$ and $\widehat{\mathsf{U}}_0 \oplus \widehat{\mathsf{U}}_1 = \mathbb{C}^{N+1}_{(d)}$, where

$$\widehat{\mathsf{U}}_1 := \Big\{ \sum_{L \in \mathtt{K}} b_L w^L : \beta = \sum_{L \in \mathtt{K}^N_d} b_L w^L \in \mathbb{C}^{N+1}_{(d)} \Big\} = \operatorname{span}\{w^L : L \in \mathtt{K}\}.$$

Then g is non-degenerate in $\widehat{\mathsf{U}}_1$. Denote by $\pi : \mathbb{C}^{N+1}_{(d)} \to \widehat{\mathsf{U}}_1$ the natural projection, induced via $(\widehat{\mathsf{U}}_0, \widehat{\mathsf{U}}_1)$. Define $\hat{\mathfrak{g}} : M \to \mathsf{U}_0$ through $\tilde{\mathfrak{g}} = \mathfrak{I} \circ \hat{\mathfrak{g}}$, with

$\mathfrak{J} : \mathsf{U}_0 \to \mathbb{C}^{N_d+1}$ the inclusion.

Then, as discussed in [3, Section 4], we have, for each $\beta \in \mathbb{C}_{(d)}^{N+1}$,

$$\beta \circ \mathfrak{g} = \beta \circ \tilde{\mathfrak{g}} = \beta \circ \mathfrak{J} \circ \hat{\mathfrak{g}} = \pi(\beta) \circ \hat{\mathfrak{g}}, \tag{3.7}$$

and

$$\eta_d(\mathfrak{y}_\lambda(\beta)) \circ \mathfrak{g} \le \eta_d(\beta) \circ \tilde{\mathfrak{g}} = \eta_d(\pi(\beta)) \circ \hat{\mathfrak{g}}. \tag{3.8}$$

Now, fix $\beta := \alpha \circ \varphi \in \mathbb{C}_{(d)}^{N+1}$ and set $\tilde{D} = \tilde{D}[\beta]$. Then

$$N_g(\tilde{D}; r, s) = N_f(D; r, s) + d\, N_{\mathsf{u}}(0; r, s), \tag{3.9}$$

since the relation $\mathsf{u}\mathfrak{f} = \varphi \circ \mathfrak{g}$ immediately leads to $\mathsf{u}^d(\alpha \circ \mathfrak{f}) = \beta \circ \mathfrak{g}$.

Write $\beta \circ \mathfrak{g} = \sum_{L \in \mathsf{K}_d^N} b_L \mathfrak{g}^L$ explicitly, with $\mathfrak{g}^L := \mathfrak{g}_0^{l_0} \mathfrak{g}_1^{l_1} \cdots \mathfrak{g}_N^{l_N}$. Then

$$N_g^{(N_d)}(\tilde{D}_L; r, s) \le d N_d \sum_{l=0}^{N} N_g^{(1)}(\tilde{H}_l; r, s) \tag{3.10}$$

with $\tilde{D}_L := \mathbf{P}((w^L)^{-1}(0))$ generated by w^L for each $L \in \mathsf{K}_d^N$.

Next, consider $\pi(\beta) = \sum_{L \in \mathsf{K}} b_L w^L \in \widehat{\mathsf{U}}_1$, and, without loss of generality, assume that $b_L \neq 0$ for each $L \in \mathsf{K}$. Denote $\tilde{\mathfrak{h}} := (\mathfrak{g}_0^{l_0} \mathfrak{g}_1^{l_1} \cdots \mathfrak{g}_N^{l_N} \mid L \in \mathsf{K}) : M \to \mathbb{C}^{T+1}$, with $T+1$ the (linear) dimension of $\widehat{\mathsf{U}}_1$ in $\mathbb{C}_{(1)}^{N_d+1} (\cong \mathbb{C}^{N_d+1})$. Clearly, $T \le N_d$. Let $\mathfrak{h}$ be a reduced representation of the meromorphic map $h := \mathbf{P} \circ \tilde{\mathfrak{h}} : M \to \mathbb{P}^T$. Then there is a function v, holomorphic on $M \setminus \mathfrak{h}^{-1}(0)$, such that $\tilde{\mathfrak{h}} = \mathsf{v}\mathfrak{h}$.

Write $z = (z_0, z_1, \ldots, z_T) \in \mathbb{C}^{T+1}$. Let $\breve{H}_L := \mathbf{P}(z_L^{-1}(0))$ be the Lth coordinate hyperplane in $\mathbb{P}^T$ for each $L \in [0, T]$, and let $\breve{H}_{T+1}$ be that associated with $\sum_{L \in \mathsf{K}} b_L z_L = 0$ in $\mathbb{P}^T$. Denote $\tilde{D}^* := \tilde{D}[\pi(\beta)]$. Then

$$\begin{aligned} T_h(r, s) &\le \sum_{L=0}^{T+1} N_h^{(T)}(\breve{H}_L; r, s) + O(\log(r T_h(r, s))) \\ &\le \sum_{L=0}^{T} N_g^{(N_d)}(\tilde{D}_L; r, s) + N_h(\breve{H}_{T+1}; r, s) + o(T_g(r, s)), \end{aligned} \tag{3.11}$$

as h is linearly non-degenerate from the preceding discussions, and

$$T_h(r, s) \le O(T_g(r, s)). \tag{3.12}$$

Moreover, from of (3.7) and $\tilde{\mathfrak{h}} = \mathsf{v}\mathfrak{h}$, it is easily seen that

$$N_h(\breve{H}_{T+1}; r, s) + N_{\mathsf{v}}(0; r, s) = N_{\hat{g}}(\tilde{D}^*; r, s) = N_g(\tilde{D}; r, s). \tag{3.13}$$

On the other hand, using the notion of "reduced representation section" (see Stoll [18, Section 5] for a detailed description, or [9]), the *Green–Jensen* *formula* yields

$$T_h(r,s)+N_{\mathsf{v}}(0;r,s)=\int\limits_{M\langle r\rangle}(\log\|\tilde{\mathfrak{h}}\|)\sigma+O(1). \tag{3.14}$$

As a consequence, together with (3.8)–(3.10) and the above estimates, we have

$$\begin{aligned}
&\int\limits_{M\langle r\rangle}[\log(\eta_d(\mathfrak{y}_\lambda(\beta))\circ\mathfrak{g})]\sigma\le\int\limits_{M\langle r\rangle}[\log(\eta_d(\pi(\beta))\circ\hat{\mathfrak{g}})]\sigma\\
&\quad\le\int\limits_{M\langle r\rangle}(\log\|\tilde{\mathfrak{h}}\|)\sigma+O(1)\le T_h(r,s)+N_{\mathsf{v}}(0;r,s)+o(T_g(r,s))\\
&\quad\le d(N_d^2+N_d)\sum_{l=0}^{N}N_g^{(1)}(\tilde{H}_l;r,s)+N_f(D;r,s)+d\,N_{\mathsf{u}}(0;r,s)+o(T_g(r,s)),
\end{aligned}$$

as $\|\tilde{\mathfrak{h}}\|=\sqrt{\sum_{L=0}^{T}|\mathsf{g}^L|^2}\ge C\eta_d(\pi(\beta))\circ\hat{\mathfrak{g}}$. By hypothesis, this finishes the proof. ■

Finally, we can derive the following main result of this paper.

THEOREM 3.6 (Second Main Theorem and Defect Relation). *Suppose that* (D1)–(D3) *hold. Then*

$$(qd-n-1)T_f(r,s)\le\sum_{j=1}^{q}N_f(D_j;r,s)+o(T_f(r,s)), \tag{3.15}$$

and

$$\sum_{j=1}^{q}\delta(D_j,f)\le\frac{n+1}{d}. \tag{3.16}$$

Proof. The proof is exactly the same as those of [2, Theorem 4.6] or [3, Theorem 5.1].

As a matter of fact, using Propositions 2.4, 3.2 and Lemma 3.5, we have

$$\begin{aligned}
&\sum_{j=1}^{q}N_f(D_j;r,s)+o(T_f(r,s))\ge\int\limits_{M\langle r\rangle}\Big[\log\Big(\|\mathfrak{g}_\mu\|^{qd-n-1}\prod_{j=0}^{n}|\mathsf{g}_{\mu(j)}|\Big)\Big]\sigma\\
&\quad\ge(qd-n-1)\int\limits_{M\langle r\rangle}(\log\|\mathfrak{g}_\mu\|)\sigma\ge(qd-n-1)T_{g_\mu}(r,s)+o(T_f(r,s))\\
&\quad=(qd-n-1)T_g(r,s)+o(T_f(r,s))=(qd-n-1)T_f(r,s)+o(T_f(r,s)),
\end{aligned}$$

which in turn yields the defect relation (3.16) in a standard manner. ■

When $N=n$ and φ is the identity map, Theorem 1.1 follows.

Final remark. After the first version of this paper was finished, I was able to get a hard copy of Biancofiore's Ph.D. thesis, *A hypersurface defect relation for a class of meromorphic maps*, University of Notre Dame, 1981; as can be seen, the proofs presented here are simpler, though again the main ideas are from his two original papers [2, 3].

REFERENCES

[1] Y. Aihara and S. Mori, *Deficiencies of meromorphic mappings for hypersurfaces*, J. Math. Soc. Japan 57 (2005), 233–258.

[2] A. Biancofiore, *A hypersurface defect relation for a class of meromorphic maps*, Trans. Amer. Math. Soc. 270 (1982), 47–60.

[3] A. Biancofiore, *An extension of the class of meromorphic maps satisfying the small Griffiths' conjecture*, Boll. Un. Mat. Ital. B 2 (1983), 867–877.

[4] J. Carlson and Ph. Griffiths, *A defect relation for equidimensional holomorphic mappings between algebraic varieties*, Ann. of Math. 95 (1972), 557–584.

[5] K. Chandler and P.-M. Wong, *Finsler geometry of holomorphic jet bundles*, in: A Sampler of Riemann–Finsler Geometry, Cambridge Univ. Press, Cambridge, 2004, 107–196.

[6] A. È. Eremenko and M. L. Sodin, *Distribution of values of meromorphic functions and meromorphic curves from the standpoint of potential theory*, St. Petersburg Math. J. 3 (1992), 109–136.

[7] Ph. Griffiths, *Holomorphic mappings: survey of some results and discussion of open problems*, Bull. Amer. Math. Soc. 78 (1972), 374–382.

[8] Ph. Griffiths and J. King, *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Math. 130 (1973), 145–220.

[9] Q. Han, *A defect relation for meromorphic maps on generalized p-parabolic manifolds intersecting hypersurfaces in complex projective algebraic varieties*, Proc. Edinburgh Math. Soc. 56 (2013), 551–574.

[10] P. Hu and C.-C. Yang, *The second main theorem of holomorphic curves into projective spaces*, Trans. Amer. Math. Soc. 363 (2011), 6465–6479.

[11] M. Ru, *A defect relation for holomorphic curves intersecting hypersurfaces*, Amer. J. Math. 126 (2004), 215–226.

[12] M. Ru, *Holomorphic curves into algebraic varieties*, Ann. of Math. 169 (2009), 255–267.

[13] B. Shiffman, *Nevanlinna defect relations for singular divisors*, Invent. Math. 31 (1975), 155–182.

[14] B. Shiffman, *Holomorphic curves in algebraic manifolds*, Bull. Amer. Math. Soc. 83 (1977), 553–568.

[15] B. Shiffman, *On holomorphic curves and meromorphic maps in projective space*, Indiana Univ. Math. J. 28 (1979), 627–641.

[16] L. Smiley, *Geometric conditions for unicity of holomorphic curves*, in: Value Distribution Theory and its Applications, Contemp. Math. 25, Amer. Math. Soc., Providence, RI, 1983, 149–154.

[17] W. Stoll, *Value Distribution on Parabolic Spaces*, Lecture Notes in Math. 600, Springer, Berlin, 1977.

[18] W. Stoll, *The Ahlfors–Weyl theory of meromorphic maps on parabolic manifolds*, in: Value Distribution Theory, Lecture Notes in Math. 981, Springer, Berlin, 1983, 101–219.

[19] P.-M. Wong, *Applications of Nevanlinna theory to geometric problems*, in: Third International Congress of Chinese Mathematicians, Part I-II, Amer. Math. Soc., Providence, RI, 2008, 523–594.

[20] P.-M. Wong and W. Stoll, *Second main theorem of Nevanlinna theory for nonequidimensional meromorphic maps*, Amer. J. Math. 116 (1994), 1031–1071.

[21] P.-M Wong and Ph. P.-M. Wong, *The second main theorem on generalized parabolic manifolds*, in: Some Topics on Value Distribution and Differentiability in Complex and p-adic Analysis, Science Press, Beijing, 2008, 3–41.

Qi Han
Department of Mathematical Sciences
Worcester Polytechnic Institute (WPI)
Worcester, MA 01609-2280, U.S.A.
E-mail: qhan@wpi.edu

Received 19 February 2013;
revised 29 November 2014 (5879)