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SOME CONGRUENCES INVOLVING BINOMIAL COEFFICIENTS

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Abstract. Binomial coefficients and central trinomial coefficients play important roles in combinatorics. Let $p>3$ be a prime. We show that

$$T_{p-1}\equiv\left(\frac{p}{3}\right)3^{p-1} \pmod{p^2},$$

where the central trinomial coefficient T_n is the constant term in the expansion of $(1+x+x^{-1})^n$. We also prove three congruences modulo p^3 conjectured by Sun, one of which is

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}((-1)^k-(-3)^{-k})\equiv\left(\frac{p}{3}\right)(3^{p-1}-1) \pmod{p^3}.$$

In addition, we get some new combinatorial identities.

1. Introduction. Throughout this paper, we set $\mathbb{N}=\{0,1,2,\ldots\}$ and $\mathbb{Z}^+=\{1,2,3,\ldots\}$.

Let $A,B\in\mathbb{Z}$. The Lucas sequences $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ are defined by

$$u_0=0,\quad u_1=1,\quad u_{n+1}=Au_n-Bu_{n-1}\quad (n\in\mathbb{Z}^+),$$
$$v_0=2,\quad v_1=A,\quad v_{n+1}=Av_n-Bv_{n-1}\quad (n\in\mathbb{Z}^+).$$

The roots of the characteristic equation $x^2-Ax+B=0$ are

$$\alpha=\frac{A+\sqrt{\Delta}}{2}\quad\text{and}\quad\beta=\frac{A-\sqrt{\Delta}}{2},$$

where $\Delta=A^2-4B$. By induction, one can easily deduce the following known formulae:

$$(\alpha-\beta)u_n=\alpha^n-\beta^n\quad\text{and}\quad v_n=\alpha^n+\beta^n\quad\text{for any } n\in\mathbb{N}.$$

(Note that in the case $\Delta=0$ we have $v_n=2(A/2)^n$ for all $n\in\mathbb{N}$.) It is well-known that

$$u_p\equiv\left(\frac{\Delta}{p}\right) \pmod{p}\quad\text{and}\quad u_{p-(\frac{\Delta}{p})}\equiv 0 \pmod{p} \tag{1.1}$$

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for any odd prime p not dividing B (see, e.g., Sun [3]), where $(-)$ denotes the Legendre symbol.

Let $p>3$ be a prime and let m be an integer not divisible by p. Recently, Sun [3, 4] established the following general congruences involving central binomial coefficients and Lucas sequences:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{\Delta}{p}\right)+u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2}\tag{1.2}$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-m)^k}&\\ \equiv\left(\frac{\Delta}{p}\right)(m-4)^{p-1}&+\left(1-\frac{m}{2}\right)u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2},\end{aligned}\tag{1.3}$$

where $\Delta=m^2-4m$. Clearly $\binom{p-1}{k}\equiv(-1)^k\pmod p$ for all $k=0,\dots,p-1$.

Note that for each $n=0,1,2,\dots$ the central binomial coefficient $\binom{2n}{n}$ is the constant term of $(1+x)^{2n}/x^n=(2+x+x^{-1})^n$. For $n\in\mathbb{N}$, the *central trinomial coefficient* T_n is the constant term in the expansion of $(1+x+x^{-1})^n$, i.e.,

$$T_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{k!k!(n-2k)!}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}.$$

Central trinomial coefficients arise naturally in enumerative combinatorics (cf. Sloane [2]), e.g., T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with the only allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$. As Andrews [1] pointed out, central trinomial coefficients were first studied by L. Euler. Recently, Sun [6] investigated congruence properties of central trinomial coefficients; for example, he proved that $\sum_{k=0}^{p-1}T_k^2\equiv\left(\frac{-1}{p}\right)\pmod p$ for any odd prime p.

Now we state our first theorem.

THEOREM 1.1. *Let $p>3$ be a prime.*

(i) *We have*

$$T_{p-1}\equiv\left(\frac{p}{3}\right)3^{p-1}\pmod{p^2}\tag{1.4}$$

and

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}((-1)^k-(-3)^{-k})\equiv\left(\frac{p}{3}\right)(3^{p-1}-1)\pmod{p^3}.\tag{1.5}$$

(ii) *If* $p \equiv \pm 1 \pmod{12}$, *then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1) \equiv (-1)^{(p-1)/2} u_{p-1}(4,1) \pmod{p^3}. \tag{1.6}$$

If $p \equiv \pm 1 \pmod 8$, *then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}\frac{u_k(4,2)}{(-2)^k} \equiv (-1)^{(p-1)/2} u_{p-1}(4,2) \pmod{p^3}. \tag{1.7}$$

Remark. (1.5) and part (ii) of Theorem 1.1 were conjectured by Sun [5, Conj. 1.3].

During our efforts to prove Theorem 1.1, we also obtain some combinatorial identities.

Theorem 1.2. *Let* n *be a positive integer.*

(i) *If* $6 \mid n$, *then*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{\left(\frac{k}{3}\right)}{4^k} = 0. \tag{1.8}$$

If $n \equiv 3 \pmod 6$, *then*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{3[3 \mid k] - 1}{4^k} = 0, \tag{1.9}$$

where $[3 \mid k]$ *is* 1 *or* 0 *according as* $3 \mid k$ *or not.*

(ii) *If* $4 \mid n$, *then*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{u_k(2,2)}{(-4)^k} = 0. \tag{1.10}$$

If $n \equiv 2 \pmod 4$, *then*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{v_k(2,2)}{(-4)^k} = 0. \tag{1.11}$$

(iii) *If* $3 \mid n$, *then*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{u_k(3,3)}{(-4)^k} = 0. \tag{1.12}$$

We will provide two lemmas in the next section and prove Theorems 1.1 and 1.2 in Section 3.

2. Two lemmas

Lemma 2.1. *Let* $A \in \mathbb{Z}^+$ *and* $B, m \in \mathbb{Z} \setminus \{0\}$ *with* $\Delta = A^2 - 4B \neq 0$. *Let* $\alpha = (A + \sqrt{\Delta})/2$ *and* $\beta = (A - \sqrt{\Delta})/2$. *Then, for every* $n \in \mathbb{N}$,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{u_k(A,B)}{m^k}=\frac{d^{n/2}(\alpha^n-(-\beta)^n)}{m^n(\alpha-\beta)}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}d^{-k}, \tag{2.1}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{v_k(A,B)}{m^k}=\frac{d^{n/2}(\alpha^n+(-\beta)^n)}{m^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}d^{-k}, \tag{2.2}$$

where $m=-4B/A$ *and* $d=4\Delta/A^2$.

Proof. For a polynomial $P(x)$ over the field of complex numbers, we use $[x^n]P(x)$ to denote the coefficient of x^n in $P(x)$. It is easy to see that

$$\begin{aligned}[x^n]((1+\alpha x)^2+mx)^n&=[x^n]\sum_{k=0}^{n}\binom{n}{k}(1+\alpha x)^{2k}(mx)^{n-k}\\&=m^n\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{\alpha^k}{m^k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}[x^n]((1+\alpha x)^2+mx)^n&=[x^n](\alpha^2x^2+(2\alpha+m)x+1)^n\\&=[x^n]\sum_{\substack{r,s,t\ge0\\r+s+t=n}}\binom{n}{r,s,t}\alpha^{2r}(2\alpha+m)^sx^{2r+s}\\&=\alpha^n\sum_{\substack{r,s\ge0\\2r+s=n}}\binom{n}{r,s,r}\left(2+\frac{m}{\alpha}\right)^s\\&=\alpha^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\left(2+\frac{m}{\alpha}\right)^{n-2k}.\end{aligned}$$

So we obtain

$$m^n\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{\alpha^k}{m^k}=\alpha^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\left(2+\frac{m}{\alpha}\right)^{n-2k}. \tag{2.3}$$

Similarly,

$$m^n\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{\beta^k}{m^k}=\beta^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\left(2+\frac{m}{\beta}\right)^{n-2k}. \tag{2.4}$$

As $4B=-mA$, we see that

$$2+\frac{2m}{A\pm\sqrt{\Delta}}=2+\frac{2m(A\mp\sqrt{\Delta})}{4B}=\pm\frac{2m}{mA}\sqrt{A^2+mA}=\pm\sqrt{d},$$

i.e., $2+m/\alpha=\sqrt{d}$ and $2+m/\beta=-\sqrt{d}$. Since $u_k=(\alpha^k-\beta^k)/(\alpha-\beta)$ and $v_k=\alpha^k+\beta^k$ for all $k\in\mathbb{N}$, combining (2.3) and (2.4) we get (2.1) and (2.2) immediately. ∎

LEMMA 2.2. *Let $p > 3$ be a prime, and let $d \in \mathbb{Z}$ with $p \nmid d$. Then*

$$\sum_{k=0}^{(p-1)/2} \binom{p-1}{k}\binom{p-1-k}{k} d^{-k} \equiv \left(\frac{D}{p}\right)\left(\frac{1-d^{p-1}}{2} + (d-4)^{p-1}\right) - \frac{d}{4} u_{p-(\frac{D}{p})}(d-2,1) \pmod{p^2}, \tag{2.5}$$

where $D = d(d-4)$.

Proof. For every $k = 0, 1, \ldots, p-1$, we clearly have

$$\binom{p-1}{k} = (-1)^k \prod_{0<j\leqslant k}\left(1 - \frac{p}{j}\right) \equiv (-1)^k(1 - pH_k) \pmod{p^2}, \tag{2.6}$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k} 1/j$. Thus

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2} \binom{p-1}{k}\binom{p-1-k}{k} d^{-k} \\
&\equiv \sum_{k=0}^{(p-1)/2} (-1)^k(1-pH_k)\binom{p-1-k}{k} d^{-k} \\
&= \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k}(-d)^{-k} - p\sum_{k=0}^{(p-1)/2} H_k\binom{p-1-k}{k}(-d)^{-k} \pmod{p^2}.
\end{aligned}$$

Since $\binom{p-1-k}{k} \equiv \binom{-1-k}{k} = (-1)^k\binom{2k}{k} \pmod{p}$ for all $k = 0, \ldots, p-1$, we obtain from the above

$$\sum_{k=0}^{(p-1)/2} \binom{p-1}{k}\binom{p-1-k}{k} d^{-k} \equiv \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k}(-d)^{-k} - p\sum_{k=0}^{(p-1)/2} H_k\binom{2k}{k} d^{-k} \pmod{p^2}. \tag{2.7}$$

It is known that

$$u_{n+1}(A,B) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} A^{n-2k}(-B)^k \quad \text{for all } n = 0, 1, 2, \ldots,$$

which can be easily proved by induction. So we have

$$\begin{aligned}
u_p(d,d) &= \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} d^{p-1-2k}(-d)^k \\
&= d^{p-1}\sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k}(-d)^{-k}.
\end{aligned}$$

By [3, Lemma 2.4],

$$2u_p(d,d)-\left(\frac{D}{p}\right)d^{p-1}\equiv u_p(d-2,1)+u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2).$$

In view of [4, (3.6)], if $p\nmid d-4$ then

$$u_p(d-2,1)-\left(\frac{D}{p}\right)\equiv\left(\frac{d}{2}-1\right)u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2).$$

This also holds when $p\mid d-4$, since $\left(\frac{D}{p}\right)=0$ and

$$u_p(d-2,1)=u_{p-(\frac{D}{p})}(d-2,1)=u_{p-(\frac{(d-2)^2-4\cdot 1}{p})}(d-2,1)\equiv 0\ (\mathrm{mod}\ p)$$

by (1.1). Combining the above two congruences we immediately get

$$u_p(d,d)\equiv\left(\frac{D}{p}\right)\frac{d^{p-1}+1}{2}+\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2).$$

Hence

$$\begin{aligned}\text{(2.8)}\qquad&\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-d)^{-k}\\&\qquad\equiv\left(\frac{D}{p}\right)\frac{d^{p-1}+1}{2d^{p-1}}+\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2)\end{aligned}$$

since $u_{p-(\frac{D}{p})}(d-2,1)\equiv 0\ (\mathrm{mod}\ p)$ and $d^{p-1}\equiv 1\ (\mathrm{mod}\ p)$.

Note that $p\mid\binom{2k}{k}$ for $k=(p+1)/2,\ldots,p-1$. By (2.6), we have

$$\begin{aligned}p\sum_{k=0}^{(p-1)/2}H_k\binom{2k}{k}d^{-k}&\equiv\sum_{k=0}^{(p-1)/2}\left(1-(-1)^k\binom{p-1}{k}\right)\binom{2k}{k}d^{-k}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{d^k}-\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{d^k}+\sum_{k=(p+1)/2}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}-\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}\\&\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{d^k}-\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}\ (\mathrm{mod}\ p^2).\end{aligned}$$

Thus, applying (1.2) and (1.3) with $m=d$ we see that $p\sum_{k=0}^{(p-1)/2}H_k\binom{2k}{k}d^{-k}$ is congruent to

$$\left(\frac{D}{p}\right)+u_{p-(\frac{D}{p})}(d-2,1)-\left(1-\frac{d}{2}\right)u_{p-(\frac{D}{p})}(d-2,1)-\left(\frac{D}{p}\right)(d-4)^{p-1}$$

modulo p^2. Hence

$$(2.9)\qquad p\sum_{k=0}^{(p-1)/2} H_k\binom{2k}{k}d^{-k} \equiv \left(\frac{D}{p}\right)(1-(d-4)^{p-1})+\frac{d}{2}u_{p-(\frac{D}{p})}(d-2,1) \pmod{p^2}.$$

Combining (2.7)–(2.9), we finally obtain

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}d^{-k}\\ &\equiv\left(\frac{D}{p}\right)\left(\frac{1-d^{p-1}}{2d^{p-1}}+(d-4)^{p-1}\right)-\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\\ &\equiv\left(\frac{D}{p}\right)\left(\frac{1-d^{p-1}}{2}+(d-4)^{p-1}\right)-\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1) \pmod{p^2}. \blacksquare\end{aligned}$$

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1(i). Denote the primitive cubic root $(-1+\sqrt{-3})/2$ by ω. For each $k=0,1,2,\ldots$, we clearly have

$$u_{3k}(-1,1)=u_{3k}(\omega+\bar\omega,\omega\bar\omega)=\frac{\omega^{3k}-\bar\omega^{3k}}{\omega-\bar\omega}=0.$$

As

$$T_{p-1}=\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k},$$

applying (2.5) with $d=1$ we get

$$T_{p-1}\equiv\left(\frac{-3}{p}\right)(-3)^{p-1}-\frac{1}{4}u_{p-(\frac{-3}{p})}(-1,1)=\left(\frac{p}{3}\right)3^{p-1} \pmod{p^2}.$$

This proves (1.4).

Note that $u_k(4,3)=(3^k-1)/(3-1)$ for all $k\in\mathbb{N}$. By Lemma 2.1 and (1.4), we have

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}\frac{u_k(4,3)}{(-3)^k}\\ &=\frac{3^{p-1}-(-1)^{p-1}}{(3-1)(-3)^{p-1}}\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}=\frac{3^{p-1}-1}{2\times 3^{p-1}}T_{p-1}\\ &\equiv\frac{3^{p-1}-1}{2\times 3^{p-1}}\left(\frac{p}{3}\right)3^{p-1} \pmod{p^3}\end{aligned}$$

and hence the desired (1.5) follows. ∎

Proof of Theorem 1.1(ii). Suppose that $p \equiv \pm 1 \pmod{12}$. In light of the second congruence in (1.1),

$$u_{p-1}(4,1) = u_{p-(\frac{4^2-4\cdot 1}{p})}(4,1) \equiv 0 \pmod{p}.$$

By Lemma 2.2,

$$\sum_{k=0}^{(p-1)/2} \binom{p-1}{k}\binom{p-1-k}{k} 3^{-k}$$
$$\equiv \left(\frac{-3}{p}\right)\left(\frac{1-3^{p-1}}{2} + (-1)^{p-1}\right) - \frac{3}{4}u_{p-(\frac{-3}{p})}(1,1) \equiv \left(\frac{p}{3}\right)\frac{3-3^{p-1}}{2} \pmod{p^2}$$

since

$$u_{3k}(1,1) = \frac{(-\omega)^{3k} - (-\bar{\omega})^{3k}}{-\omega - (-\bar{\omega})} = 0 \quad \text{for all } k \in \mathbb{N}.$$

Combining this with Lemma 2.1 we get

$$\begin{aligned}
&\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1) \\
&\qquad = \frac{3^{(p-1)/2}}{(-1)^{p-1}} u_{p-1}(4,1) \sum_{k=0}^{(p-1)/2} \binom{p-1}{k}\binom{p-1-k}{k} 3^{-k} \\
&\qquad \equiv 3^{(p-1)/2} u_{p-1}(4,1) \left(\frac{p}{3}\right) \frac{3-3^{p-1}}{2} \pmod{p^3}.
\end{aligned}$$

Note that $3^{p-1} \equiv 2\cdot 3^{(p-1)/2} - 1 \pmod{p^2}$ since $3^{(p-1)/2} \equiv (\frac{3}{p}) = 1 \pmod{p}$. So we have

$$\begin{aligned}
\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1) &\equiv 3^{(p-1)/2}\left(\frac{-3}{p}\right)\frac{3-3^{p-1}}{2} u_{p-1}(4,1) \\
&\equiv (-1)^{(p-1)/2} 3^{(p-1)/2}(2 - 3^{(p-1)/2}) u_{p-1}(4,1) \\
&\equiv (-1)^{(p-1)/2} u_{p-1}(4,1) \pmod{p^3}.
\end{aligned}$$

This proves (1.6).

Now assume that $p \equiv \pm 1 \pmod 8$. By the second congruence in (1.1),

$$u_{p-1}(4,2) = u_{p-(\frac{4^2-4\cdot 2}{p})}(4,2) \equiv 0 \pmod{p}.$$

By Lemma 2.2,

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}&\binom{p-1}{k}\binom{p-1-k}{k}2^{-k}\\&\equiv\left(\frac{-4}{p}\right)\left(\frac{1-2^{p-1}}{2}+(-2)^{p-1}\right)-\frac{2}{4}u_{p-(\frac{-4}{p})}(0,1)\\&=\left(\frac{-1}{p}\right)\frac{1+2^{p-1}}{2}\pmod{p^2}\end{aligned}$$

since $u_{2k}(0,1)=0$ for all $k\in\mathbb{N}$. Combining this with Lemma 2.1 we get

$$\begin{aligned}\sum_{k=0}^{p-1}&\binom{p-1}{k}\binom{2k}{k}\frac{u_k(4,2)}{(-2)^k}\\&=\frac{2^{(p-1)/2}}{(-2)^{p-1}}u_{p-1}(4,2)\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}2^{-k}\\&\equiv\frac{u_{p-1}(4,2)}{2^{(p-1)/2}}\left(\frac{-1}{p}\right)\frac{1+2^{p-1}}{2}\pmod{p^3}.\end{aligned}$$

This is equivalent to (1.7) since $2^{p-1}+1-2\cdot 2^{(p-1)/2}=(2^{(p-1)/2}-1)^2\equiv 0\pmod{p^2}$. ∎

Proof of Theorem 1.2. (i) As $-\omega-\bar\omega=1$ and $(-\omega)(-\bar\omega)=1$, for any $k\in\mathbb{Z}$ we have

$$u_k(1,1)=\frac{(-\omega)^k-(-\bar\omega)^k}{-\omega-(-\bar\omega)}=(-1)^{k-1}\left(\frac{k}{3}\right),$$
$$v_k(1,1)=(-\omega)^k+(-\bar\omega)^k=(-1)^k(3[3\mid k]-1).$$

If $6\mid n$, then $(-\omega)^n=1=\bar\omega^n$ and hence by (2.1) we deduce

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{u_k(1,1)}{(-4)^k}=0,$$

which is equivalent to (1.8). If $n\equiv 3\pmod 6$, then $(-\omega)^n=-1=-\bar\omega^n$ and hence by (2.2) we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{v_k(1,1)}{(-4)^k}=0,$$

which is equivalent to (1.9).

(ii) Clearly $(1+i)+(1-i)=(1+i)(1-i)=2$. When n is even,

$$(1+i)^n=i^n(1-i)^n=(-1)^{n/2}(1-i)^n=\begin{cases}(i-1)^n & \text{if } 4\mid n,\\ -(i-1)^n & \text{if } n\equiv 2\pmod 4.\end{cases}$$

So we get the desired result in Theorem 1.2(ii) by applying Lemma 2.1.

(iii) Let $\alpha = (3+\sqrt{-3})/2$ and $\beta = (3-\sqrt{-3})/2$. Then $\alpha+\beta = \alpha\beta = 3$. Observe that

$$\alpha^2 - \alpha\beta + \beta^2 = (\alpha+\beta)^2 - 3\alpha\beta = 0$$

and so $\alpha^3 = (-\beta)^3$. If $3 \mid n$, then $\alpha^n = (-\beta)^n$ and hence (1.12) holds by (2.1). ■

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REFERENCES

[1] G. E. Andrews, *Euler's "exemplum memorabile inductionis fallacis" and q-trinomial coefficients*, J. Amer. Math. Soc. 3 (1990), 653–669.

[2] N. J. A. Sloane, Sequence A002426 in OEIS (*On-Line Encyclopedia of Integer Sequences*), http://oeis.org.

[3] Z.-W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients*, Sci. China Math. 53 (2010), 2473–2488.

[4] Z.-W. Sun, *On sums of binomial coefficients modulo p^2*, Colloq. Math. 127 (2012), 39–54.

[5] Z.-W. Sun, *On harmonic numbers and Lucas sequences*, Publ. Math. Debrecen 80 (2012), 25–41.

[6] Z.-W. Sun, *Congruences involving generalized central binomial coefficients*, Sci. China Math. 57 (2014), 1375–1400.

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