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NONALIQUOTS AND ROBBINS NUMBERS

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Abstract. Let $\varphi(\cdot)$ and $\sigma(\cdot)$ denote the Euler function and the sum of divisors function, respectively. We give a lower bound for the number of $m \le x$ for which the equation $m = \sigma(n) - n$ has no solution. We also show that the set of positive integers m not of the form $(p-1)/2 - \varphi(p-1)$ for some prime number p has a positive lower asymptotic density.

1. Introduction. Let $\varphi(\cdot)$ denote the Euler function, whose value at the positive integer n is

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right),$$

and let $\sigma(\cdot)$ denote the sum of divisors function, whose value at the positive integer n is

$$\sigma(n) = \sum_{d \mid n} d = \prod_{p^a \| n} \frac{p^{a+1} - 1}{p - 1}.$$

An integer in the image of the function $f_a(n) = \sigma(n) - n$ is called an *aliquot number*. If m is a positive integer for which the equation $f_a(n) = m$ has no solution, then m is said to be *nonaliquot*. Erdős [1] showed that the collection of nonaliquot numbers has a positive lower asymptotic density, but no numerical lower bound on this density was given. In Theorem 1 (Section 2), we show that the lower bound $\#\mathcal{N}_a(x) \ge \frac{1}{48} x\,(1 + o(1))$ holds, where

$$\mathcal{N}_a(x) = \{1 \le m \le x : m \ne f_a(n) \text{ for every positive integer } n\}.$$

For an odd prime p, let $f_r(p) = (p-1)/2 - \varphi(p-1)$. Note that $f_r(p)$ counts the number of quadratic nonresidues modulo p which are not primitive roots. At the 2002 Western Number Theory Conference in San Francisco, Neville Robbins asked whether there exist infinitely many positive integers m for

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which $f_r(p) = m$ has no solution; let us refer to such integers as *Robbins numbers*. The existence of infinitely many Robbins numbers has been shown recently by Luca and Walsh [4], who proved that for every odd integer $w \geq 3$, there exist infinitely many integers $\ell \geq 1$ such that $2^\ell w$ is a Robbins number. In Theorem 2 (Section 3), we show that the set of Robbins numbers has a positive density; more precisely, if

$$\mathcal{N}_r(x) = \{1 \leq m \leq x : m \neq f_r(p) \text{ for every odd prime } p\},$$

then the lower bound $\#\mathcal{N}_r(x) \geq \frac{1}{3}x\,(1 + o(1))$ holds.

Notation. Throughout the paper, the letters p and q are used to denote prime numbers. As usual, $\pi(x)$ denotes the number of primes $p \leq x$, and if $a, b > 0$ are coprime integers, $\pi(x; b, a)$ denotes the number of primes $p \leq x$ such that $p \equiv a \pmod{b}$. For any set $\mathcal{A}$ and real number $x \geq 1$, we denote by $\mathcal{A}(x)$ the set $\mathcal{A} \cap [1, x]$. For a real number $x > 0$, we put $\log x = \max\{\ln x, 1\}$, where $\ln x$ is the natural logarithm, and $\log_2 x = \log(\log x)$. Finally, we use the Vinogradov symbols $\ll$ and $\gg$, as well as the Landau symbols O and o, with their usual meanings.

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2. Nonaliquots

THEOREM 1. *The inequality*

$$\#\mathcal{N}_a(x) \geq \frac{x}{48}\,(1 + o(1))$$

holds as $x \to \infty$.

Proof. Let $\mathcal{K}$ be the set of positive integers $k \equiv 0 \pmod{12}$. Clearly,

$$\#\mathcal{K}(x) = \frac{x}{12} + O(1). \tag{1}$$

We first determine an upper bound for the cardinality of $(\mathcal{K} \setminus \mathcal{N}_a)(x)$. Let $k \in (\mathcal{K} \setminus \mathcal{N}_a)(x)$; then there exists a positive integer n such that

$$f_a(n) = \sigma(n) - n = k.$$

Since $k \in \mathcal{K}$, it follows that

$$n \equiv \sigma(n) \pmod{12}. \tag{2}$$

Assume first that n is odd. Then $\sigma(n)$ is odd as well, and therefore n is a perfect square. If $n = p^2$ for some prime p, then

$$x \geq k = \sigma(p^2) - p^2 = p + 1;$$

hence, the number of such integers k is at most $\pi(x-1) = o(x)$. On the other hand, if n is not the square of a prime, then n has at least four prime factors (counted with multiplicity). Let p_1 be the smallest prime dividing n; then $p_1 \leq n^{1/4}$, and therefore

$$n^{3/4} \leq \frac{n}{p_1} \leq \sigma(n) - n = k \leq x;$$

hence, $n \leq x^{4/3}$. Since n is a perfect square, the number of integers k is at most $x^{2/3} = o(x)$ in this case.

The above arguments show that all but $o(x)$ integers $k \in (\mathcal{K} \setminus \mathcal{N}_a)(x)$ satisfy an equation of the form

$$f_a(n) = \sigma(n) - n = k$$

for some *even* positive integer n. For such k, we have

$$\frac{n}{2} \leq \sigma(n) - n = k \leq x;$$

that is, $n \leq 2x$. It follows from the work of [2] (see, for example, the discussion on page 196 of [3]) that $12 \mid \sigma(n)$ for all but at most $o(x)$ positive integers $n \leq 2x$. Hence, using (2), we see that every integer $k \in (\mathcal{K} \setminus \mathcal{N}_a)(x)$, with at most $o(x)$ exceptions, can be represented in the form $k = f_a(n)$ for some $n \equiv 0 \pmod{12}$. For such k, we have

$$x \geq k = \sigma(n) - n = n\left(\frac{\sigma(n)}{n} - 1\right) \geq n\left(\frac{\sigma(12)}{12} - 1\right) = \frac{4n}{3},$$

therefore $n \leq \frac{3}{4}x$. Since n is a multiple of 12, it follows that

$$\#(\mathcal{K} \setminus \mathcal{N}_a)(x) \leq \frac{x}{16}\,(1 + o(1)).$$

Combining this estimate with (1), we derive that

$$\begin{aligned}\#\mathcal{N}_a(x) &\geq \#(\mathcal{K} \cap \mathcal{N}_a)(x) = \#\mathcal{K}(x) - \#(\mathcal{K} \setminus \mathcal{N}_a)(x)\\ &\geq \left(\frac{x}{12} - \frac{x}{16}\right)(1 + o(1)) = \frac{x}{48}\,(1 + o(1)),\end{aligned}$$

which completes the proof. ■

3. Robbins numbers

THEOREM 2. *The inequality*

$$\#\mathcal{N}_r(x) \geq \frac{x}{3}\,(1 + o(1))$$

holds as $x \to \infty$.

Proof. Let

$$\mathcal{M}_1 = \{2^\alpha k : k \equiv 3 \pmod 6 \text{ and } \alpha \equiv 0 \pmod 2\},$$
$$\mathcal{M}_2 = \{2^\alpha k : k \equiv 5 \pmod 6 \text{ and } \alpha \equiv 1 \pmod 2\},$$

and let $\mathcal{M}$ be the (disjoint) union $\mathcal{M}_1 \cup \mathcal{M}_2$. It is easy to see that

$$\#\mathcal{M}_1(x) = \frac{2x}{9}\,(1 + o(1)) \quad \text{and} \quad \#\mathcal{M}_2(x) = \frac{x}{9}\,(1 + o(1))$$

as $x \to \infty$; therefore,

$$\#\mathcal{M}(x) = \frac{x}{3}\,(1 + o(1)).$$

Hence, it suffices to show that all but $o(x)$ numbers in $\mathcal{M}(x)$ also lie in $\mathcal{N}_r(x)$.

Let $m \in \mathcal{M}(x)$, and suppose that $f_r(p) = m$ for some odd prime p. If $m = 2^\alpha k$ and $p - 1 = 2^\beta w$, where k and w are positive and odd, then

$$2^{\beta-1}(w - \varphi(w)) = \frac{p-1}{2} - \varphi(p-1) = f_r(p) = m = 2^\alpha k.$$

If $w = 1$, then $w - \varphi(w) = 0$, and thus $m = 0$, which is not possible. Hence, $w \ge 3$, which implies that $\varphi(w)$ is even, and $w - \varphi(w)$ is odd. We conclude that $\beta = \alpha + 1$ and $w - \varphi(w) = k$.

Let us first treat the case that $q^2 \mid w$ for some odd prime q. In this case, we have

$$k = w - \varphi(w) \ge \frac{w}{q},$$

and therefore $w \le qk \le qm \le qx$. Since $q^2 \mid w$ and $w \mid (p-1)$, it follows that $p \equiv 1 \pmod{q^2}$. Note that $q^2 \le w \le qx$; hence, $q \le x$. Since

$$p = 2^{\alpha+1} w + 1 \le 2^{\alpha+1} qk + 1 = 2qm + 1 \le 3qx,$$

the number of such primes p is at most $\pi(3qx; q^2, 1)$. Put $y = \exp(\sqrt{\log x}\,)$. If $q < x/y$, we use the well known result of Montgomery and Vaughan [5] to derive that

$$\pi(3qx; q^2, 1) \le \frac{6qx}{\varphi(q^2)\log(3x/q)} < \frac{6x}{(q-1)\log y} < \frac{9x}{q\sqrt{\log x}}$$

(in the last step, we used the fact that $q \ge 3$), while for $q \ge x/y$, we have the trivial estimate

$$\pi(3qx; q^2, 1) \le \frac{3qx}{q^2} = \frac{3x}{q}.$$

Summing over q, we see that the total number of possibilities for the prime p is at most

$$\frac{9x}{\sqrt{\log x}} \sum_{q < x/y} \frac{1}{q} + 3x \sum_{x/y \le q \le x} \frac{1}{q}.$$

Since

$$\sum_{q<x/y} \frac{1}{q} \ll \log_2(x/y) \le \log_2 x,$$

and

$$\begin{aligned}\sum_{x/y\le q\le x} \frac{1}{q} &= \log_2 x - \log_2(x/y) + O\left(\frac{1}{\log x}\right)\\ &= \log\left(1 + \frac{\log y}{\log x - \log y}\right) + O\left(\frac{1}{\log x}\right) \ll \frac{1}{\sqrt{\log x}},\end{aligned}$$

the number of possibilities for p (hence, also for $m = f_r(p)$) is at most

$$O\left(\frac{x\log_2 x}{\sqrt{\log x}}\right) = o(x).$$

Thus, for the remainder of the proof, we can assume that w is squarefree.

We claim that $3 \mid w$. Indeed, suppose that this is not the case. As w is squarefree and coprime to 3, it follows that $\varphi(w) \not\equiv 2 \pmod 3$ (if $q \mid w$ for some prime $q \equiv 1 \pmod 3$, then $3 \mid (q-1) \mid \varphi(w)$; otherwise $q \equiv 2 \pmod 3$ for all $q \mid w$; hence, $\varphi(w) = \prod_{q\mid w}(q-1) \equiv 1 \pmod 3$). In the case that $m \in \mathcal{M}_1$, we have $p = 2^{\alpha+1}w + 1 \equiv 2w + 1 \pmod 3$, thus $w \not\equiv 1 \pmod 3$ (otherwise, $p = 3$ and $m = 0$); then $w \equiv 2 \pmod 3$. However, since $\varphi(w) \not\equiv 2 \pmod 3$, it follows that 3 cannot divide $k = w - \varphi(w)$, which contradicts the fact that $k \equiv 3 \pmod 6$. Similarly, in the case that $m \in \mathcal{M}_2$, we have $p = 2^{\alpha+1}w + 1 \equiv w + 1 \pmod 3$, thus $w \not\equiv 2 \pmod 3$; then $w \equiv 1 \pmod 3$. However, since $\varphi(w) \not\equiv 2 \pmod 3$, it follows that $k = w - \varphi(w) \equiv 0$ or $1 \pmod 3$, which contradicts the fact that $k \equiv 5 \pmod 6$. These contradictions establish our claim that $3 \mid w$.

From the preceding result, we have

$$k = w - \varphi(w) \ge \frac{w}{3},$$

which implies that $p = 2^{\alpha+1}w + 1 = 2^{\alpha+1} \cdot 3k + 1 \le 6m + 1 \le 7x$. As $\pi(7x) \ll x/\log x$, the number of integers $m \in \mathcal{M}(x)$ such that $m = f_r(p)$ for some prime p of this form is at most $o(x)$, and this completes the proof. ■

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