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A CONSTRUCTION OF COMPLEX SYZYGY PERIODIC MODULES OVER SYMMETRIC ALGEBRAS

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Abstract. We construct arbitrarily complicated indecomposable finite-dimensional modules with periodic syzygies over symmetric algebras.

Introduction. Throughout the paper, by an algebra we mean a finite-dimensional associative K-algebra with an identity over a fixed (commutative) field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules and by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$. For a module M in $\operatorname{mod} A$, consider its minimal projective resolution

$$\to P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

and call $\Omega_A^i M = \operatorname{Ker} d_{i-1}$ the ith *syzygy* of M. Then M is said to be *periodic* if $\Omega_A^i M \cong M$ for some $i \geq 1$, and the minimal i with this property will be called the *period* of M. Note that if $\operatorname{mod} A$ admits a periodic module then A is of infinite global dimension.

An algebra A is called *self-injective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective. Further, A is called *symmetric* if A and $D(A)$ are isomorphic as A-A-bimodules. The classical examples of self-injective algebras are provided by the blocks of group algebras of finite groups and Hopf algebras. Further, it is known that if all simple modules over an algebra A are periodic, then A is self-injective [14, Theorem 1.4]. An interesting open problem concerns the classification (up to Morita equivalence) of the self-injective algebras all of whose nonprojective indecomposable finite-dimensional modules are periodic. These are quite exceptional self-injective algebras and the distinguished classes of such algebras are provided by the self-injective algebras of finite representation type ([15], [16], [8]), the self-injective algebras of tubular type and their socle deformations (see [5]–[7]), the algebras of quaternion type (see [9]), and the deformed preprojective algebras of generalized Dynkin type (see [2], [4], [11]). On the

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other hand, there are many self-injective algebras without periodic modules. For example, this is the case for all self-injective algebras of wild tilted type (see [10]).

The aim of this paper is to prove a quite surprising fact: for every finite-dimensional module M over an arbitrary algebra A, there exist indecomposable symmetric algebras Λ such that A is a factor algebra of Λ and M is a subfactor of many indecomposable periodic Λ-modules with given sequences of even periods (see Theorem 2.1). We present explicit constructions of such symmetric algebras and periodic modules, invoking the trivial extension algebras [12] and generalized canonical algebras introduced in [21].

For an algebra A, we denote by Γ_A the Auslander–Reiten quiver of A and by τ_A the Auslander–Reiten translation $D\,\mathrm{Tr}$ in Γ_A. We do not distinguish between an indecomposable module in $\operatorname{mod} A$ and the vertex of Γ_A corresponding to it. Further, for a module X in $\operatorname{mod} A$, we denote by $\operatorname{pd}_A X$ the projective dimension of X in $\operatorname{mod} A$. Finally, we denote by $\mathbb{N}_1$ the set of all positive integers and by $\mathbb{P}_1(K)$ the projective line over K.

For basic background on representation theory we refer to [1], [3], [13] and [17].

1. Trivial extensions of algebras. Let A be an algebra. We denote by $\mathrm{T}(A)$ the trivial extension $A \ltimes D(A)$ of A by its injective cogenerator $D(A)$. Recall that $\mathrm{T}(A) = A \oplus D(A)$ as K-vector spaces, and the multiplication in $\mathrm{T}(A)$ is defined by

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in A$ and $f, g \in {}_A D(A)_A$. It is known that $\mathrm{T}(A)$ is a symmetric algebra. The category $\operatorname{mod} \mathrm{T}(A)$ is equivalent to the category

$$\operatorname{mod} A \ltimes (- \otimes_A D(A))$$

whose objects are morphisms in $\operatorname{mod} A$ of the form $\alpha : X \otimes_A D(A) \to X$, with X from $\operatorname{mod} A$, satisfying the condition $\alpha(\alpha \otimes 1) = 0$. Moreover, if $\alpha : X \otimes_A D(A) \to X$ and $\beta : Y \otimes_A D(A) \to Y$ are objects of $\operatorname{mod} A \ltimes (- \otimes_A D(A))$ then a morphism $f : \alpha \to \beta$ in $\operatorname{mod} A \ltimes (- \otimes_A D(A))$ consists of a morphism $f : X \to Y$ in $\operatorname{mod} A$ such that $\beta(f \otimes 1) = f\alpha$, and the morphism composition in $\operatorname{mod} A \ltimes (- \otimes_A D(A))$ is the morphism composition in $\operatorname{mod} A$ (see [12, Section 1]). We identify $\operatorname{mod} \mathrm{T}(A)$ with $\operatorname{mod} A \ltimes (- \otimes_A D(A))$. Then the category $\operatorname{mod} A$ is identified with the full subcategory of $\operatorname{mod} \mathrm{T}(A)$ consisting of the zero homomorphisms $0 : X \otimes D(A) \to X$, where X runs through modules in $\operatorname{mod} A$.

We need the following proposition whose proof in the hereditary case can be found in [22]–[24].

PROPOSITION 1.1. *Let A be an algebra. Then $\tau_{\mathrm{T}(A)}X \cong \tau_A X$ for any module X in* $\operatorname{mod} A$ *with* $\operatorname{pd}_A X = 1$.

Proof. For modules X, Y in $\operatorname{mod} A$ and an A-homomorphism $\alpha : X \otimes_A D(A) \to Y$, we denote by (X, Y, α) the induced $T(A)$-module

$$(X \otimes_A D(A)) \oplus (Y \otimes_A D(A)) = (X \oplus Y) \otimes_A D(A) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right)} X \oplus Y.$$

Observe that then a module X from $\operatorname{mod} A$ is identified with the triple $(X, 0, 0) = (0, X, 0)$. Since $\mathrm{T}(A)$ is a symmetric algebra, the projective modules in $\operatorname{mod} \mathrm{T}(A)$ are injective, and are of the form $\widetilde{P} = (P, P \otimes_A D(A), 1)$, where P is a projective module in $\operatorname{mod} A$ and $1 : P \otimes_A D(A) \to P \otimes_A D(A)$ is the identity homomorphism. Moreover, we have $\tau_{T(A)} \cong \Omega^2_{T(A)}$ [3, (IV.3.8)].

Consider the Nakayama functor

$$\nu_A = D \operatorname{Hom}_A(-, A) : \operatorname{mod} A \to \operatorname{mod} A$$

on $\operatorname{mod} A$, equivalent to the functor $- \otimes_A D(A)$ (see [13, (2.1)]).

Let X be a module in $\operatorname{mod} A$ with $\operatorname{pd}_A X = 1$. Then X admits a minimal projective resolution in $\operatorname{mod} A$ of the form

$$0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$$

and we have in $\operatorname{mod} A$ an exact sequence

$$0 \to \tau_A X \to \nu_A(P_1) \xrightarrow{\nu_A(d_1)} \nu_A(P_0) \xrightarrow{\nu_A(d_0)} \nu_A(X) \to 0$$

(see [13, (2.5)]). Invoking the equivalence $\nu_A \cong - \otimes_A D(A)$ we get an exact sequence in $\operatorname{mod} A$ of the form

$$0 \to \tau_A X \to P_1 \otimes_A D(A) \xrightarrow{d_1 \otimes 1} P_0 \otimes_A D(A) \xrightarrow{d_0 \otimes 1} X \otimes_A D(A) \to 0.$$

We now calculate the second syzygy $\Omega^2_{\mathrm{T}(A)}X$ of the module $X = (X, 0, 0)$ in $\operatorname{mod} \mathrm{T}(A)$. Observe that the projective cover of X in $\operatorname{mod} \mathrm{T}(A)$ is of the form

$$\widetilde{P}_0 = (P_0, P_0 \otimes_A D(A), 1) \xrightarrow{(d_0, 0)} (X, 0, 0) \to 0$$

and hence $\Omega^1_{\mathrm{T}(A)}X \cong (P_1, P_0 \otimes_A D(A), d_1 \otimes 1)$. Then the projective cover of $\Omega^1_{\mathrm{T}(A)}X$ in $\operatorname{mod} \mathrm{T}(A)$ is of the form

$$\widetilde{P}_1 = (P_1, P_1 \otimes_A D(A), 1) \xrightarrow{\left(\begin{smallmatrix} 1_{P_1} & 0 \\ \alpha & d_1 \otimes 1 \end{smallmatrix}\right)} (P_1, P_0 \otimes_A D(A), d_1 \otimes 1) \to 0$$

and hence $\Omega^2_{T(A)}X \cong (0, \operatorname{Ker}(d_1 \otimes 1), 0) \cong (\tau_A X, 0, 0)$. Therefore, we obtain $\tau_A X \cong \Omega^2_{\mathrm{T}(A)}X \cong \tau_{\mathrm{T}(A)}X$. ■

Recall that a connected component $\mathscr{C}$ of an Auslander–Reiten quiver Γ_A is called *regular* if $\mathscr{C}$ contains neither a projective module nor an injective module.

As an immediate consequence of the above proposition we obtain the following useful fact.

COROLLARY 1.2. *Let A be an algebra and $\mathscr{C}$ a regular connected component in Γ_A consisting entirely of modules of projective dimension one. Then $\mathscr{C}$ is a regular connected component in $\Gamma_{\mathrm{T}(A)}$.*

2. The main result. Let A be an algebra. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, is said to be a *stable tube* of rank r. Therefore, a stable tube of rank r in Γ_A is an infinite connected component consisting of τ_A-periodic indecomposable A-modules having period r. Further, two connected components $\mathscr{C}$ and $\mathscr{D}$ in Γ_A are said to be *orthogonal* if $\mathrm{Hom}_A(X, Y) = 0$ and $\mathrm{Hom}_A(Y, X) = 0$ for all modules X in $\mathscr{C}$ and Y in $\mathscr{D}$. Following [19], a family $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ of connected components in Γ_A is said to be *generalized standard* if the restriction of the infinite radical $\mathrm{rad}^\infty(\mathrm{mod}\, A)$ of $\mathrm{mod}\, A$ to $\mathscr{C}$ is zero. We note that then the components $\mathscr{C}_i$, $i \in I$, are pairwise orthogonal.

Let A be an algebra. For modules M, N in $\mathrm{mod}\, A$ and a positive integer m, M is said to an *m-multiple subfactor* of N if there is a chain of submodules of N

$$X_0 \subset X_1 \subset \cdots \subset X_{m-1} \subset X_m \subset N$$

such that $X_i/X_{i-1} \cong M$ for $i \in \{1, \ldots, m\}$. A module X in $\mathrm{mod}\, A$ is said to be *sincere* if every simple A-module occurs as a composition factor (1-multiple subfactor) of X. A connected component $\mathscr{C}$ in Γ_A is called *sincere* if it contains an indecomposable sincere module. It is known that a stable tube $\mathcal{T}$ in Γ_A is sincere if and only if all but finitely many modules in $\mathcal{T}$ are sincere (see [17], [20]). In fact, a sincere stable tube $\mathcal{T}$ in Γ_A has the following stronger property: for any positive integer m, all but finitely modules X in $\mathcal{T}$ are *m-sincere*, that is, every simple A-module is an m-multiple subfactor of X.

The following theorem is the main result of the paper.

THEOREM 2.1. *Let A be an arbitrary algebra and M be a finite-dimensional right A-module. Let $p : \mathbb{P}_1(K) \to \mathbb{N}_1$ be a function with $p(\lambda) = 1$ for all but finitely many $\lambda \in \mathbb{P}_1(K)$. Then there exists an indecomposable symmetric algebra Λ and a generalized standard family $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, of sincere stable tubes in Γ_Λ such that the following statements hold*:

(i) *A is a factor algebra of Λ.*
(ii) *For each $\lambda \in \mathbb{P}_1(K)$, $\mathcal{T}_\lambda$ consists of indecomposable periodic modules having period $2p(\lambda)$.*
(iii) *For any $m \in \mathbb{N}_1$ and $\lambda \in \mathbb{P}_1(K)$, the A-module M is an m-multiple subfactor of all but finitely many modules in $\mathcal{T}_\lambda$.*

Proof. We identify $\mathbb{P}_1(K) = K \cup \{\infty\}$. Denote by $\Sigma(p)$ the set of all $\lambda \in \mathbb{P}_1(K)$ with $p(\lambda) > 1$. By assumption $\Sigma(p)$ is finite. We may assume that $\Sigma(p) = \{\lambda_1, \dots, \lambda_r\}$ is a subset of K and $\lambda_1 = 0$ (if $\Sigma(p)$ is nonempty). Moreover, we set $p_i = p(\lambda_i)$ for $i \in \{1, \dots, r\}$.

We assign to the A-module M the faithful right A-module $M_0 = A \oplus M$. Consider the one-point extension

$$A' = A[M_0] = \begin{bmatrix} K & M_0 \\ 0 & A \end{bmatrix}$$

of A by M_0. Since M_0 is a faithful A-module, A' admits a unique indecomposable projective faithful module Q' with $\operatorname{rad} Q' \cong M_0$. Next consider the one-point coextension

$$A_0 = [Q']A' = \begin{bmatrix} A' & D(Q') \\ 0 & K \end{bmatrix}$$

of A' by Q'. Then A_0 admits a unique indecomposable projective-injective faithful module P_0 with $P_0/\operatorname{soc} P_0 \cong Q'$. Observe that A_0 is the matrix algebra of the form

$$A_0 = \begin{bmatrix} K_0 & M_0 & K_0 \\ 0 & B_0 & D(M_0) \\ 0 & 0 & K_0 \end{bmatrix},$$

where $K_0 = K$, $B_0 = A$, and multiplication is given by the K-A-bimodule structure of M_0, A-K-bimodule structure of $D(M_0)$, the canonical K-K-bimodule structure of K, and the K-linear map $\varphi_0 : M_0 \otimes_A D(M_0) \to K$ given by $\varphi_0(m_0 \otimes f_0) = f_0(m_0)$ for $m_0 \in M_0$, $f_0 \in D(M_0)$.

For an integer $n \geq 2$, denote by $\mathrm{T}_n(K)$ the algebra of $n \times n$ upper triangular matrices over K. We note that $\mathrm{T}_n(K)$ is isomorphic to the path algebra of the equioriented quiver

$$1 \to 2 \to \cdots \to n-1 \to n.$$

A construction. Now, we construct a generalized canonical algebra

$$C = C(A, M, p) \tag{2.2}$$

using the algebra A_0, the module M and upper triangular matrix algebras depending on the function $p : \mathbb{P}_1(K) \to \mathbb{N}_1$. This is a minor extension of the construction presented in [21], taking into account that the algebra A (and hence A_0) is possibly not basic.

CASE 1. Assume $\Sigma(p)$ is empty. Then $C(A,M,p)$ is defined as

$$C(A,M,p)=\begin{bmatrix} K & M_0 & K_0\oplus K_1 \\ 0 & B_0 & D(M_0) \\ 0 & 0 & K \end{bmatrix},$$

where $K_0=K_1=K$, and the algebra structure is given by the algebra structure of A_0 and the canonical K-K-bimodule structure of $K_0\oplus K_1=K^2$.

CASE 2. Assume $\Sigma(p)$ is not empty. For $i\in\{1,\ldots,r\}$, we denote by A_i the upper triangular matrix algebra $\mathrm{T}_{p_i+1}(K)$. Observe that

$$A_i=\begin{bmatrix} K_i & M_i & K_i \\ 0 & B_i & D(M_i) \\ 0 & 0 & K_i \end{bmatrix},$$

where $K_i=K$, $B_i=\mathrm{T}_{p_i-1}(K)$ for $p_i\geq 3$ and $B_i=K$ for $p_i=2$, M_i is the unique indecomposable projective-injective B_i-module, and multiplication is given by the K-B_i-bimodule structure of M_i, the B_i-K-bimodule structure of $D(M_i)$, the K_i-K_i-bimodule structure of K_i, and the canonical K-linear map $\varphi_i: M_i\otimes_{B_i}D(M_i)\to K_i$ given by $\varphi_i(m_i\otimes f_i)=f_i(m_i)$ for $m_i\in M_i$, $f_i\in D(M_i)$. Consider the matrix algebra

$$R=\begin{bmatrix} K & M_0\oplus M_1\oplus\cdots\oplus M_r & K_0\oplus K_1\oplus\cdots\oplus K_r \\ 0 & B_0\oplus B_1\oplus\cdots\oplus B_r & D(M_0\oplus M_1\oplus\cdots\oplus M_r) \\ 0 & 0 & K \end{bmatrix},$$

where $K_0=K_1=\cdots=K_r=K$, and the algebra structure is given by the product algebra structure of $B=B_0\oplus B_1\oplus\cdots\oplus B_r$, the product K-B-bimodule structure of $M_0\oplus M_1\oplus\cdots\oplus M_r$, the product B-K-bimodule structure of $D(M_0\oplus M_1\oplus\cdots\oplus M_r)=D(M_0)\oplus D(M_1)\oplus\cdots\oplus D(M_r)$, the canonical K-K-bimodule structure of $K_0\oplus K_1\oplus\cdots\oplus K_r=K^{r+1}$, and the canonical K-bilinear map

$$\varphi_0:(M_0\oplus M_1\oplus\cdots\oplus M_r)\otimes_B D(M_0\oplus M_1\oplus\cdots\oplus M_r)\to K_0\oplus K_1\oplus\cdots\oplus K_r$$

induced by the maps $\varphi_i: M_i\otimes_{B_i}D(M_i)\to K_i$, $i=1,\ldots,r$.

CASE 2.1. If $r=1$, we put $C(A,M,p)=R$.

CASE 2.2. Assume $r\geq 2$. For $i\in\{0,1,\ldots,r\}$, denote by u_i the canonical basis vector of $K_0\oplus K_1\oplus\cdots\oplus K_r=K^{r+1}$ having 1 on the ith coordinate and 0 otherwise. Further, denote by U the subspace of $K_0\oplus K_1\oplus\cdots\oplus K_r$ generated by the elements

$$u_i+u_0+\lambda_i u_1,\qquad i=2,\ldots,r.$$

Then the vector space

$$I = \begin{bmatrix} 0 & 0 & U \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a two-sided ideal of R, and $C(A, M, p) = R/I$ is the required generalized canonical algebra.

Note that A is a factor algebra of $C(A, M, p)$.

If A (and hence A_0) is a basic algebra then $C = C(A, M, p)$ is a generalized canonical algebra as defined in [21, Section 2]. In general, C is Morita equivalent to the suitable generalized canonical algebra using the basic algebra of A. Applying [21, Theorem 2.1] (the proof works for an arbitrary field K) we conclude that the Auslander–Reiten quiver Γ_C of C admits a family $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal stable tubes such that:

(1) For each $\lambda_i \in \Sigma(p)$, $\mathcal{T}_{\lambda_i}$ is a stable tube of rank $p_i = p(\lambda_i)$.

(2) For each $\lambda \in \mathbb{P}_1(K) \setminus \Sigma(p)$, $\mathcal{T}_\lambda$ is a stable tube of rank $1 = p(\lambda)$.

Moreover, since by the above construction, the M_i are faithful B_i-modules for all $i \in \{0, 1, \ldots, r\}$, applying [20, Theorem 2.1] again, we also have

(3) For each $\lambda \in \mathbb{P}_1(K)$, $\mathcal{T}_\lambda$ is a faithful stable tube of Γ_C.

Recall that a component $\mathscr{C}$ of Γ_C is called *faithful* if the intersection $\operatorname{ann}\mathscr{C}$ of the annihilators of all modules in $\mathscr{C}$ is zero. In particular, every faithful component of Γ_C is sincere. We proved in [19, Lemma 5.3] that every faithful generalized standard component of an Auslander–Reiten quiver consists entirely of modules of projective dimension one. Therefore, we also have

(4) For each $\lambda \in \mathbb{P}_1(K)$, $\mathcal{T}_\lambda$ consists of indecomposable C-modules of projective dimension one.

Recall also that the *mouth* of a stable tube $\mathcal{T}$ in Γ_C is the unique τ_C-orbit of $\mathcal{T}$ formed by the modules having exactly one direct predecessor (and exactly one direct successor). In [20, Section 2], we presented an explicit description of modules forming the mouth of the considered family $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$. In particular, invoking the construction of the algebra A_0, we obtain

(5) For each $\lambda \in \mathbb{P}_1(K)$, the mouth of $\mathcal{T}_\lambda$ contains an indecomposable module having the module M as a subfactor. Moreover, for $\lambda_i \in \Sigma(p)$, the remaining modules on the mouth of $\mathcal{T}_{\lambda_i}$ are the $p_i - 1$ simple modules of the algebra B_i.

Since all modules of a stable tube $\mathcal{T}$ in Γ_C are obtained by suitable iterated extensions of modules lying on its mouth (see [17], [20]), applying (5) we conclude that

(6) For any $m \in \mathbb{N}_1$ and $\lambda \in \mathbb{P}_1(K)$, the module M is an m-multiple subfactor of all but finitely many modules in $\mathcal{T}$.

Finally, we define Λ to be the trivial extension algebra

$$\Lambda = C \ltimes D(C). \tag{2.3}$$

Then Λ is a symmetric algebra, C is a factor algebra of Λ, and the simple Λ-modules are the simple C-modules. Moreover, A is a factor algebra of Λ, because A is a factor algebra of C, and hence (i) holds. Further, it follows from Corollary 1.2 and the property (4) that $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, is a generalized standard family of stable tubes of Γ_Λ. Moreover, each $\mathcal{T}_\lambda$ is a sincere tube of Γ_Λ, because $\mathcal{T}_\lambda$ is a faithful (hence sincere) tube of Γ_C. We also know that $\tau_\Lambda = \Omega_\Lambda^2$, because Λ is symmetric. Invoking now properties (1) and (2) we conclude that, for each $\lambda \in \mathbb{P}_1(K)$, $\mathcal{T}_\lambda$ consists of indecomposable periodic modules of τ_Λ-period $p(\lambda)$ and the required Ω_Λ-period $2p(\lambda)$. Hence (ii) holds. Statement (iii) of the theorem follows from (6). ■

REMARK 2.4. If the field K is finite then we have a bound on the number of different periods $2p(\lambda)$ of periodic modules in the tubular family $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, occurring in Theorem 2.1. In [18], Ringel introduces the class of canonical algebras over an arbitrary field K, where the projective line $\mathbb{P}_1(K)$ is replaced by the (infinite) set $S({}_FM_G)$ of the isomorphism classes of simple regular representations of a (tame) bimodule ${}_FM_G$ over division rings F and G having K in the center and with $(\dim {}_FM)(\dim M_G) = 4$. One can extend in an obvious way our construction (2.2) of algebras $C = C(A, M, p)$ with $p : S({}_FM_G) \to \mathbb{N}_1$ and prove the corresponding version of Theorem 2.1.

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