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ISOMETRIES OF SPACES OF CONVEX COMPACT SUBSETS OF GLOBALLY NON-POSITIVELY BUSEMANN CURVED SPACES

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Abstract. We consider the Hausdorff metric on the space of compact convex subsets of a proper, geodesically complete metric space of globally non-positive Busemann curvature in which geodesics do not split, and characterize their surjective isometries. Moreover, an analogous characterization of the surjective isometries of the space of compact subsets of a proper, uniquely geodesic, geodesically complete metric space in which geodesics do not split is given.

1. Introduction. Let (X, d) be a metric space. For $A \subset X$, $r > 0$ we define the closed tubular neighborhood $N_r(A)$ of A of radius r as

$$N_r(A) := \{x \in X \mid \exists a \in A \text{ with } d(a, x) \le r\}.$$

For $p \in X$ we also write $B_r(p) := N_r(p)$. The sphere $S_r(p)$ of radius r around p is defined via $S_r(p) := \{x \in X \mid d(x, p) = r\}$. On the set $\mathcal{B} = \mathcal{B}(X, d)$ of closed, bounded subsets of X the map $d_{\mathrm{H}} : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_0^+$ given via

$$(1) \qquad d_{\mathrm{H}}(A, B) := \inf\{r \mid A \subset N_r(B) \wedge B \subset N_r(A)\}$$

defines the so-called *Hausdorff metric*.

In the late 70's and early 80's several authors started to investigate the relations of isometries of the Euclidean space $\mathbb{E}^n$ and those of the space $\mathcal{C}(\mathbb{E}^n)$ of its compact convex subsets endowed with the Hausdorff metric. Of course, given an isometry i of the Euclidean space, one derives an isometry I of the space $(\mathcal{C}(\mathbb{E}^n), d_{\mathrm{H}})$ by setting

$$I(C) := i(C) \quad \forall C \in \mathcal{C}(\mathbb{E}^n).$$

In [9] Schneider showed that these are the only surjective isometries of $(\mathcal{C}(\mathbb{E}^n), d_{\mathrm{H}})$. In [5] Gruber proved the same for the surjective isometries of $(\mathfrak{C}(\mathbb{E}^n), d_{\mathrm{H}})$, where $\mathfrak{C}(\mathbb{E}^n)$ denotes the set of compact subsets of $\mathbb{E}^n$; [7] generalizes these observations to certain non-Euclidean cases and raises the

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question whether a similar statement also holds for real hyperbolic spaces. To the best of our knowledge, this question has not been considered so far.

Recall that for a metric space (X, d) a *midpoint map* $m : X \times X \to X$ is a symmetric map satisfying $d(m(x,y),x) = \frac{1}{2}d(x,y) = d(m(x,y),y)$ for all $x, y \in X$.

The main purpose of this paper is to prove the following broad generalizations of the above-mentioned theorems of Schneider and Gruber:

THEOREM 1. *Let (X, d) be a proper, uniquely geodesic, geodesically complete metric space in which geodesics do not split and assume that the unique midpoint map m of (X, d) is convex. Let further I be a surjective isometry of $(\mathcal{C}(X, d), d_{\mathrm{H}})$. Then there exists an isometry $i \in \mathrm{Isom}(X, d)$ such that*

$$I(C) = i(C) \qquad \forall C \in \mathcal{C}(X, d).$$

THEOREM 2. *Let (X, d) be a proper, uniquely geodesic, geodesically complete metric space in which geodesics do not split, and let I be an isometry of $(\mathfrak{C}(X, d), d_{\mathrm{H}})$ onto itself. Then there exists an isometry $i \in \mathrm{Isom}(X, d)$ such that*

$$I(C) = i(C) \qquad \forall C \in \mathfrak{C}(X, d).$$

For the precise definitions of the notions involved in Theorems 1 and 2 we refer the reader to Sections 2.1 and 2.2. Note, however, that our theorems in particular apply to all proper, geodesically complete $CAT(0)$-spaces in which geodesics do not split, therefore for instance to all complete, connected, simply connected Riemannian manifolds of non-positive curvature and, moreover, to all finite-dimensional Banach spaces with strictly convex norm balls.

Outline of the paper. In Section 2.1 we recall some definitions and set up the notation we frequently use. In Section 2.2 convex midpoint maps are introduced, examples of which will be given in Section 2.3, where we also observe that one consequence of our Theorem 1 is the existence of a certain class of geodesics, which is invariant under isometries of the spaces considered. Here we also point out that this can be interpreted as a Mazur–Ulam type theorem for metric spaces.

Then, in Section 3, we prove Theorem 1, while the proof of Theorem 2 is the subject of Section 4.

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2. Preliminaries

2.1. *Basic definitions and notation.* Recall that a metric space is called *proper* if all its closed metric balls are compact. Further recall that for $a, b \in \mathbb{R}$ and $I \in \{(-\infty, b], [a, b], [a, \infty)\}$ an isometric embedding $\gamma : (I, |\cdot|) \to$

(X,d) is called a *geodesic* of (X,d). In case $I=[a,b]$ we say that γ connects $\gamma(a)$ to $\gamma(b)$.

A metric space (X,d) is said to be *geodesic* if for each $x,y\in X$ there exists a geodesic of (X,d) connecting x to y. Any such geodesic will be denoted by γ_{xy}. In general, it might not be unique.

We call a geodesic metric space (X,d) *geodesically complete* if each geodesic γ_{xy} connecting $x\in X$ to $y\in X$ has a biinfinite extension, i.e. a geodesic $\gamma:(-\infty,\infty)\to X$ such that $\mathrm{im}\{\gamma_{xy}\}\subset\mathrm{im}\{\gamma\}$. If for each geodesic γ_{xy} in (X,d), $x\neq y$, the image of this biinfinite extension is unique, we say that *geodesics do not split*.

A subset $C\subset X$ of a metric space (X,d) is called *convex* if together with any two points $a,b\in C$ it also contains the images of all geodesics connecting a to b. We denote by $\mathcal{C}(X,d)$ the set of convex, compact subsets of (X,d), by $\mathfrak{C}(X,d)$ the set of compact subsets of (X,d), and by $\mathrm{Isom}(X,d)$ the group of isometries of (X,d) onto itself.

Note that for $p\in X$ we have $\{p\}\in\mathcal{C}(X,d),\mathfrak{C}(X,d)$ and by a slight abuse of notation we will also write $p=\{p\}$.

Finally, recall that a $CAT(0)$*-space* is a geodesic metric space (X,d) such that any points $a\in\mathrm{im}\{\gamma_{xy}\}$ and $b\in\mathrm{im}\{\gamma_{xz}\}$ on a geodesic triangle $\Delta(\gamma_{xy},\gamma_{xz},\gamma_{yz})$ with vertices $x,y,z\in X$ lie not further apart than their corresponding comparison points $\overline{a},\overline{b}\in\mathbb{E}^2$ in a comparison triangle $\Delta(\gamma_{\overline{x}\overline{y}},\gamma_{\overline{x}\overline{z}},\gamma_{\overline{y}\overline{z}})$ in $\mathbb{E}^2$. Here a *comparison triangle* for $\Delta(\gamma_{xy},\gamma_{xz},\gamma_{yz})$ is a geodesic triangle in $\mathbb{E}^2=(\mathbb{R}^2,d_e)$ with vertices $\overline{x},\overline{y},\overline{z}\in\mathbb{E}^2$ such that $d(x,y)=d_e(\overline{x},\overline{y})$, $d(x,z)=d_e(\overline{x},\overline{z})$ and $d(y,z)=d_e(\overline{y},\overline{z})$, and the comparison points $\overline{a}\in\mathrm{im}\{\gamma_{\overline{x}\overline{y}}\}$ and $\overline{b}\in\mathrm{im}\{\gamma_{\overline{x}\overline{z}}\}$ are determined via $d(x,a)=d_e(\overline{x},\overline{a})$ and $d(x,b)=d_e(\overline{x},\overline{b})$.

Note that in a $CAT(0)$-space geodesics connecting two points are unique and metric balls are convex (see e.g. Proposition II.1.4 in [1]).

2.2. *Convex midpoint maps.* In this section we introduce the notion of (convex) midpoint maps in a metric space (X,d).

Definition 1. Let (X,d) be a metric space. A symmetric map $m:X\times X\to X$ is called a *midpoint map* for (X,d) if

$$d(m(x,y),x)=\tfrac{1}{2}d(x,y)=d(m(x,y),y)\quad\forall x,y\in X.$$

Furthermore, the midpoint map m is called *convex* if

$$d(m(x_1,y_1),m(x_2,y_2))\le\tfrac{1}{2}[d(x_1,x_2)+d(y_1,y_2)]\quad\forall x_1,x_2,y_1,y_2\in X.$$

Remark 1. Assuming that the underlying metric space (X,d) is complete, such a midpoint map corresponds to a certain class of geodesics in (X,d): Given two points $x,y\in X$, in the first step we add the point $m(x,y)$. In the second step we add the two points $m(x,m(x,y))$ and $m(m(x,y),y)$. Proceeding like that, in the nth step we add 2^{n-1} points. Since the metric space is complete, this procedure determines a distinguished geodesic segment connecting x to y. We will refer to such a segment as an *m-geodesic segment*.

Definition 2. Let (X,d) be a metric space and $m : X \times X \to X$ be a midpoint map for (X,d). Then (X,d) is said to be

(i) *m-distance convex* if

$$d(m(x,y),z) \leq \tfrac{1}{2}[d(x,z)+d(y,z)] \quad \forall x,y,z \in X;$$

(ii) *m-global non-positively Busemann curved* (m-global NPBC) if

$$d(m(z,x),m(z,y)) \leq \tfrac{1}{2}d(x,y) \quad \forall x,y,z \in X.$$

For an investigation of the notion of distance convexity we refer the reader to [4].

The following lemma is a simple consequence of Definitions 1 and 2:

Lemma 1. *Let (X,d) be a metric space and $m : X \times X \to X$ be a convex midpoint map. Then*

(1) *m is continuous,*
(2) *(X,d) is m-distance convex,*
(3) *(X,d) is m-global NPBC.*

In fact, (X,d) being m-global NPBC is a sufficient condition for the midpoint map m to be convex:

Lemma 2. *Let (X,d) be a metric space and $m : X \times X \to X$ be a midpoint map for (X,d). Then m is a convex midpoint map if and only if (X,d) is m-global NPBC.*

Proof. Due to Lemma 1 we only have to show that (X,d) being m-global NPBC is a sufficient condition for m being convex. Let therefore $x_1, x_2, y_1, y_2 \in X$. Then one has

$$\begin{aligned} d(m(x_1,y_1),m(x_2,y_2)) &\leq d(m(x_1,y_1),m(x_2,y_1)) + d(m(x_2,y_1),m(x_2,y_2)) \\ &\leq \tfrac{1}{2}[d(x_1,x_2)+d(y_1,y_2)]. \end{aligned}$$

Thus m is indeed convex. ∎

Example 1. Let $(V, \|\cdot\|)$ be a normed vector space. Then

$$\mathfrak{m}(x,y) := \frac{x+y}{2} \quad \forall x,y \in V$$

is a convex midpoint map.

If V is finite-dimensional, then it is not hard to see that $\mathfrak{m}$ as defined above is the only convex midpoint map in $(V, \|\cdot\|)$. Whether or not this generalizes to infinite dimensions is not known to the author.

Given a convex midpoint map m in a metric space (X,d) and an isometry $I \in \mathrm{Isom}(X,d)$, obviously $(I \circ m)(I^{-1}(\cdot), I^{-1}(\cdot))$ is again a convex midpoint map. Thus, establishing the uniqueness of a convex midpoint map in a complete metric space (X,d) gives rise to a class of distinguished geodesics (compare Remark 1) which is invariant under any isometry $I \in \mathrm{Isom}(X,d)$. Unfortunately the author is not aware of a metric space admitting two different convex midpoint maps. However, if two such midpoint maps exist in a metric space, then there are infinitely many:

LEMMA 3. *Let (X,d) be a metric space and $m_1, m_2 : X \times X \to X$ be two convex midpoint maps for (X,d). Then the map $\widetilde{m} : X \times X \to X$ defined via*

$$\widetilde{m}(x,y) := m_1(m_1(x,y), m_2(x,y)) \quad \forall x,y \in X$$

is also a convex midpoint map for (X,d).

Proof. That $\widetilde{m}$ is a midpoint map follows simply from the m_1-distance convexity of (X,d) (Lemma 1). The convexity of $\widetilde{m}$ follows from

$$\begin{aligned} d(\widetilde{m}(x_1,y_1), \widetilde{m}(x_2,y_2)) & \\ &\leq \tfrac{1}{2}[d(m_1(x_1,y_1), m_1(x_2,y_2)) + d(m_2(x_1,y_1), m_2(x_2,y_2))] \\ &\leq [d(x_1,x_2) + d(y_1,y_2)] \quad \forall x_1,x_2,y_1,y_2 \in X. \ \blacksquare \end{aligned}$$

2.3. *Spaces of closed, bounded, convex sets.* Note that for a proper metric space we have $\mathcal{B} = \mathfrak{C}(X,d)$.

Given a midpoint map m for (X,d), we call a set $A \subset X$ *m-convex* if together with any two points $a, a' \in A$ it also contains their m-midpoint: $m(a,a') \in A$. The *m-convex hull*, $\mathrm{conv}_m(A)$, of a set $A \subset X$ is defined via

$$\mathrm{conv}_m(A) := \bigcap \{C \mid C \text{ is closed and } m\text{-convex} \wedge A \subset C\}.$$

Denoting by $\mathcal{C}_m$ the set of m-convex elements of $\mathcal{B}$, we write $\mathrm{conv}_m : \mathcal{B} \to \mathcal{C}_m$ for the map which associates to an $A \in \mathcal{B}$ its m-convex hull $\mathrm{conv}_m(A)$. A function $f : X \to \mathbb{R}$ is called *m-convex* if

$$f(m(x,y)) \leq \tfrac{1}{2}[f(x) + f(y)] \quad \forall x,y \in X.$$

With this terminology it is easy to prove

LEMMA 4. *Let (X,d) be a metric space and $m : X \times X \to X$ be a convex midpoint map for (X,d). Let further $C \subset X$ be a closed m-convex set in (X,d). Then the map*

$$d_C : X \to \mathbb{R}_0^+, \quad x \mapsto \mathrm{dist}(x,C),$$

is m-convex.

Furthermore one obtains

Lemma 5 ([3]). *The map* $\operatorname{conv}_m : \mathcal{B} \to \mathcal{C}_m$ *is* 1*-Lipschitz and does not change the diameter.*

Proof. Connecting $b, b' \in B \in \mathcal{B}$ by an m-geodesic segment (see Remark 1) increases neither $\operatorname{diam} B$ nor the Hausdorff distance to any $B' \in \mathcal{B}$ by convexity of the distance function. The claim follows since $\operatorname{conv}_m(B)$ coincides with the closure of $\bigcup_n B_n$, where $B_0 := B$ and B_{n+1} is obtained from B_n by connecting each pair of points $b, b' \in B_n$ by an m-geodesic segment. ∎

Proposition 1. *Let* (X, d) *be a metric space and* $m : X \times X \to X$ *be a convex midpoint map for* (X, d)*. Then the map* $M : \mathcal{C}_m \times \mathcal{C}_m \to \mathcal{C}_m$ *defined via*

$$M(A, A') := \operatorname{conv}_m(\{x \in X \mid \exists a \in A, a' \in A' \textit{ such that } x = m(a, a')\}) \qquad \forall A, A' \in \mathcal{C}_m$$

is a convex midpoint map for $(\mathcal{C}_m, d_{\mathrm{H}})$*.*

Proof. (1) M *is a midpoint map*: Let $\widetilde{M} \subset X$ be the set of midpoints $m(a, a')$ for all $a \in A$, $a' \in A'$. We set $\lambda := \frac{1}{2} d_{\mathrm{H}}(A, A')$ and assume that there exists $b \in \widetilde{M}$ with $\operatorname{dist}(b, A) > \lambda$. Then $b = m(a, a')$ with $a \in A$ and $a' \in A'$. Since A is m-convex, the distance function d_A is m-convex (Lemma 4). Thus $\operatorname{dist}(a', A) \geq 2 \operatorname{dist}(b, A)$, because $\operatorname{dist}(a, A) = 0$. Hence, $\operatorname{dist}(a', A) > d_{\mathrm{H}}(A, A')$, contradicting the definition of $d_{\mathrm{H}}(A, A')$. This shows that $\widetilde{M}$ lies in the closed λ-neighborhood of A, $N_\lambda(A)$.

On the other hand, for each $a \in A$ there is $b \in \widetilde{M}$ with $d(b, a) \leq \lambda$: let $b := m(a, a')$, where $a' \in A'$ is the closest point to a, thus $d(a, a') \leq 2\lambda$. This shows that $A \subset N_\lambda(\widetilde{M})$. Thus $d_{\mathrm{H}}(\widetilde{M}, A) \leq \lambda$ and, similarly, $d_{\mathrm{H}}(\widetilde{M}, A') \leq \lambda$. By the triangle inequality we have $2\lambda \leq d_{\mathrm{H}}(A, \widetilde{M}) + d_{\mathrm{H}}(\widetilde{M}, A') \leq 2\lambda$ and hence

$$d_{\mathrm{H}}(\widetilde{M}, A) = \lambda = d_{\mathrm{H}}(\widetilde{M}, A').$$

For the m-convex hull $M = \operatorname{conv}_m(\widetilde{M})$ we have $d_{\mathrm{H}}(M, A), d_{\mathrm{H}}(M, A') \leq \lambda$ by Lemma 5. Hence, $d_{\mathrm{H}}(M, A) = \lambda = d_{\mathrm{H}}(M, A')$ and M is indeed a midpoint map for $(\mathcal{C}_m, d_{\mathrm{H}})$.

(2) M *is convex*: We need to show that

$$d_{\mathrm{H}}(M(A, A'), M(B, B')) \leq \tfrac{1}{2}[d_{\mathrm{H}}(A, B) + d_{\mathrm{H}}(A', B')] \quad \forall A, B, A', B' \in \mathcal{C}_m.$$

Let $A, A', B, B' \in \mathcal{C}_m$ and set $r_0 := d_{\mathrm{H}}(A, B)$ and $r_0' := d_{\mathrm{H}}(A', B')$. All we have to prove is that given $x \in M(A, A')$, there exists a $y \in M(B, B')$ such that $d(x, y) \leq (r_0 + r_0')/2$.

(i) Suppose first that $x = m(a, a')$ for some $a \in A$ and $a' \in A'$. Due to the definition of r_0 and r_0' there exist $b \in B$ and $b' \in B'$ such that $d(a, b) \leq r_0$ and $d(a', b') \leq r_0'$. Now m is a convex midpoint map for (X, d), and for

$y := m(b, b') \in M(B, B')$ we find

$$d(y, x) = d(m(b, b'), m(a, a')) \leq \tfrac{1}{2}[d(b, a) + d(b', a')] \leq \frac{r_0 + r_0'}{2}.$$

(ii) For a general $x \in M(A, A')$ the existence of $y \in M(B, B')$ with $d(x, y) \leq (r_0 + r_0')/2$ just follows by induction and the fact that the convex midpoint map is continuous (see Lemma 1): In order to make this more precise, write $M_0 := \widetilde{M}$ and let M_{n+1} be obtained from M_n by adding to M_n the midpoints of all pair of points in M_n. By induction we find that for each $n \in \mathbb{N}$ every point in M_n is at distance less than or equal to $(r_0 + r_0')/2$ from a point in $M(B, B')$, due to the convexity of the midpoint map. Thus, since M is the closure of $\bigcup_n M_n$ and m is continuous, the same already holds for $M(A, A')$. ∎

PROPOSITION 2. *Let (X, d) be a proper metric space and $m : X \times X \to X$ be a convex midpoint map for (X, d). Then the map $\mathfrak{M} : \mathcal{C}_m \times \mathcal{C}_m \to \mathcal{C}_m$ defined via*

$$\mathfrak{M}(A, B) := N_{d_{\mathrm{H}}(A,B)/2} A \cap N_{d_{\mathrm{H}}(A,B)/2} B \quad \forall A, B \in \mathcal{C}_m \tag{2}$$

is a midpoint map for $(\mathcal{C}_m, d_{\mathrm{H}})$.

Proof. From (2) it follows that

$$\mathfrak{M}(A, B) \subset N_{d_{\mathrm{H}}(A,B)/2}(A) \quad \wedge \quad \mathfrak{M}(A, B) \subset N_{d_{\mathrm{H}}(A,B)/2}(B). \tag{3}$$

With M as in Proposition 1 we have

$$d_{\mathrm{H}}(A, M(A, B)) = d_{\mathrm{H}}(A, B)/2 = d_{\mathrm{H}}(B, M(A, B)).$$

Thus we find

$$M(A, B) \subset N_{d_{\mathrm{H}}(A,B)/2}(A) \quad \wedge \quad M(A, B) \subset N_{d_{\mathrm{H}}(A,B)/2}(B) \tag{4}$$

as well as

$$A \subset N_{d_{\mathrm{H}}(A,B)/2}(M(A, B)) \quad \wedge \quad B \subset N_{d_{\mathrm{H}}(A,B)/2}(M(A, B)). \tag{5}$$

Now (2) and (4) yield $M(A, B) \subset \mathfrak{M}(A, B)$. This together with (5) implies

$$A \subset N_{d_{\mathrm{H}}(A,B)/2}(\mathfrak{M}(A, B)) \quad \wedge \quad B \subset N_{d_{\mathrm{H}}(A,B)/2}(\mathfrak{M}(A, B)),$$

which, combined with (3), yields

$$d_{\mathrm{H}}(A, \mathfrak{M}(A, B)), d_{\mathrm{H}}(B, \mathfrak{M}(A, B)) \leq d_{\mathrm{H}}(A, B)/2,$$

so that the triangle inequality for d_{H} implies

$$d_{\mathrm{H}}(A, \mathfrak{M}(A, B)) = d_{\mathrm{H}}(A, B)/2 = d_{\mathrm{H}}(B, \mathfrak{M}(A, B)).$$

Finally, the facts that (X, d) is proper and m is convex imply that $\mathfrak{M}(A, B) \in \mathcal{C}_m$. ∎

It is easy to see that, in contrast to M, $\mathfrak{M}$ is not convex in general:

Example 2. Consider Cartesian coordinates in $\mathbb{E}^2$ and define $A, B, C, D \in \mathcal{C}_m(\mathbb{E}^2)$ as follows:

$$A := \{(-1,1)\}, \quad B := \{(1,1)\}, \quad C := B_1((-1,0)), \quad D := B_1((1,0)).$$

Then $\mathfrak{M}(A,B) = \{(0,1)\}$ and $\mathfrak{M}(C,D) = B_2((-1,0)) \cap B_2((1,0))$, and thus

$$d_{\mathrm{H}}(\mathfrak{M}(A,B), \mathfrak{M}(C,D)) = 1 + \sqrt{3} > 2 = \tfrac{1}{2}[d_{\mathrm{H}}(A,C) + d_{\mathrm{H}}(B,D)].$$

Along the lines of the proof of Proposition 2 one also obtains

Proposition 3. *Let (X,d) be a proper metric space and $m : X \times X \to X$ be a midpoint map for (X,d). Then the map $\widetilde{\mathfrak{M}} : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ defined via*

$$\widetilde{\mathfrak{M}}(A,B) := N_{d_{\mathrm{H}}(A,B)/2} A \cap N_{d_{\mathrm{H}}(A,B)/2} B \quad \forall A, B \in \mathfrak{C}$$

is a midpoint map for $(\mathfrak{C}(X,d), d_{\mathrm{H}})$.

Note that Theorem 1 implies that the class of distinguished geodesics in $(\mathcal{C}(X,d), d_{\mathrm{H}})$ determined by the midpoint map M (compare Remark 1) is invariant under any isometry of $(\mathcal{C}(X,d), d_{\mathrm{H}})$ onto itself.

This can be interpreted as a Mazur–Ulam type statement for these metric spaces. Recall that the famous Mazur–Ulam Theorem (see [8]) states that the surjective isometries from a normed vector space onto itself are linear up to translations, i.e. that they map straight lines onto straight lines, thus leaving invariant a certain class of geodesics determined by the convex midpoint map as given in Example 1. (For an astonishingly nice and simple proof of the Mazur–Ulam Theorem see also [10].)

3. The proof of Theorem 1. In this section we prove Theorem 1. The strategy of this proof is the same as those given in [9] and [5] for the Euclidean case: First we establish that images of points are points, i.e. $i \in \mathrm{Isom}(X,d)$ given via $i(p) := I(p)$ is well defined. Then we prove that the isometry $J \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$ given via $J(C) := (i^{-1} \circ I)(C)$ for all $C \in \mathcal{C}(X,d)$ is the identity.

Lemma 6. *Let (X,d) be a geodesic metric space such that geodesics do not split, and let $p \in X$ and $A, B \in \mathfrak{C}(X,d)$ be such that*

$$d_{\mathrm{H}}(p,A) = \tfrac{1}{2} d_{\mathrm{H}}(A,B) = d_{\mathrm{H}}(p,B).$$

Then $\min\{\#A, \#B\} = 1$.

Proof. Let $h := d_{\mathrm{H}}(A,p) = d_{\mathrm{H}}(B,p)$. Since A and B are compact, there exist $a \in A$, $b \in B$ such that $d(a,b) = 2h$ as well as $d(a,b') \geq 2h$ for all $b' \in B$ or $d(a',b) \geq 2h$ for all $a' \in A$. Without loss of generality we assume that $d(a,b') \geq 2h$ for all $b' \in B$. Then $\#B = 1$, since for $a \in S_h(p)$ there exists a unique $b \in B_h(p)$ with $d(a,b) \geq 2h$ and thus $B = \{b\}$. ∎

LEMMA 7. *Let (X,d) be a uniquely geodesic, geodesically complete metric space in which geodesics do not split, and let $I \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$ be such that there exists $p \in X$ with $\#I(p) = 1$. Then $\#I(q) = 1$ for all $q \in X$.*

Proof. Let $q \in X$ and choose $\widetilde{p} \in X$ such that

$$d(\widetilde{p}, p) = \tfrac{1}{2} d(\widetilde{p}, q) = d(p,q).$$

Then p is the unique midpoint between $\widetilde{p}$ and q in $(\mathcal{C}(X,d), d_{\mathrm{H}})$ and, since $I \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$, $I(p)$ is clearly the unique midpoint between $I(\widetilde{p})$ and $I(q)$ in $(\mathcal{C}(X,d), d_{\mathrm{H}})$. Therefore $M(I(\widetilde{p}), I(q)) = I(p) \in X$ with M as defined in Proposition 1, which yields $\#I(q) = 1$. ∎

LEMMA 8. *Let (X,d) be a proper, uniquely geodesic, geodesically complete metric space such that geodesics do not split and the midpoint map is convex. Let further $I \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$. Then $\#I(p) = 1$ for all $p \in X$.*

Proof. Suppose there exists $A \in \mathcal{C}(X,d)$ with $\#A > 1$ and $\#I(A) = 1$, i.e. $I(A) \in X$. Let $r := \operatorname{diam} A \neq 0$ and $q \in A$ be such that there exists $\widetilde{q} \in A$ with $d(q, \widetilde{q}) = r$.

For each $x \in I(q)$ we choose $y(x)$ such that

$$d(I(A), x) = d(x, y(x)) = \tfrac{1}{2} d(I(A), y(x)),$$

set $\widetilde{Q} = \bigcup_{x \in I(q)} \{y(x)\}$ and write Q for the closed convex hull of $\widetilde{Q}$. It immediately follows that $d_{\mathrm{H}}(I(A), I(q)) = \frac{1}{2} d_{\mathrm{H}}(I(A), Q)$ and thus

$$d_{\mathrm{H}}(Q, I(q)) \geq d_{\mathrm{H}}(Q, I(A)) - d_{\mathrm{H}}(I(A), I(q)) = \tfrac{1}{2} d_{\mathrm{H}}(Q, I(A)).$$

To prove the opposite inequality, $d_{\mathrm{H}}(Q, I(q)) \leq \frac{1}{2} d_{\mathrm{H}}(Q, I(A))$, we have to show that for all $z \in Q$ there exists $z' \in I(q)$ such that

$$d(z, z') \leq \tfrac{1}{2} d_{\mathrm{H}}(Q, I(A)) = d_{\mathrm{H}}(I(A), I(q)).$$

This is obviously true for all $z \in \widetilde{Q}$. Next let $z \in Q$ be such that there exist $y_1, y_2 \in \widetilde{Q}$ with $d(y_1, z) = d(y_2, z) = \frac{1}{2} d(y_1, y_2)$. Then there exist $x_1, x_2, x \in I(q)$ such that $d(x_1, y_1), d(x_2, y_2) \leq d_{\mathrm{H}}(I(A), I(q))$ and $d(x, x_1) = d(x, x_2) = \frac{1}{2} d(x_1, x_2)$. Since m is convex, we derive

$$d(x, z) \leq \tfrac{1}{2} d(x_1, y_1) + \tfrac{1}{2} d(x_2, y_2) \leq d_{\mathrm{H}}(I(A), I(q)).$$

The claim for general $z \in Q$ now follows by induction, applying the same argument again and again, the definition of Q and the fact that (X,d) is complete.

Thus we find

$$d_{\mathrm{H}}(A, q) = d_{\mathrm{H}}(q, I^{-1}(Q)) = \tfrac{1}{2} d_{\mathrm{H}}(A, I^{-1}(Q))$$

and it follows from Lemma 6 that $\#I^{-1}(Q) = 1$.

Let $p := I^{-1}(Q) \in X$. Then there exists $z \in A$ such that $d(z,q) = d(q,p) = \frac{1}{2} d(z,p)$ and q is the unique midpoint between p and z in $(\mathcal{C}(X,d), d_{\mathrm{H}})$.

Therefore $I(q)$ is also the unique midpoint between $I(p)$ and $I(z)$ in $(\mathcal{C}(X,d), d_{\mathrm{H}})$. From Propositions 1 and 2 it follows that

$$I(q) = M(I(z), I(p)) = \mathfrak{M}(I(z), I(p))$$

with M and $\mathfrak{M}$ as defined in Propositions 1 and 2.

Next we prove

(i) $I(A) \in I(z)$: Since $d_{\mathrm{H}}(I(A), I(q)) = d_{\mathrm{H}}(A, q) = r$, it follows that there exists $x \in I(q)$ with $d(I(A), x) = r$. From $I(q) = \mathfrak{M}(I(z), I(p))$ we deduce that there exists $z' \in I(z)$ with $z' \in B_r(x)$. Since geodesics do not split we also know that for all $\widetilde{z} \in B_r(x) \setminus I(A)$ we have $d(\widetilde{z}, y(x)) < 2r$ for $y(x)$ as in the definition of $Q = I(p)$. Thus $z' = I(A)$, for otherwise $I(q) = M(I(z), I(p))$ yields the existence of an $x' \in I(q)$ with $d(x', I(A)) > r$, a contradiction.

Now we establish

(ii) $I(A) \in I(q)$: Without loss of generality $\# I(q) > 1$, for otherwise the claim of the lemma follows from Lemma 7. Thus, since $I(q)$ is convex and $B_r(I(A))$ is strictly convex, there exists $x' \in I(q)$ such that $d(I(A), x') = \mathrm{dist}(I(A), I(q)) =: r - \varepsilon < r$. Suppose now $I(A) \notin I(q)$, i.e. $\mathrm{dist}(I(A), I(q)) > 0$, and denote the midpoint between $a, b \in X$ by $m(a,b)$. Then, since $I(q) = M(I(z), I(p))$ and $I(A) \in I(z)$, we have $m(I(A), y(x')) \in I(q)$; but $d(I(A), m(I(A), y(x'))) = r - 2\varepsilon$, contradicting $\mathrm{dist}(I(A), I(q)) = r - \varepsilon$.

(iii) Now $I(A) \in I(q)$ of course implies $I(A) \in I(p)$. On the other hand, since $r = d_{\mathrm{H}}(z, A) = d_{\mathrm{H}}(I(z), I(A))$, there exists $z_0 \in I(z)$ with $d(z_0, I(A)) = r$. Now $m(z_0, I(A)) \in I(q)$, from which we conclude $z_0 \in I(p)$ and thus $z_0 = m(z_0, z_0) \in I(q)$, due to $I(q) = M(I(z), I(p))$. But then $y(z_0) \in I(p)$ and, once again due to $I(q) = M(I(z), I(p))$, we have $m(y(z_0), z_0) \in I(q)$. This, however, contradicts $d_{\mathrm{H}}(I(A), I(q)) = r$, since $d(y(z_0), I(A)) = \frac{3}{2}r$. ∎

LEMMA 9. *Let* (X,d) *be as in Theorem* 1, $I \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$, $i \in \mathrm{Isom}(X,d)$ *defined via* $i(p) := I(p)$ *for all* $p \in X$, *and* $J := i^{-1} \circ I \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$. *Then for all* $p \in X$ *and* $r > 0$,

$$J(B_r(p)) = B_r(p).$$

Proof. From the definition of J it follows that $J(p) = p$ for all $p \in X$. Thus we find

$$r = d_{\mathrm{H}}(p, B_r(p)) = d_{\mathrm{H}}(J(p), J(B_r(p))) = d_{\mathrm{H}}(p, J(B_r(p))),$$

which yields $J(B_r(p)) \subset B_r(p)$. It now suffices to prove that $S_r(p) \subset J(B_r(p))$. Under our assumptions, for all $q \in S_r(p)$ there exists a unique $\widetilde{q} \in B_r(p)$ such that $d(q, \widetilde{q}) = 2r$. Then

$$2r = d_{\mathrm{H}}(\widetilde{q}, B_r(p)) = d_{\mathrm{H}}(\widetilde{q}, J(B_r(p)))$$

and thus $q \in J(B_r(p))$. ∎

Now we are ready to provide

Proof of Theorem 1. Let $I \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$, and let $i \in \mathrm{Isom}(X,d)$ and $J \in \mathrm{Isom}(\mathcal{C}(X,d), d_{\mathrm{H}})$ be as defined in Lemma 9. All we have to prove is that $J(C) = C$ for all $C \in \mathcal{C}(X,d)$.

Suppose that there exists $p \in C \setminus J(C)$. Since $J(C)$ is compact, there exists a $q \in J(C)$ such that $\mathrm{dist}(p, J(C)) = d(p,q)$. Now let n be such that $C, J(C) \subset B_{(2^n-1)d(p,q)}(p)$. Then

$$d_{\mathrm{H}}(J(C), B_{(2^n-1)d(p,q)}(p)) = d_{\mathrm{H}}(C, B_{(2^n-1)d(p,q)}(p)) \leq (2^n-1)d(p,q). \tag{6}$$

Let $p_n \in S_{(2^n-1)d(p,q)}(p)$ be such that $d(p_n,q) = d(p_n,p)+d(p,q) = 2^n d(p,q)$. From Lemma 4 we know that $\mathrm{dist}(J(C),\cdot)$ is m-convex. Thus

$$\begin{aligned}\mathrm{dist}(p_n, J(C)) &\geq 2\,\mathrm{dist}(m(p_n,q), J(C)) \\ &\geq 2^2\,\mathrm{dist}(m(m(p_n,q),q), J(C)) \\ &\geq \cdots \geq 2^n d(p,q).\end{aligned}$$

and therefore

$$\mathrm{dist}(J(C), p_n) = 2^n d(p,q),$$

contradicting (6). This proves $C \subset J(C)$. Of course, the same argument with J replaced by J^{-1} yields $J(C) \subset C$ and therefore $J(C) = C$. ∎

4. The proof of Theorem 2. In this section we prove Theorem 2. This proof is even more closely modeled on the classical one dealing with the Euclidean space. In fact, once Lemma 12 is established in our more general setting, Gruber's original proof essentially also works in this setting (see [6]).

LEMMA 10 (see (2) in [5]). *Let (X,d) be a proper, uniquely geodesic metric space, $I \in \mathrm{Isom}(\mathfrak{C}(X,d), d_{\mathrm{H}})$ and $p,q \in X$, $p \neq q$. Then*

$$I(p) \subset \partial N_{d(p,q)}(I(q)).$$

Proof. Suppose to the contrary $I(p) \not\subset \partial N_{d(p,q)}(I(q))$. We have

$$d_{\mathrm{H}}(I(p), I(q)) = d(p,q)$$

and the compactness of $I(p)$ and $I(q)$ yields $I(p) \subset N_{d(p,q)}(I(q))$. Thus for $\widetilde{\mathfrak{M}}(I(p), I(q))$ as in Proposition 3 we deduce $\widetilde{\mathfrak{M}}(I(p), I(q))^\circ \neq \emptyset$, i.e., the interior of $\widetilde{\mathfrak{M}}(I(p), I(q))$ is not-empty.

From $\widetilde{\mathfrak{M}}(I(p), I(q))$ we remove a non-empty open subset contained in $\widetilde{\mathfrak{M}}(I(p), I(q))^\circ$ of diameter $< d(p,q)/2$, obtaining a set D. Now it is easy to see that $D \neq \widetilde{\mathfrak{M}}(I(p), I(q))$ is also a midpoint between $I(p)$ and $I(q)$ in $(\mathfrak{C}(X,d), d_{\mathrm{H}})$, contradicting the uniqueness of the midpoint. ∎

Lemma 11. *Let (X, d) be a proper, uniquely geodesic, geodesically complete metric space in which geodesics do not split. If $I \in \mathrm{Isom}(\mathfrak{C}(X,d), d_{\mathrm{H}})$, then $\#I(p) = 1$ for all $p \in X$.*

Proof. Suppose there exists $A \in \mathfrak{C}(X,d)$ with $\#A > 1$ and $\#I(A) = 1$. Then with the notation as in the proof of Lemma 8 we find that $I(q)$ is a midpoint between $I(A)$ and $\widetilde{Q}$ and there exists $p \in X$ such that $I(p) = \widetilde{Q}$. Lemma 10 applied to z and p as well as to z and q yields $I(A) \in I(z)$, from which together with Lemma 10 it follows that $I(q) \in S_r(I(A))$. The same argument, of course, yields $I(z) \in S_r(I(A))$, which clearly contradicts $I(A) \in I(z)$. ∎

Lemma 12. *Let $S \subset S_r(p)$ with $\#S < \infty$. Then, with J as defined in Lemma 9, $J(S) = S$.*

Proof. Since $d_{\mathrm{H}}(p, S) = d_{\mathrm{H}}(J(p), J(S)) = d_{\mathrm{H}}(p, J(S))$, we find on the one hand

$$J(S) \subset B_r(p). \tag{7}$$

On the other hand,

$$J(S) \cap S_r(p) = S. \tag{8}$$

In order to see this, let $q \in S$. Then there exists a unique $\widetilde{q} \in B_r(p)$ with $d(q, \widetilde{q}) = 2r$. Moreover, $d(\widetilde{q}, q') < 2r$ for all $q' \in B_r(p)$, $q' \neq q$. But $2r = d_{\mathrm{H}}(\widetilde{q}, S) = d_{\mathrm{H}}(\widetilde{q}, J(S))$, hence the inclusion (7) implies $q \in J(S)$, which yields $S \subset J(S)$. The opposite inclusion just follows by an analogous argument interchanging the roles of S and $J(S)$.

Furthermore, the same argument yields

$$S_R(p) \subset J(B_R(p)) \subset B_R(p) \quad \forall R \geq 0. \tag{9}$$

From (7) and (8) the claim obviously follows, once we establish that

$$J(S) \cap B_r^\circ(p) = \emptyset.$$

Suppose to the contrary that there exists $q \in J(S) \cap B_r^\circ(p)$ and let $\mu := (r - d(p,q))/2 > 0$. Since $\mu \leq r/2$, we find $N_{r-\mu}(S_\mu(p)) = B_r(p)$. From this and the inclusion (7) it follows that

$$N_{r-\mu}(S_\mu(p)) = B_r(p) \supset J(S), \tag{10}$$

while from $0 \leq d(p,q) = r - 2\mu$ and $q \in J(S)$ we deduce that

$$N_{r-\mu}(J(S)) \supset B_{r-\mu}(q) \supset B_\mu(p). \tag{11}$$

Now (9)–(11) imply

$$d_{\mathrm{H}}(J(S), J(B_\mu(p))) \leq r - \mu,$$

which contradicts $d_{\mathrm{H}}(S, B_\mu(p)) \geq r$, since J is an isometry of $(\mathfrak{C}(X,d), d_{\mathrm{H}})$. ∎

Proof of Theorem 2. We only have to show that J as in Lemma 12 satisfies $J(A) = A$ for all $A \in \mathfrak{C}(X,d)$.

Suppose there exists $p \in A \setminus J(A)$. Since $J(A)$ is compact, we have

$$\mu := \frac{1}{2} \inf\{d(p,q) \mid q \in J(A)\} > 0.$$

Thus $U := \bigcup_{q \in J(A)} B^{\circ}_{d(p,q)-\mu}(q)$ is an open covering of $J(A)$. Since $J(A)$ is compact, there exists a finite subcovering of U, say

$$\bigcup_{k=1}^{n} B^{\circ}_{d(p,q_k)-\mu}(q_k) \supset J(A).$$

Fix $\lambda > \operatorname{dist}(p, J(A)) + \operatorname{diam} J(A)$ and let $y_1, \dots, y_k$ be such that $d(y_k,p) = d(p,q_k) + d(q_k,y_k)$, $k = 1,\dots,n$, and $d(y_1,p) = d(y_2,p) = \dots = d(y_n,p) = \lambda$. Then $\bigcup_{k=1}^{n} B^{\circ}_{\lambda-\mu}(y_k)$ is an open covering of $J(A)$.

We set $S := \{y_1,\dots,y_k\} \subset S_\lambda(p)$ and obtain, on the one hand,

$$N_{\lambda-\mu} S \supset J(A). \tag{12}$$

On the other hand, $d(q_k,p) \ge 2\mu$ and thus $d(y_k,q_k) = d(y_k,p) - d(p,q_k) \le \lambda - 2\mu$, which yields

$$S \subset N_{\lambda-2\mu}(J(A)). \tag{13}$$

From Lemma 12 we know that $S = J(S)$, which together with the inclusions (12) and (13) yields

$$d_{\mathrm{H}}(J(S), J(A)) \le \lambda - \mu. \tag{14}$$

But, since $p \in A$ and $S \subset S_\lambda(p)$, we also have $d_{\mathrm{H}}(S,A) \ge \lambda$, contradicting inequality (14), due to the fact that J is an isometry of $(\mathfrak{C}(X,d), d_{\mathrm{H}})$. Hence $A \subset J(A)$ and the same argument with J replaced by J^{-1} yields $J(A) \subset A$, hence $A = J(A)$ and thus the claim. ∎

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