COLLOQUIUM MATHEMATICUM

VOL. 103 2005 NO. 1

STRICHARTZ'S CONJECTURE ON HARDY–SOBOLEV SPACES

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Abstract. We prove Strichartz's conjecture regarding a characterization of Hardy–Sobolev spaces.

Introduction. Hardy–Sobolev spaces arise as an alternative of L^p Sobolev spaces. To describe this notion, let H^p denote the real-variable Hardy spaces on $\mathbb{R}^n$ for $p > 0$ and I_α the Riesz potential operators of order $\alpha > 0$ defined via the Fourier transform formula $(I_\alpha f)\widehat{\ }(\xi) = |\xi|^{-\alpha}\widehat{f}(\xi)$ on the class of tempered distributions modulo polynomials. The image spaces of H^p under I_α, denoted by $I_\alpha(H^p)$, are called the *homogeneous Hardy–Sobolev spaces*. For each $f \in I_\alpha(H^p)$ there exists a unique $g \in H^p$ with $f = I_\alpha(g)$ and we define a quasi-norm

$$\|f\|_{I_\alpha(H^p)} = \|g\|_{H^p} = \|\Lambda_\alpha f\|_{H^p} \quad (0 < p < \infty)$$

in which Λ_α stands for the inverse operator of I_α. For $p > 1$, the $I_\alpha(H^p)$ are identical to the homogeneous L^p Sobolev spaces $I_\alpha(L^p)$. For $0 < p \le 1$, it is well known that the H^p provide an ideal alternative of the L^p and thus the $I_\alpha(H^p)$ may be thought of as a natural generalization of the $I_\alpha(L^p)$. As usual, we may define the inhomogeneous Hardy–Sobolev spaces as $H^p \cap I_\alpha(H^p)$.

As for characterizing $I_\alpha(H^p)$, let us recall the work of Strichartz which gives us the main motivation. Given a positive integer m and a point $y \in \mathbb{R}^n$, let Δ_y^m be the mth forward difference operator defined inductively as

$$\Delta_y^m f(x) = \Delta_y[\Delta_y^{m-1} f](x), \qquad \Delta_y f(x) = f(x+y) - f(x)$$

for each locally integrable function f and consider

$$D_{m,\alpha}(f)(x) = \left(\int_0^\infty \left[\int_B |\Delta_{ry}^m f(x)|\, dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right)^{1/2} \tag{1}$$

2000 *Mathematics Subject Classification*: Primary 42B35, 46E35.

Key words and phrases: Hardy–Sobolev space, Littlewood–Paley g-function, Lusin S-function, Strichartz's conjecture.

This research was supported by Korea Research Foundation KRF-2003-013-C00005.

where B denotes the unit ball in $\mathbb{R}^n$. A classical theorem of Strichartz and Bagby states that a function $f \in \bigcup_{1\le q<\infty} L^q$ belongs to $I_\alpha(L^p)$ if and only if $D_{m,\alpha}(f) \in L^p$ for $p > 1$ and $0 < \alpha < m$ (see [Sz1], [Ba]).

In the case when $n/(n+\alpha) < p \le 1$, each distribution in $I_\alpha(H^p)$ coincides with a locally integrable function in view of the Sobolev embedding inequalities (see [Ch] or [K]). In this range of p, it is shown in [Sz2] that if $f \in I_\alpha(H^p)$ and $0 < \alpha < m$, then $D_{m,\alpha}(f) \in L^p$ with $\|D_{m,\alpha}(f)\|_p \approx \|f\|_{I_\alpha(H^p)}$ ([1]). In addition, it is also shown that if $f \in I_\alpha(H^p)$ and $m-1 < \alpha < m$, then

$$(2)\qquad T_{m,\alpha}(f)(x) = \left(\int_0^\infty \left[\int_B |Q^m_{ry} f(x)|\,dy\right]^2 \frac{dr}{r^{1+2\alpha}}\right)^{1/2},$$

where $Q^m_y f(x) = f(x+y) - \sum_{|\sigma|<m} (\partial^\sigma f)(x) y^\sigma/\sigma!$, belongs to L^p with $\|T_{m,\alpha}(f)\|_p \approx \|f\|_{I_\alpha(H^p)}$. In his work, however, Strichartz left the following reverse direction as an open conjecture.

CONJECTURE. *If either $D_{m,\alpha}(f) \in L^p$ with $0 < \alpha < m$ or $T_{m,\alpha}(f) \in L^p$ with $m-1 < \alpha < m$, then $f \in I_\alpha(H^p)$ for $n/(n+\alpha) < p \le 1$.*

Our primary aim in this paper is to prove Strichartz's conjecture so as to establish a characterization of $I_\alpha(H^p)$ via $D_{m,\alpha}$ or $T_{m,\alpha}$ in the stated range of p and α. To accomplish our aim, we shall exploit a set of different characterizations for $I_\alpha(H^p)$ which are interesting in their own right. Identifying $I_\alpha(H^p)$ as particular instances of Triebel–Lizorkin spaces, it is shown in the work of Bui *et al.* [BPT] that a certain variant of Littlewood–Paley g-functions characterizes $I_\alpha(H^p)$ spaces. In a similar fashion, it will be shown that a modification of Lusin S-functions characterizes $I_\alpha(H^p)$. Dominating appropriate characterizing means in terms of $D_{m,\alpha}(f)$ or $T_{m,\alpha}(f)$, we shall obtain the desired proofs.

It turns out that Strichartz's characterization provides effective means in a number of problems on Sobolev spaces. In dealing with pointwise multiplier problems, for example, Strichartz used the aforementioned results to prove that $I_{n/p}(H^p)$ forms an algebra for $0 < p \le 1$ and also observed that his conjecture, if affirmative, would imply that the characteristic function χ_Ω of a Lipschitz domain Ω is a multiplier on $I_\alpha(H^p)$ for $n(1/p-1) < \alpha < n/p$.

Acknowledgements. The current research has been completed during the author's visit at the University of Edinburgh in 2004. The author wishes to express his sincere gratitude to the staff members of School of Mathematics for their warm hospitality. Especially, the author is deeply indebted to James Wright for his generous time sharing and help on this project,

([1]) This means, as usual, that $C_1\|f\|_{I_\alpha(H^p)} \le \|D_{m,\alpha}(f)\|_p \le C_2\|f\|_{I_\alpha(H^p)}$ for some positive constants C_1, C_2 not depending on f.

in particular, on Lemma A2. Finally, the author is grateful to the anonymous referee who pointed out the work of Bui *et al.* in the present form of Theorem C1.

A. Preliminaries. For a tempered distribution f and a Schwartz function φ on $\mathbb{R}^n$, set $u(x,t) = (f * \varphi_t)(x)$ with $\varphi_t(x) = t^{-n}\varphi(x/t)$ and define

$$u^+(x) = \sup_{t>0} |u(x,t)|.$$

We recall from [FS] that $f \in H^p$ for $0 < p \le \infty$ if and only if $u^+ \in L^p$ with any choice of φ satisfying $\widehat{\varphi}(0) \neq 0$ and $\|f\|_{H^p} = \|u^+\|_p$.

It is well known that the Lusin S-functions defined by

$$S(u)(x) = \Big(\iint_{\Gamma(x)} |u(y,t)|^2 t^{-n}\, dy\, \frac{dt}{t} \Big)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |y-x| < t\}$, provide another characterizing means of H^p under certain conditions on φ and f. The required condition on φ comes mainly from the L^2 estimates. In fact, with $\omega_n = |B|$,

$$\|S(u)\|_2^2 = \omega_n \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \Big[\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} \Big] d\xi \qquad (f \in L^2)$$

so that the *a priori* inequality $\|S(u)\|_2 \le C\|f\|_2$ holds if and only if

$$\sup_{\xi \in \mathbb{R}^n} \int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} < \infty, \tag{3}$$

and the reverse *a priori* inequality $\|S(u)\|_2 \ge C\|f\|_2$ holds if and only if

$$\inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} > 0. \tag{4}$$

In a more general setting, these conditions can be formulated in terms of other equivalent ones.

LEMMA A1. *Let φ be a Schwartz function on $\mathbb{R}^n$ and $\alpha \ge 0$.*

(i) *The condition* (3) *is equivalent to the condition $\widehat{\varphi}(0) = 0$.*

(ii) *If $\int x^\sigma \varphi(x)\, dx = 0$ for all multi-indices σ with $|\sigma| \le [\alpha]$, then*

$$\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t^{1+2\alpha}} \le C_\alpha |\xi|^{2\alpha} \qquad (\xi \in \mathbb{R}^n).$$

Proof. Assume (3) holds but $\widehat{\varphi}(0) \neq 0$. Use the continuity of $\widehat{\varphi}$ to choose $\delta > 0$ such that $|\widehat{\varphi}(\xi)| \ge |\widehat{\varphi}(0)|/2$ for all $|\xi| \le \delta$. Then

$$\sup_{\xi\in\mathbb{R}^n}\int_0^\infty |\widehat{\varphi}(t\xi)|^2\,\frac{dt}{t} \geq \sup_{|\xi|\leq\delta}\int_0^1 |\widehat{\varphi}(t\xi)|^2\,\frac{dt}{t} \geq \frac{|\widehat{\varphi}(0)|^2}{4}\int_0^1 \frac{dt}{t} = +\infty,$$

a contradiction. Thus $\widehat{\varphi}(0) = 0$. The vanishing moment condition of (ii) implies that $|\widehat{\varphi}(\xi)| \leq C_N|\xi|^{[\alpha]+1}(1+|\xi|^2)^{-N}$ for any $N > 0$, from which the conclusion of (ii) as well as the converse of (i) follow immediately. ∎

LEMMA A2. *For a Schwartz function* φ *on* $\mathbb{R}^n$ *and* $\alpha \geq 0$, *the following statements are equivalent.*

(i)

$$\inf_{\xi\neq 0}\left[|\xi|^{-2\alpha}\int_0^\infty |\widehat{\varphi}(t\xi)|^2\,\frac{dt}{t^{1+2\alpha}}\right] = c_\alpha > 0. \tag{5}$$

(ii) $|\widehat{\varphi}(t\xi)|$ *does not vanish identically as a function of* $t > 0$ *for* $\xi \neq 0$, *that is*, $\sup_{t>0}|\widehat{\varphi}(t\xi)| > 0$ *for* $\xi \neq 0$.

(iii) *There exists a Schwartz function* ζ *such that* $\widehat{\zeta}$ *has compact support away from the origin and*

$$\int_0^\infty \widehat{\varphi}(t\xi)\widehat{\zeta}(t\xi)\,\frac{dt}{t^{1+2\alpha}} = |\xi|^{2\alpha} \quad (\xi\neq 0). \tag{6}$$

Proof. Evidently, (i) or (iii) implies (ii). To prove (ii) implies (i), it suffices to show $\inf_{\xi\in S^{n-1}}\Omega(\xi) = c_\alpha > 0$, where S^{n-1} denotes the unit sphere and

$$\Omega(\xi) = \int_0^\infty |\widehat{\varphi}(t\xi)|^2\,\frac{dt}{t^{1+2\alpha}}.$$

We first observe that (ii) implies that $\Omega(\xi) > 0$ for all $\xi \in S^{n-1}$. Indeed, for a fixed $\xi \in S^{n-1}$, there exists a $t_0 > 0$ with $|\widehat{\varphi}(t_0\xi)| > 0$. By continuity, there is an open interval I such that $t_0 \in I \subset (0,\infty)$ and $|\widehat{\varphi}(t\xi)| > 0$ for each $t \in I$. It follows that

$$\Omega(\xi) \geq \int_I |\varphi(t\xi)|^2\,\frac{dt}{t^{1+2\alpha}} > 0.$$

We now choose a sequence $(\xi_k) \subset S^{n-1}$ with $\Omega(\xi_k) \to c_\alpha$. As S^{n-1} is compact, there exist $k_1 < k_2 < \cdots$ and $\xi_0 \in S^{n-1}$ such that $\xi_{k_j} \to \xi_0$. By Fatou's lemma and the continuity of $\widehat{\varphi}$, we have

$$\begin{aligned}
c_\alpha = \lim_{j\to\infty}\Omega(\xi_{k_j}) &= \lim_{j\to\infty}\int_0^\infty |\widehat{\varphi}(t\xi_{k_j})|^2\,\frac{dt}{t^{1+2\alpha}} \\
&\geq \int_0^\infty [\liminf_{j\to\infty}|\widehat{\varphi}(t\xi_{k_j})|^2]\,\frac{dt}{t^{1+2\alpha}} = \int_0^\infty |\widehat{\varphi}(t\xi_0)|^2\,\frac{dt}{t^{1+2\alpha}} \\
&= \Omega(\xi_0) > 0,
\end{aligned}$$

which proves that (ii) implies (i).

To prove (ii) implies (iii), we let $0 < \varepsilon < 1$ and take a nonnegative C^∞ function θ on $(0, \infty)$ such that $\theta = 1$ on $(\varepsilon, 1/\varepsilon)$ and its support is contained in $(\varepsilon/2, 2/\varepsilon)$. Choosing ε so small that $\sup_{t>0}[|\widehat{\varphi}(t\xi)|\theta(t)] > 0$ for $\xi \neq 0$, we have as in the preceding case

$$\inf_{\xi \in S^{n-1}} \left[\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \theta(t) \, \frac{dt}{t^{1+2\alpha}} \right] > 0.$$

Defining ζ through the Fourier transform formula

$$\widehat{\zeta}(\xi) = \overline{\widehat{\varphi}(\xi)} \, \theta(|\xi|) \left[\int_0^\infty |\widehat{\varphi}(t\xi/|\xi|)|^2 \theta(t) \, \frac{dt}{t^{1+2\alpha}} \right]^{-1},$$

it is plain to check that ζ has the stated properties of (iii). ■

REMARK 1. When $\alpha = 0$, the equivalence of (i) and (ii) is due to James Wright and the equivalence of (ii) and (iii) is discussed in Stein's book [St, pp. 185–186]. The proof that (ii) implies (iii) is a slight modification of that of Lemma 4.1 in [CT].

For $\alpha \geq 0$, let $\mathcal{O}_\alpha$ be the class of Schwartz functions φ on $\mathbb{R}^n$ such that

(i) $\int x^\sigma \varphi(x) \, dx = 0 = (\partial^\sigma \widehat{\varphi})(0)$ for all $|\sigma| \leq [\alpha]$,
(ii) $\sup_{t>0} |\widehat{\varphi}(t\xi)| > 0$ $(\xi \neq 0)$.

With $\mathcal{O}_0 = \mathcal{O}$, the preceding lemmas show that $\|S(u)\|_2 \approx \|f\|_2$ if and only if $\varphi \in \mathcal{O}$. Introduced by Bui *et al.* [BPT], each $\mathcal{O}_\alpha$ will serve as the admissible class of Schwartz functions in our characterization of $I_\alpha(H^p)$. As for the admissible distributions in our characterizations, given $\alpha \geq 0$, we denote by $\mathcal{A}_\alpha$ the class of distributions f on $\mathbb{R}^n$ such that $\widehat{f}$ coincides with a function satisfying

$$\widehat{f}(\xi)|\xi|^\alpha(1 + |\xi|^2)^{-\delta} \in L^2 \quad \text{for some } \delta \geq 0$$

and $\mathcal{L}_\alpha = \bigcup_{1 \leq q < \infty} I_\alpha(L^p)$. Write $\mathcal{A}_0 = \mathcal{A}$, $I_0 = I$, and $\mathcal{L}_0 = \mathcal{L}$ for simplicity. Evidently, we have $\mathcal{A} \subset \mathcal{A}_\alpha \subset \mathcal{A}_\beta$ for any $0 \leq \alpha \leq \beta$.

LEMMA A3. *Given $\alpha \geq 0$, if $f \in I_\alpha(H^p)$ for $0 < p \leq 2$, then*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, |\xi|^{2\alpha} \, (1 + |\xi|^2)^{-2\delta} \, d\xi \;\leq\; C \, \|f\|^2_{I_\alpha(H^p)} \tag{7}$$

with any $\delta > (1/p - 1/2)n/2$. Consequently, $I_\alpha(H^p) \subset \mathcal{A}_\alpha$ for $0 < p \leq 2$ and $I_\alpha(H^p) \subset \mathcal{L}_\alpha$ for $1 < p < \infty$.

Proof. It suffices to treat the case $\alpha = 0$. The inequality (7) follows from the estimate $|\widehat{f}(\xi)| \leq C|\xi|^{n(1/p-1)}\|f\|_{H^p}$ for $0 < p \leq 1$ and from the Hölder and Hausdorff–Young inequalities for $1 < p \leq 2$. ■

A characterization of H^p via Lusin S-functions is established by Calderón and Torchinsky (see Theorems 6.7, 6.9, 6.10 in [CT]).

THEOREM A4. *Let f be a tempered distribution on $\mathbb{R}^n$. For $\varphi \in \mathcal{O}$, put $u(x,t) = (f * \varphi_t)(x)$.*

(i) *If $f \in H^p$ for $0 < p < \infty$, then $S(u) \in L^p$ with $\|S(u)\|_p \approx \|f\|_{H^p}$.*
(ii) *Assume $S(u) \in L^p$ for $0 < p < \infty$. If $f \in \mathcal{A} \cup \mathcal{L}$, then $f \in H^p$ with $\|f\|_{H^p} \le C_p \|S(u)\|_p$. For a general f, there exists a polynomial P such that $g = f - P \in H^p$ and $\|g\|_{H^p} \le C_p \|S(u)\|_p$.*

REMARK 2. The presence of a polynomial factor in (ii) is partly due to the fact that $S(u) = S(f * \varphi_t)$, as a function of f for a fixed $\varphi \in \mathcal{O}$, annihilates any polynomial of degree less than the order of zero of $\widehat{\varphi}$ at the origin. For $b > 0$, consider

$$S_b(u)(x) = \biggl(\iint_{\Gamma_b(x)} |u(y,t)|^2 (bt)^{-n}\, dy\, \frac{dt}{t} \biggr)^{1/2},$$

where $\Gamma_b(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |y - x| < bt\}$. As it is shown in [CT] that $\|S_b(u)\|_p \approx \|S_d(u)\|_p$ for $0 < p < \infty$ and for any $b, d > 0$, we may replace $S(u)$ by $S_b(u)$ in Theorem A4 without altering any conclusions.

The Littlewood–Paley g-functions are defined by

$$g_\varphi(f)(x) = \biggl(\int_0^\infty |(f * \varphi_t)(x)|^2\, \frac{dt}{t} \biggr)^{1/2}.$$

As before, $\|g_\varphi(f)\|_2 \approx \|f\|_2$ if and only if $\varphi \in \mathcal{O}$, and the following characterization result is due to Uchiyama ([U1], [U2]).

THEOREM A5. *Let f be a tempered distribution on $\mathbb{R}^n$ and $\varphi \in \mathcal{O}$.*

(i) *If $f \in H^p$ for $0 < p < \infty$, then $g_\varphi(f) \in L^p$ with $\|g_\varphi(f)\|_p \approx \|f\|_{H^p}$.*
(ii) *Assume $g_\varphi(f) \in L^p$ for $0 < p < \infty$. If $f \in \mathcal{L}$, then $f \in H^p$ with $\|f\|_{H^p} \le C_p \|g_\varphi(f)\|_p$. For a general f, there exists a polynomial P such that $h = f - P \in H^p$ and $\|h\|_{H^p} \le C_p \|g_\varphi(f)\|_p$.*

B. A variant of Lusin S-functions. As the first characterizing means for $I_\alpha(H^p)$, we introduce

$$S_b^\alpha(u)(x) = \biggl(\iint_{\Gamma_b(x)} |u(y,t)|^2\, (bt)^{-n}\, dy\, \frac{dt}{t^{1+2\alpha}} \biggr)^{1/2}. \tag{8}$$

A simple computation shows that $\|S_b^\alpha(u)\|_2 \approx \|f\|_{I_\alpha(L^2)}$ for $\varphi \in \mathcal{O}_\alpha$ according to Lemmas A1 and A2. The purpose of this section is to prove that $f \in I_\alpha(H^p)$ if and only if $S_b^\alpha(u) \in L^p$ and $f \in \mathcal{A}_\alpha \cup \mathcal{L}_\alpha$ with any choice $\varphi \in \mathcal{O}_{2\alpha}$, a bit more restrictive class of admissible Schwartz functions. Our methods of proof will be standard as in [CT] or [BPT].

We begin by showing that a different choice of b results in equivalent L^p norms for any Schwartz function φ.

LEMMA B1. *For $b, d > 0$, if $S_b^\alpha(u) \in L^p$, then $S_d^\alpha(u) \in L^p$ with*

$$\|S_d^\alpha(u)\|_p \leq \begin{cases} C_p(1+d/b)^{n(1/p-1/2)}\|S_b^\alpha(u)\|_p & (0 < p \leq 2), \\ C_p(1+b/d)^{n/2}\|S_b^\alpha(u)\|_p & (2 < p < \infty). \end{cases}$$

Proof. For $0 < p < 2$, we shall follow the same reasoning as in [CT, pp. 17–19]. For each $s > 0$, set $D_s = \{S_b^\alpha(u)(x) > s\}$ and $\widehat{D}_s = \{M_d(x) > 1/2\}$ where

$$M_d(x) = \sup\left\{\frac{|B(y,bt) \cap D_s|}{|B(y,bt)|} : (y,t) \in \Gamma_d(x)\right\}.$$

Here $B(y, bt)$ denotes the open ball with center y and radius bt. By a maximal theorem in [CT], $D_s \subset \widehat{D}_s$ with $|\widehat{D}_s| \leq c_n(1+d/b)^n|D_s|$. Using

$$\int_{\widehat{D}_s^c} [S_d^\alpha(u)(x)]^2\,dx \leq 2\int_{D_s^c} [S_b^\alpha(u)(x)]^2\,dx,$$

a readily verifiable inequality, we have the estimate

$$\begin{aligned} |\{S_d^\alpha(u) > rs\}| &\leq |\widehat{D}_s^c \cap \{S_d^\alpha(u) > rs\}| + |\widehat{D}_s| \\ &\leq \frac{4}{(rs)^2}\int_0^s t|\{S_b^\alpha(u) > t\}|\,dt + c_n(1+d/b)^n|D_s| \end{aligned}$$

for all $r, s > 0$. It follows plainly that

$$\int [S_d^\alpha(u)(x)]^p\,dx \leq \left[\frac{4r^{p-2}}{2-p} + c_n(1+d/b)^n r^p\right]\int [S_b^\alpha(u)(x)]^p\,dx.$$

Choosing r that optimizes this bound, we obtain the desired inequality.

For $p = 2$, it is trivial to see $\|S_d^\alpha(u)\|_2 = \|S_b^\alpha(u)\|_2$. In the case $p > 2$,

$$\|S_d^\alpha(u)\|_p^2 = \|[S_d^\alpha(u)]^2\|_{p/2} = \sup_{\|g\|_q=1}\left|\int [S_d^\alpha(u)(x)]^2 g(x)\,dx\right|$$

with q determined by $2/p + 1/q = 1$. For a fixed g with $\|g\|_q = 1$, the last absolute value of integral is bounded by

$$\iint |u(y,t)|^2\left[(dt)^{-n}\int_{B(y,dt)} |g(x)|\,dx\right]dy\,\frac{dt}{t^{1+2\alpha}}. \tag{10}$$

For each $z \in B(y, bt)$, note that $B(y, dt) \subset B(z, (b+d)t)$ and thus

$$(dt)^{-n}\int_{B(y,dt)} |g(x)|\,dx \leq \omega_n(1+b/d)^n g^*(z)$$

where g^* denotes the Hardy–Littlewood maximal function of g. Integrating this inequality over the ball $B(y, bt)$ with respect to z, we get

$$(dt)^{-n} \int_{B(y,dt)} |g(x)|\, dx \le (1 + b/d)^n (bt)^{-n} \int_{B(y,bt)} g^*(z)\, dz.$$

A simple algebra shows then (10) is bounded by $(1 + b/d)^n$ times

$$\int [S_b^\alpha(u)(z)]^2\, g^*(z)\, dz \le \|[S_b^\alpha(u)]^2\|_{p/2} \|g^*\|_q \le C_p \|S_b^\alpha(u)\|_p^2$$

by the maximal theorem. This completes the proof for the case $p > 2$. ∎

Focusing on $S_1^\alpha = S^\alpha$, we now proceed to prove the L^p norm equivalence of S^α for different choices of Schwartz functions in the class $\mathcal{O}_{2\alpha}$. For $\lambda > 0$, consider a variant of Littlewood–Paley g_λ-functions in the form

$$g_\lambda^\alpha(u)(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} |u(y,t)|^2 \left[1 + \frac{|y - x|}{t}\right]^{-2\lambda} t^{-n-2\alpha}\, dy\, \frac{dt}{t} \right)^{1/2}. \tag{11}$$

It follows from a slight modification of Theorem 3.5 in [CT] that

$$2^{-2\lambda} [S^\alpha(u)(x)]^2 \le [g_\lambda^\alpha(u)(x)]^2 \le \sum_{k=1}^\infty 2^{-k(1-k/2\lambda)} [S^\alpha_{2^{k/2\lambda}}(u)(x)]^2.$$

Combining this with Lemma B1, we have $\|g_\lambda^\alpha(u)\|_p \approx \|S^\alpha(u)\|_p$ valid for every $0 < p < \infty$ provided $\lambda > \max(n/p, n/2)$.

LEMMA B2. *Let ζ, ψ be Schwartz functions on $\mathbb{R}^n$ such that $\widehat{\zeta}$ has compact support away from the origin and $(\partial^\sigma \widehat{\psi})(0) = 0$ for all $|\sigma| \le \ell$. For $s, t, \lambda > 0$, let*

$$J_\lambda(s,t) = \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{s}\right)^{2\lambda} |\zeta_s * \psi_t(z)|^2\, dz.$$

Then for any $N > 0$ there exists a constant $C_N > 0$ such that

$$J_\lambda(s,t) \le C_N\, s^{-n} \left(\frac{t}{s}\right)^{2(\ell+1)} \left(1 + \frac{t}{s}\right)^{-2N}. \tag{12}$$

Proof. Set $\mu = [\lambda] + 1$. It follows from Plancherel's theorem that

$$\begin{aligned} J_\lambda(s,t) &= s^n \int (1 + |z|)^{2\lambda} |\zeta_s * \psi_t(sz)|^2\, dz \\ &\le C s^n \sum_{|\beta| \le \mu} \int |\partial^\beta [s^{-n} (\zeta_s * \psi_t)\widehat{\ }(\xi/s)]|^2\, d\xi \\ &= C s^{-n} \sum_{|\beta| \le \mu} \int |\partial^\beta [\widehat{\zeta}(\xi) \widehat{\psi}(t\xi/s)]|^2\, d\xi. \end{aligned}$$

Since $\widehat{\zeta}$ has compact support away from the origin, (12) follows from the evident Fourier transform estimates

$$|\widehat{\psi}(\xi)| \le C_N |\xi|^{\ell+1}(1+|\xi|)^{-N}$$

and

$$|\partial^\beta \widehat{\psi}(\xi)| \le C_N (1+|\xi|)^{-N} \quad \text{for each } \beta. \ \blacksquare$$

REMARK 3. Referred to as size estimates of Heideman type, more extensive and precise L^1 estimates can be found in Lemma 2.1 of [BPT].

We shall need a version of Calderón's reproducing formula in the following form whose proof is a minor modification of those in Theorems 4.6 and 5.1 of [CT]. (For a reference to its historical developments, see the remark after Lemma 2.3 of [BPT].)

LEMMA B3. *Let φ be a Schwartz function with $\sup_{t>0} |\widehat{\varphi}(t\xi)| > 0$ for $\xi \neq 0$ and let ζ, as in Lemma A2, be a Schwartz function such that $\widehat{\zeta}$ has compact support away from the origin and*

$$\int_0^\infty \widehat{\varphi}(s\xi)\widehat{\zeta}(s\xi)\,\frac{ds}{s} = 1 \quad (\xi \neq 0).$$

(i) *Let ψ be a Schwartz function such that $(\partial^\sigma \widehat{\psi})(0) = 0$ for all $|\sigma| \le \ell$. If $f \in \mathcal{A}_{\ell+1}$ or $f \in \mathcal{L}_\alpha$ with $0 \le \alpha \le \ell+1$, then the identity*

$$(f * \psi_t)(x) = \int_0^\infty (f * \varphi_s * \zeta_s * \psi_t)(x)\,\frac{ds}{s} \tag{13}$$

holds for almost every $x \in \mathbb{R}^n$ and for each $t > 0$, where the integral converges absolutely.

(ii) *If $\widehat{\psi}$ has support away from the origin, then the identity* (13) *remains valid for a general tempered distribution f.*

LEMMA B4. *Assume $f \in \mathcal{A}_{[2\alpha]+1}$ or $f \in \mathcal{L}_\beta$ for some $0 \le \beta \le [2\alpha]+1$. Let $u(x,t) = (f * \varphi_t)(x), v(x,t) = (f * \psi_t)(x)$ with $\varphi, \psi \in \mathcal{O}_{2\alpha}$. Then*

$$\|S^\alpha(u)\|_p \approx \|S^\alpha(v)\|_p \quad (0 < p < \infty).$$

Proof. In view of the equivalence $\|S^\alpha(u)\|_p \approx \|g^\alpha_\lambda(u)\|_p$ for $0 < p < \infty$ and $\lambda > \max(n/p, n/2)$, it will be sufficient to show that

$$S^\alpha(v)(x) \le C_{\alpha,\lambda}\, g^\alpha_\lambda(u)(x) \quad (\lambda > 0). \tag{14}$$

Let ζ be as in Lemma B3. By (i) of Lemma B3, since $\psi \in \mathcal{O}_{2\alpha}$, we have

$$v(y,t) = \int_0^\infty \int_{\mathbb{R}^n} u(z,s)(\zeta_s * \psi_t)(y-z)\,dz\,\frac{ds}{s}$$

under the stated hypothesis on f. Fix $x \in \mathbb{R}^n$. For $y \in B(x,t)$, we have

$$\begin{aligned}|v(y,t)| \le \int_0^\infty \Big[\int |u(z,s)|^2\Big(1+\frac{|z-x|}{s}\Big)^{-2\lambda} dz\Big]^{1/2} \\ \times \Big(1+\frac{t}{s}\Big)^{\lambda}\Big[\int\Big(1+\frac{|z|}{s}\Big)^{2\lambda}|\zeta_s * \psi_t(z)|^2\,dz\Big]^{1/2}\frac{ds}{s} \\ \le C\int_0^\infty \Big[\int |u(z,s)|^2\Big(1+\frac{|z-x|}{s}\Big)^{-2\lambda} dz\Big]^{1/2} \\ \times s^{-n/2}\Big(1+\frac{t}{s}\Big)^{\lambda}\Big(\frac{t}{s}\Big)^{\ell}\Big(1+\frac{t}{s}\Big)^{-N}\frac{ds}{s}\end{aligned}$$

where $\ell = [2\alpha]+1$ in view of Lemma B2 and the fact $\psi \in \mathcal{O}_{2\alpha}$. Applying the Cauchy–Schwarz inequality, we see that $|v(y,t)|$ is bounded by

$$C\Big[\int_0^\infty\int |u(z,s)|^2\Big(1+\frac{|z-x|}{s}\Big)^{-2\lambda}s^{-n}\Big(\frac{t}{s}\Big)^{\ell}\Big(1+\frac{t}{s}\Big)^{-N+\lambda}dz\,\frac{ds}{s}\Big]^{1/2} \times \Big[\int_0^\infty\Big(\frac{t}{s}\Big)^{\ell}\Big(1+\frac{t}{s}\Big)^{-N+\lambda}\frac{ds}{s}\Big]^{1/2}.$$

Choosing $N > \ell + \lambda$, observe that the quantity on the second line is a constant independent of t. Therefore,

$$\begin{aligned}[S^\alpha(v)(x)]^2 &= \int_0^\infty \int_{B(x,t)} |v(y,t)|^2\,t^{-n-2\alpha}\,dy\,\frac{dt}{t} \\ &\le C\int_0^\infty\int_{\mathbb{R}^n}|u(z,s)|^2\Big(1+\frac{|z-x|}{s}\Big)^{-2\lambda}s^{-n-2\alpha} \\ &\quad\times\Big[\int_0^\infty\Big(\frac{t}{s}\Big)^{\ell-2\alpha}\Big(1+\frac{t}{s}\Big)^{-N+\lambda}\frac{dt}{t}\Big]dz\,\frac{ds}{s} \\ &\le C[g_\lambda^\alpha(u)(x)]^2\end{aligned}$$

because $\ell - 2\alpha = 1 + [2\alpha] - 2\alpha > 0$ so that the quantity inside the bracket is again a constant independent of s. This completes the proof. ∎

The following is the main result of this section.

THEOREM B5. *Let f be a tempered distribution on $\mathbb{R}^n$. Given $\alpha \ge 0$, let $u(x,t) = (f * \varphi_t)(x)$ with $\varphi \in \mathcal{O}_{2\alpha}$ and $0 < p < \infty$.*

(i) *If $f \in I_\alpha(H^p)$, then $S^\alpha(u) \in L^p$ with $\|S^\alpha(u)\|_p \approx \|f\|_{I_\alpha(H^p)}$.*
(ii) *Assume that $S^\alpha(u) \in L^p$. If $f \in \mathcal{A}_\alpha \cup \mathcal{L}_\alpha$, then $f \in I_\alpha(H^p)$ with $\|f\|_{I_\alpha(H^p)} \le C_{\alpha,p}\|S^\alpha(u)\|_p$. For a general f, there exist a polynomial P and $g \in H^p$ such that $f = I_\alpha(g+P)$ and $\|g\|_{H^p} \le C_{\alpha,p}\|S^\alpha(u)\|_p$.*

Proof. Assume $f = I_\alpha(g)$ with a unique $g \in H^p$. Choose $\psi \in \mathcal{O}$ such that $\widehat{\psi}$ has support away from the origin. Set

$$\Psi = I_\alpha(\psi), \quad U(x,t) = (g * \Psi_t)(x), \quad V(x,t) = (f * \psi_t)(x).$$

As is easily verified, $U = t^{-\alpha} V$ and so $S(U)(x) = S^\alpha(V)(x)$. According to Theorem A4 and Lemma B4,

$$\|S^\alpha(u)\|_p \approx \|S^\alpha(V)\|_p = \|S(U)\|_p \approx \|g\|_{H^p} = \|f\|_{I_\alpha(H^p)}$$

since $V \in \mathcal{O}_{2\alpha}$ and $f \in \mathcal{A}_\alpha \cup \mathcal{L}_\alpha$ by Lemma A3. This proves (i).

Assume now $S^\alpha(u) \in L^p$. With the same Schwartz functions ψ, Ψ as above, we set $W(x,t) = (\Lambda_\alpha f * \Psi_t)(x)$ this time. Then $W = t^{-\alpha} V$ and $S(W)(x) = S^\alpha(V)(x)$. Since ψ has support away from the origin, the identity (13) holds and so does the inequality (14). It follows that

$$\|S(W)\|_p = \|S^\alpha(v)\|_p \le C_{\alpha,p}\, \|S^\alpha(u)\|_p < \infty.$$

The stated properties of (ii) follow now from Theorem A4. ■

REMARK 4. If we replace $S^\alpha(u)$ by $S_b^\alpha(u)$ with any $b > 0$ or by $g_\lambda^\alpha(u)$ with any $\lambda > \max(n/p, n/2)$, Theorem B5 remains unchanged. When $\alpha = 0$, Theorem B5 reduces to Theorem A4. As we shall see in the next section, assertion (ii) turns out to be valid with the minimal class $\mathcal{O}_\alpha$ of admissible Schwartz functions.

COROLLARY B6. *Let $H_\alpha^p = H^p \cap I_\alpha(H^p)$ with $\|f\|_{H_\alpha^p} = \|f\|_{H^p} + \|f\|_{I_\alpha(H^p)}$. For a tempered distribution f and $\varphi \in \mathcal{O}_{2\alpha}$, let $u(x,t) = (f * \varphi_t)(x)$. Then $f \in H_\alpha^p$ for $0 < p \le 2$ if and only if $f \in H^p$ and $S^\alpha(u) \in L^p$. Moreover,*

$$\|f\|_{H_\alpha^p} \approx \|f\|_{H^p} + \|S^\alpha(u)\|_p \qquad (0 < p \le 2).$$

C. A variant of Littlewood–Paley g-functions. It is known that Hardy–Sobolev spaces may be realized as particular instances of Triebel–Lizorkin spaces. To be more precise, given a smoothing index $\alpha \in \mathbb{R}$ and scale indices $0 < p < \infty$, $0 < q \le \infty$, let $\dot{F}_{p,q}^\alpha$ denote the associated homogeneous Triebel–Lizorkin space, the set of all tempered distributions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{\dot{F}_{p,q}^\alpha} = \Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} |f * \psi_{2^{-j}}|)^q \Big]^{1/q} \Big\|_p < \infty \tag{15}$$

where ψ is any Schwartz function on $\mathbb{R}^n$ such that the support of $\widehat{\psi}$ is contained in $\{1/2 \le |\xi| \le 2\}$ and $|\widehat{\psi}(\xi)| \ge c > 0$ for $3/5 \le |\xi| \le 5/3$. According to Triebel's books [T1], [T2], we have the special identification

$$\dot{F}_{p,2}^0 = H^p, \quad \dot{F}_{p,2}^\alpha = I_\alpha(H^p) \quad (\alpha > 0,\ 0 < p < \infty).$$

In [BPT], Bui, Paluszyński and Taibleson established a list of characterizations for $\dot{F}_{p,q}^\alpha$ spaces. To single out what we need from their work, consider a modification of the Littlewood–Paley g-function defined by

$$g_\varphi^\alpha(f)(x) = \left(\int_0^\infty |(f*\varphi_t)(x)|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}. \tag{16}$$

A close inspection of Theorems 3.1 and 6.1 of [BPT] and Theorem A5 yields the following characterization result for $I_\alpha(H^p)$ spaces.

THEOREM C1 (Bui, Paluszyński and Taibleson). *Given $\alpha \geq 0$ and $0 < p < \infty$, let f be a tempered distribution on $\mathbb{R}^n$ and $\varphi \in \mathcal{O}_\alpha$.*

(i) *If $f \in I_\alpha(H^p)$, then $g_\varphi^\alpha(f) \in L^p$ with $\|g_\varphi^\alpha(f)\|_p \approx \|f\|_{I_\alpha(H^p)}$.*

(ii) *If $g_\varphi^\alpha(f) \in L^p$, then there exist a polynomial P and $h \in H^p$ such that $f = I_\alpha(h+P)$ and $\|h\|_{H^p} \leq C_{\alpha,p}\|g_\varphi^\alpha(f)\|_p$. If f satisfies the extra condition $f \in \mathcal{L}_\alpha$, then $f \in I_\alpha(H^p)$ with $\|f\|_{I_\alpha(H^p)} \leq C_{\alpha,p}\|g_\varphi^\alpha(f)\|_p$.*

REMARK 5. In view of Lemmas A1 and A2, the admissible class $\mathcal{O}_\alpha$ is optimal in the sense that the above results are no longer valid if φ has vanishing moments of order less than $[\alpha]$ or if it violates a condition in Lemma A2. For a Schwartz function φ on $\mathbb{R}^n$, put $u(x,t) = (f*\varphi_t)(x)$. As $u(x,t)$ is continuous in $t>0$, we have

$$|(f*\varphi_t)(x)|^2 = \lim_{b\to 0}\frac{1}{|B(x,bt)|}\int_{B(x,bt)} |u(y,t)|^2\,dy.$$

It follows from Fatou's lemma and a simple computation that

$$g_\varphi^\alpha(f)(x) \leq \frac{1}{\sqrt{\omega_n}}\liminf_{b\to 0}[S_b^\alpha(u)(x)]. \tag{17}$$

Since $\mathcal{O}_{2\alpha} \subset \mathcal{O}_\alpha$, combined with Theorem C1, this pointwise inequality implies directly the second part of Theorem B5 with the optimal class $\mathcal{O}_\alpha$ of admissible Schwartz functions. Although plausible, however, we do not know if the first part of Theorem B5 remains valid with $\mathcal{O}_\alpha$ in place of $\mathcal{O}_{2\alpha}$.

D. Proofs of Strichartz's conjecture. To begin with, we make use of our characterization of $I_\alpha(H^p)$ by variants of Lusin S-functions to prove Strichartz's conjecture on $T_{m,\alpha}$.

THEOREM D1. *Let f be a locally integrable function on $\mathbb{R}^n$. For a positive integer m with $m > \alpha$, suppose that $T_{m,\alpha}(f) \in L^p$ for $n/(n+\alpha) < p < \infty$. Then there exist a polynomial P and $g \in H^p$ such that*

$$f = I_\alpha(g+P) \quad \textit{with} \quad \|g\|_{H^p} \leq C_{\alpha,p}\|T_{m,\alpha}(f)\|_p. \tag{18}$$

Proof. Choose $\varphi \in C_0^\infty(B(0,1/2))$ which has vanishing moments at least up to the order $\max(m-1,[2\alpha])$ and $|\widehat{\varphi}(t\xi)|$ does not vanish identically as a function of $t>0$ for $\xi \neq 0$. Put $u(x,t) = (f*\varphi_t)(x)$. Clearly, $\varphi \in \mathcal{O}_{2\alpha}$.

Fix $x \in \mathbb{R}^n$. For $y \in B(0,1/2)$, it results from the cancelation condition of φ that

$$u(x+ty,t) = \int \Big[f(x+t(y-w)) - \sum_{|\sigma|<m} \frac{(\partial^\sigma f)(x)}{\sigma!} [t(y-w)]^\sigma \Big] \varphi(w)\, dw.$$

Thus $|u(x+ty,t)|$ is bounded by a constant times

$$\int_B \Big| f(x+tz) - \sum_{|\sigma|<m} \frac{(\partial^\sigma f)(x)}{\sigma!} (tz)^\sigma \Big| \, dz = \int_B |Q^m_{tz}(x)|\, dz$$

and consequently

$$\int_{B(0,1/2)} |u(x+ty,t)|^2\, dy \le C \Big[\int_B |Q^m_{tz}(x)|\, dz \Big]^2.$$

Inserting this estimate in the definition of

$$S^\alpha_{1/2}(u)(x) = 2^n \bigg(\int_0^\infty \int_{B(0,1/2)} |u(x+ty,t)|^2\, dy \, \frac{dt}{t^{1+2\alpha}} \bigg)^{1/2},$$

we obtain the pointwise estimate $S^\alpha_{1/2}(u)(x) \le C\, T_{m,\alpha}(f)(x)$ from which the desired conclusion follows immediately by Theorem B5. ∎

REMARK 6. Since $T_{m,\alpha}$ annihilates any polynomial of degree less than m, it would be inevitable to have a polynomial factor in the theorem without imposing an additional condition on f such as $f \in \mathcal{A}_\alpha \cup \mathcal{L}_\alpha$.

COROLLARY D2. *Let f be a locally integrable function on $\mathbb{R}^n$ and let m be a positive integer with $m-1 < \alpha < m$. Then $f \in H^p_\alpha$ for $n/(n+\alpha) < p \le 2$ if and only if $f \in H^p$ and $T_{m,\alpha}(f) \in L^p$. Moreover,*

$$\|f\|_{H^p_\alpha} \approx \|f\|_{H^p} + \|T_{m,\alpha}(f)\|_p.$$

Dealing with $D_{m,\alpha}$, we shall exploit variants of Littlewood–Paley g-functions in proving Strichartz's conjecture. When α is an integer, however, it is not necessary to appeal to a characterization of $I_\alpha(H^p)$.

THEOREM D3. *Let f be a locally integrable function on $\mathbb{R}^n$. For integers $1 \le k < m$, if $D_{m,k}(f) \in L^p$ for $n/(n+k) < p < \infty$, then $f \in I_k(H^p)$ and*

$$\|f\|_{I_k(H^p)} \approx \sum_{|\sigma|=k} \|\partial^\sigma f\|_{H^p} \le C_{k,p} \|D_{m,k}(f)\|_p. \tag{19}$$

Proof. Fix a multi-index σ with $|\sigma| = k$. Choose a Schwartz function ζ with $\widehat{\zeta}(0) \ne 0$ and define

$$\psi(x) = (\partial^\sigma \zeta)(x), \qquad \Psi(x) = \sum_{j=1}^m (-1)^{m-j} \binom{m}{j} \frac{(-j)^k}{j^n}\, \zeta\Big(-\frac{x}{j}\Big).$$

Note that $\widehat{\Psi}(0) \neq 0$. Setting $U(x,t) = [(\partial^\sigma f) * \Psi_t](x)$, we shall derive

$$U^+(x) = \sup_{t>0} |U(x,t)| \leq C_k\, D_{m,k}(f)(x) \tag{20}$$

from which the result follows immediately. On account of the identity

$$\Delta_y^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+jy) \tag{21}$$

and $\widehat{\psi}(0) = 0$, we obtain

$$\begin{aligned} U(x,t) &= t^{-k}[f * (\partial^\sigma \Psi)_t](x) \\ &= t^{-k} \int f(x-y) \Big\{ \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} (jt)^{-n} \psi\Big(-\frac{y}{jt}\Big) \Big\}\, dy \\ &= t^{-k} \int (\Delta_y^m f)(x) \psi_t(y)\, dy. \end{aligned}$$

For any nonnegative measurable function g, we may write

$$\int_{\mathbb{R}^n} g(x)\, dx = \log 2 \int_0^\infty \int_A g(ry) r^n\, dy\, \frac{dr}{r}, \tag{22}$$

where $A = \{1/2 \leq |y| \leq 1\}$ (see [Sz2]). It follows that

$$|U(x,t)| \leq Ct^{-k} \int_0^\infty \int_A |\Delta_{ry}^m f(x)|\, |\psi_t(ry)| r^n\, dy\, \frac{dr}{r}.$$

Since $|\psi_t(ry)| \leq C_n\, t^{-n}\, (1 + r/t)^{-N}$ for $y \in A$ and any $N > 0$, we get

$$\begin{aligned} |U(x,t)| &\leq Ct^{-k} \int_0^\infty \int_A |\Delta_{ry}^m f(x)|\, dy \Big(\frac{r}{t}\Big)^n \Big(1 + \frac{r}{t}\Big)^{-N} \frac{dr}{r} \\ &\leq C\bigg(\int_0^\infty \Big[\int_B |\Delta_{ry}^m f(x)|\, dy \Big]^2 \frac{dr}{r^{1+2k}} \bigg)^{1/2} \\ &\quad \times \bigg(\int_0^\infty \Big(\frac{r}{t}\Big)^{2n+2k} \Big(1 + \frac{r}{t}\Big)^{-2N} \frac{dr}{r} \bigg)^{1/2} = CD_{m,k}(f)(x) \end{aligned}$$

where we take $N > n + k$, which proves the inequality (20). ∎

THEOREM D4. *Let f be a locally integrable function on $\mathbb{R}^n$. For a positive integer m with $m > \alpha$, suppose that $D_{m,\alpha}(f) \in L^p$ for $n/(n+\alpha) < p < \infty$. Then there exist a polynomial P and $g \in H^p$ such that*

$$f = I_\alpha(g + P) \quad \textit{with} \quad \|g\|_{H^p} \leq C_{\alpha,p} \|D_{m,\alpha}(f)\|_p. \tag{23}$$

Proof. Choose a radial Schwartz function θ such that $\widehat{\theta}$ has compact support away from the origin and define

$$\varphi(x)=\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}j^{-n}\theta\Big(-\frac{x}{j}\Big).$$

In view of Theorem C1, it will be sufficient to show that

$$g_{\varphi}^{\alpha}(f)(x)\le C_\alpha D_{m,\alpha}(f)(x). \tag{24}$$

As is easily verified, $(f*\varphi_t)(x)=\int(\Delta_y^m f)(x)\theta_t(y)\,dy$. From this point on, we reproduce the arguments in [Sz2]. Using (22), we get

$$|(f*\varphi_t)(x)|\le C\int_0^\infty\int_A|\Delta_{ry}^m f(x)|\,|\theta_t(ry)|r^n\,dy\,\frac{dr}{r}.$$

Setting

$$K(r)=r^{n+\alpha-1/2}\sup_{y\in\mathbb{R}^n}|\theta(ry)|,\qquad u(r)=r^{-\alpha-1/2}\int_A|\Delta_{ry}^m f(x)|\,dy,$$

it is straightforward to derive the estimate

$$t^{-\alpha-1/2}|(f*\varphi_t)(x)|\le\frac{C}{t}\int_0^\infty K\Big(\frac{r}{t}\Big)u(r)\,dr=C(Hu)(t). \tag{25}$$

A classical theorem of Hardy–Littlewood–Pólya states that the operator H maps $L^p(0,\infty)$ boundedly into itself for $1<p<\infty$ if

$$\|H\|_{p\to p}=\int_0^\infty|K(r)|r^{-1/p}\,dr<\infty.$$

Since this condition certainly holds for $p=2$, taking the $L^2(0,\infty)$ norm on both sides of (25), we obtain (24). ∎

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Received 12 July 2004;
revised 27 January 2005 (4482)