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UNIVERSAL COMPLETELY REGULAR DENDRITES

BY

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Dedicated to the memory of Professor Janusz J. Charatonik

Abstract. We define a dendrite $E_{\{n\}}$ which is universal in the class of all completely regular dendrites with order of points not greater than n. In particular, the dendrite $E_{\{\omega\}}$ is universal in the class of all completely regular dendrites. The construction starts with the standard universal dendrite $D_{\{n\}}$ of order n described by J. J. Charatonik.

We use the term *continuum* to mean any nonempty, compact and connected metrizable space. A continuum X is said to be:

- *regular* if X has a basis of open sets with finite boundaries;
- *completely regular* if each nondegenerate subcontinuum of X has nonempty interior (in X);
- a *dendrite* if X is locally connected and contains no simple closed curve.

It is well known that any dendrite is regular ([8, §51, VI, p. 301]), any planar, completely regular continuum is regular and every regular continuum is hereditarily locally connected ([8, §51, IV]). Thus any completely regular continuum that contains no simple closed curve is a dendrite.

For more results concerning the properties of dendrites and their behavior under some special mappings we refer the reader to [3].

A space X is said to be *universal* for a class $\mathcal{F}$ of spaces provided that $X \in \mathcal{F}$ and each member of $\mathcal{F}$ can be homeomorphically imbedded in X. Note that the definition of a universal space does not guarantee its uniqueness.

It is known that:

(1) There exists a universal dendrite ([13]).
(2) There is no universal regular continuum ([12], compare [9, Th. 1.6]).
(3) There exists a universal completely regular continuum ([5]).
(4) There is no universal planar regular continuum ([6], compare [10, Th. 4.2]).

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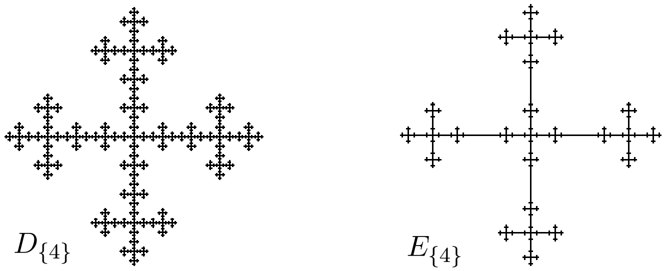
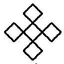
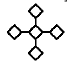
The problem of existence of a universal element in the class of all planar completely regular continua raised by J. Krasinkiewicz ([7]) is still open.

The universal dendrite, first constructed by Ważewski, is described in [1] (denoted as D_ω). In [2], [4] there are descriptions of dendrites $D_{\{n\}}$ which are universal in the class of all dendrites with order of points not greater than n.

In this paper we define dendrites $E_{\{n\}}$ with similar properties in the class of completely regular dendrites.



$D_{\{4\}}$ $E_{\{4\}}$

For the reader's convenience we picture the dendrites $D_{\{4\}}$ and $E_{\{4\}}$ ($D_{\{4\}}$ is sometimes called the Janiszewski cemetery). They are limits of the following spaces. Starting with a square we inductively replace each square with a small copy of the appropriate pattern:  for $D_{\{4\}}$ and  for $E_{\{4\}}$. The dendrites $E_{\{n\}}$ will be defined axiomatically.

First we recall the concept of order of a point (see [8, §51]). By the *order of a point p in a space X*, written $\operatorname{ord}(p, X)$, is meant the least cardinal number n such that p has an arbitrarily small neighborhood in X with boundary of cardinality $\le n$. We say that p is of order ω in X if p has arbitrarily small neighborhoods in X with finite boundaries but $\operatorname{ord}(p, X) > n$ for any natural number n.

We put $\operatorname{Ord}_n X$ for the set of all points of X of order n.

A point of order 2 (resp. > 2) is called an *ordinary point* (resp. a *branch point*).

The set of all ordinary points is a dense subset of a dendrite and the set of all branch points of a dendrite is at most countable ([8, §51, VI, Theorems 7, 8].

The symbol pq stands for the arc with end points p and q. An arc pq is said to be *free* in a space X if $pq \setminus \{p, q\}$ is an open subset of X. For dendrites this is equivalent to $pq \setminus \{p, q\}$ not containing any branch points.

Note that the above definition of order of a point in a regular continuum coincides with the definition of order of a point p as the number of arcs intersecting exactly in their common end point p (see [8, §51, I, 8, and the following remark]).

We use the following concept of arc-density.

DEFINITION 1. We say that a set $Q \subset X$ is *arcwise dense at* $a \in X$ if $Q \cap ab \setminus \{a\} \neq \emptyset$ for any arc $ab \subset X$; Q is *arcwise dense in* X if Q is arcwise dense at each $a \in X$.

In [2] and [4] there are some generalizations concerning the uniqueness of some special universal dendrites. They may be summarized in the following theorem.

THEOREM 2 ([2], [4]). *Let* $\emptyset \neq S \subset \{3, 4, \ldots, \omega\}$ *be given. There exists a unique* D_S *with the following two properties*:

($\mathcal{D}'_S$) *the order of any branch point of* D_S *belongs to* S,
($\mathcal{D}''_S$) $\mathrm{Ord}_s D_S$ *is arcwise dense in* D_S *for any* $s \in S$.

Moreover, *if* $m = \max S$, *then* D_S *and* $D_{\{m\}}$ *are universal in the class of dendrites having orders at most* m. *In particular* $D_{\{\omega\}}$ *is a universal dendrite.*

We define axiomatically completely regular dendrites E_S with similar properties.

DEFINITION 3. Let $\emptyset \neq S \subset \{3, 4, \ldots, \omega\}$ be given. We denote by E_S any dendrite X satisfying the following three conditions:

($\mathcal{E}'_S$) the order of any branch point of X belongs to S,
($\mathcal{E}''_S$) $\mathrm{Ord}_s X$ is arcwise dense at any non-ordinary point of X for any $s \in S$,
($\mathcal{E}'''$) for any arc $ab \subset X$ there is a free arc $a'b'$ in X contained in ab.

We show the existence and uniqueness of E_S, and that E_S is universal in the class of completely regular dendrites with branch points having orders in S. But first we describe some details of a method of replacing points with arcs.

Let q be a separating point of a continuum X and let $X \setminus \{q\} = C^0 \cup C^1$ be the union of disjoint open sets. We can replace q with an arc by attaching its end points to C^0 and C^1. Formally, we can define this new space $\mathcal{A}(X, \{q\})$ as the subspace of the product $X \times [0, 1]$:

$$\mathcal{A}(X, \{q\}) = (C^0 \times \{0\}) \cup (\{q\} \times [0, 1]) \cup (C^1 \times \{1\}).$$

Note that if q separates X into exactly two components, then $\mathcal{A}(X, \{q\})$ is uniquely defined (up to homeomorphism).

For a countable set $Q = \{q_1, q_2, \ldots\}$ of separating points of a continuum X we may replace these points with arcs inductively. In short, if $X \setminus \{q_n\} = C^0_n \cup C^1_n$, where C^0_n and C^1_n are open and disjoint for $n = 1, 2, \ldots$, then we put

$$\mathcal{A}(X, Q) = \{(x, t_1, t_2, \ldots) \in X \times [0, 1]^\omega : x \in C^i_n \Rightarrow t_n = i\}.$$

Note that if for any $q_n \in Q$ the set $X \setminus \{q_n\}$ has exactly two components, then the space $\mathcal{A}(X,Q)$ is uniquely defined (up to homeomorphism) and it does not depend on the enumeration of elements of Q.

Observe that the projection $\pi : \mathcal{A}(X,Q) \to X$ is monotone since $\pi^{-1}(q)$ is a free arc of $\mathcal{A}(X,Q)$ for $q \in Q$, and $\pi^{-1}(x)$ is a singleton for $x \notin Q$.

PROPOSITION 4. *Let Q be a countable set of ordinary points of a dendrite X.*

(i) *Then $\mathcal{A}(X,Q)$ is a dendrite.*
(ii) *If Q is arcwise dense in X, then $\mathcal{A}(X,Q)$ is completely regular.*

Proof. (i) Of course $\mathcal{A}(X,\{q_1\})$ is a dendrite and the projection $f_1 : \mathcal{A}(X,\{q_1\}) \to X$ is monotone. Observe that for $n = 1,2,\ldots$ we have by induction

$$\mathcal{A}(X,\{q_1,\ldots,q_n,q_{n+1}\}) = \mathcal{A}(\mathcal{A}(X,\{q_1,\ldots,q_n\}),(f_1 \circ \cdots \circ f_n)^{-1}(q_{n+1})),$$

hence the space $\mathcal{A}(X,\{q_1,\ldots,q_n,q_{n+1}\})$ is a dendrite and the natural projection $f_{n+1} : \mathcal{A}(X,\{q_1,\ldots,q_{n+1}\}) \to \mathcal{A}(X,\{q_1,\ldots,q_n\})$ is monotone.

So $\mathcal{A}(X,Q)$ is homeomorphic to the inverse limit

$$\varprojlim\{\mathcal{A}(X,\{q_1,\ldots,q_n\}),f_n\}$$

of the system of dendrites with monotone bonding mappings, hence it is a dendrite (see [11, Theorem 10.36]).

(ii) As Q is arcwise dense in X, the sets $\pi^{-1}(X\setminus Q)$ and $\mathrm{cl}(\pi^{-1}(X \setminus Q))$ are zero-dimensional. Therefore each nondegenerate subcontinuum of $\mathcal{A}(X,Q)$ contains an interior point of some free arc $\pi^{-1}(q_n)$, hence it has nonempty interior.

THEOREM 5. *Let $\emptyset \neq S \subset \{3,4,\ldots,\omega\}$ be given.*

(i) *If Q is a countable set of ordinary points of the dendrite D_S which is arcwise dense in D_S, then the dendrite $\mathcal{A}(D_S,Q)$ has properties $(\mathcal{E}'_S)$, $(\mathcal{E}''_S)$ and $(\mathcal{E}''')$.*
(ii) *If a dendrite X has properties $(\mathcal{E}'_S)$, $(\mathcal{E}''_S)$ and $(\mathcal{E}''')$, then it is homeomorphic to $\mathcal{A}(D_S,Q)$ for some countable arcwise dense set Q of ordinary points in D_S.*
(iii) *Let Q', Q'' be countable sets of ordinary points of the dendrite D_S which are arcwise dense in D_S. Then there exists an autohomeomorphism h of D_S such that $h(Q') = Q''$. Moreover, $\mathcal{A}(D_S,Q')$ is homeomorphic to $\mathcal{A}(D_S,Q'')$.*

Proof. (i) By Proposition 4 the space $\mathcal{A}(D_S,Q)$ is a completely regular dendrite, hence it has property $(\mathcal{E}''')$.

It follows easily from the construction of $\mathcal{A}(D_S, Q)$ that $\operatorname{ord}(z, \mathcal{A}(D_S, Q)) = \operatorname{ord}(\pi(z), D_S)$ for each $z \in \mathcal{A}(D_S, Q)$, and therefore properties $(\mathcal{D}'_S)$ and $(\mathcal{D}''_S)$ of D_S yield properties $(\mathcal{E}'_S)$ and $(\mathcal{E}''_S)$ of $\mathcal{A}(D_S, Q)$.

(ii) First, notice that $(\mathcal{E}''_S)$ implies that the end points of any free arc in X are ordinary. Therefore maximal free arcs are pairwise disjoint. Now we identify points of free arcs; formally, we define $x \approx y$ iff $x = y$ or xy is a free arc in X.

We shall prove that the quotient space $X/\approx$ is the dendrite D_S.

Since the natural projection $p : X \to X/\approx$ is monotone, the space $X/\approx$ is a dendrite and for any nonfree arc ab of X the projection $p(ab)$ is an arc of $X/\approx$. One can easily verify that $\operatorname{ord}(x, X) = \operatorname{ord}(p(x), X/\approx)$ for each $x \in X$. Therefore, since X satisfies conditions $(\mathcal{E}'_S)$ and $(\mathcal{E}''_S)$, the dendrite $X/\approx$ satisfies conditions $(\mathcal{D}'_S)$ and $(\mathcal{D}''_S)$ of Theorem 2. Thus $X/\approx$ is the dendrite D_S.

Let Q denote the set of all nondegenerate equivalence classes of $\approx$. Since for any $q \in Q$ the set $p^{-1}(q)$ is a maximal free arc of X, Q is a countable set of ordinary points of $X/\approx$. From $(\mathcal{E}''')$ it follows that Q is arcwise dense in $X/\approx$.

It is easy to see that $\mathcal{A}(X/\approx, Q)$ is homeomorphic to X.

(iii) The proof of the existence of the homeomorphism h is similar to the proof of Theorem 6.2 of [4] (cf. Lemma 6.13 of [4]), therefore it is omitted. Of course h induces a natural homeomorphism between $\mathcal{A}(D_S, Q')$ and $\mathcal{A}(D_S, Q'')$.

THEOREM 6. *Let $\emptyset \neq S \subset \{3, 4, \ldots, \omega\}$ be given. There exists a unique dendrite E_S with properties $(\mathcal{E}'_S)$, $(\mathcal{E}''_S)$ and $(\mathcal{E}''')$. The space E_S is universal in the class of completely regular dendrites with branch points having orders in S, i.e. in the class of dendrites which satisfy $(\mathcal{E}'_S)$ and $(\mathcal{E}''')$.*

Proof. The existence and uniqueness of E_S follow from Theorem 5. To prove its universality let a completely regular dendrite X have orders of its branch points in S. We can assume that $X \subset D_S$ (see Theorems 6.6–6.8 of [4]). Since $\operatorname{Ord}_2 D_S$ is arcwise dense in D_S the set $\operatorname{Ord}_2 D_S \cap X$ is arcwise dense in X. Since X satisfies $(\mathcal{E}''')$ we can find a countable set $Q_1 \subset \operatorname{Ord}_2 D_S \cap X$ arcwise dense in X which is contained in the union of the interiors of all free arcs in X. Of course $Q_1 \subset \operatorname{Ord}_2 X$. One can easily verify that $\mathcal{A}(X, Q_1)$ is homeomorphic to X.

Further let Q_2 be a countable set arcwise dense in $D_S \setminus X$ such that $Q_2 \subset \operatorname{Ord}_2 D_S$. Observe that for any arc $ab \subset D_S$ we have $(Q_1 \cup Q_2) \cap ab \neq \emptyset$, i.e. $Q_1 \cup Q_2$ is arcwise dense in D_S. Theorem 5 shows that $\mathcal{A}(D_S, Q_1 \cup Q_2)$ is homeomorphic to E_S. Obviously $\mathcal{A}(X, Q_1)$ is homeomorphic to a subspace of $\mathcal{A}(D_S, Q_1 \cup Q_2)$.

The proof is complete.

COROLLARY 7. *Let $\emptyset \neq S \subset \{3, 4, \ldots, \omega\}$ be given and suppose $m = \max S$ exists. Then E_S and $E_{\{m\}}$ are universal in the class of completely regular dendrites with branch points of orders at most m. In particular, $E_{\{\omega\}}$ is universal in the class of all completely regular dendrites.*

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