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# ON SELFINJECTIVE ARTIN ALGEBRAS HAVING NONPERIODIC GENERALIZED STANDARD AUSLANDER–REITEN COMPONENTS

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*Dedicated to Otto Kerner on the occasion of his sixtieth birthday*

**Abstract.** We describe the structure of all selfinjective artin algebras having at least three nonperiodic generalized standard Auslander–Reiten components. In particular, all selfinjective artin algebras having a generalized standard Auslander–Reiten component of Euclidean type are described.

Throughout the paper, by an *algebra* is meant a basic, connected, artin algebra (associative, with an identity) over a fixed commutative artinian ring $k$. For an algebra $A$, we denote by $\operatorname{mod} A$ the category of finitely generated right $A$-modules, and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_k(-, E)$, where $E$ is a minimal injective cogenerator in $\operatorname{mod} k$. An algebra $A$ is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective $A$-modules are injective. If $A$ is a selfinjective algebra, then the left and the right socles of $A$ coincide, and we denote them by $\operatorname{soc} A$. Two selfinjective algebras $A$ and $\Lambda$ are said to be *socle equivalent* if the factor algebras $A/\operatorname{soc} A$ and $\Lambda/\operatorname{soc} \Lambda$ are isomorphic.

An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$ where $\widehat{B}$ is the *repetitive algebra* (see [9]) (locally bounded, without identity)

$$\widehat{B} = \bigoplus_{r \in \mathbb{Z}} (B_r \oplus (DB)_r)$$

of an algebra $B$, where $B_r = B$ and $(DB)_r = DB$ for all $r \in \mathbb{Z}$, the multiplication in $\widehat{B}$ is defined by

$$(a_r, f_r)_r \cdot (b_r, g_r)_r = (a_r b_r, a_r g_r + f_r b_{r+1})_r$$

for $a_r, b_r \in B_r$, $f_r, g_r \in (DB)_r$, and $G$ is an admissible group of automorphisms of $\widehat{B}$. More precisely, for a fixed set $\mathcal{E} = \{e_i \mid 1 \le i \le m\}$ of primitive orthogonal idempotents of $B$ with $1_B = e_1 + \ldots + e_m$, consider

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the canonical set $\widehat{\mathcal{E}} = \{e_{j,r} \mid 1 \leq j \leq m,\ r \in \mathbb{Z}\}$ of primitive orthogonal idempotents of $\widehat{B}$ such that $e_{j,r}\widehat{B} = (e_jB)_r \oplus (e_jDB)_r$ for $1 \leq j \leq m$ and $r \in \mathbb{Z}$. By an *automorphism* of $\widehat{B}$ we mean a $k$-algebra automorphism of $\widehat{B}$ which fixes $\widehat{\mathcal{E}}$. A group $G$ of automorphisms of $\widehat{B}$ is said to be *admissible* if the induced action of $G$ on $\widehat{\mathcal{E}}$ is free and has finitely many orbits. Then the *orbit algebra* $\widehat{B}/G$ is a selfinjective algebra and the $G$-orbits in $\widehat{\mathcal{E}}$ form a canonical set of primitive orthogonal idempotents of $\widehat{B}/G$ whose sum is the identity of $\widehat{B}/G$ (see [7]). We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ whose restriction to each copy $B_r \oplus (DB)_r$ is the identity map $B_r \oplus (DB)_r \to B_{r+1} \oplus (DB)_{r+1}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the *trivial extension* $B \ltimes D(B)$ of $B$ by $DB$. An automorphism $\varphi$ of $\widehat{B}$ is said to be *positive* (respectively, *rigid*) when $\varphi(B_r) \subseteq \sum_{i\geq r}(B_i \oplus (DB)_i)$ (respectively, $\varphi(B_r) = B_r$) for any $r \in \mathbb{Z}$. Moreover, $\varphi$ is said to be *strictly positive* if it is positive but not rigid. We refer to [12] for more information on the automorphisms of repetitive algebras, and to [29] for general background on selfinjective algebras.

For an algebra $A$, we denote by $\Gamma_A$ the *Auslander–Reiten quiver* of $A$, and by $\tau_A$ and $\tau_A^-$ the Auslander–Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, respectively. We shall identify the vertices of $\Gamma_A$ with the corresponding indecomposable $A$-modules. By a *component* of $\Gamma_A$ we mean a connected component of the translation quiver $\Gamma_A$. A component $\mathcal{C}$ of $\Gamma_A$ is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module. A subquiver $\Gamma$ of $\Gamma_A$ is said to be *right stable* (respectively, *left stable*) if $\tau_A^-$ (respectively, $\tau_A$) is defined on all modules in $\Gamma$. A component $\mathcal{C}$ of $\Gamma_A$ is called *nonperiodic* if $\mathcal{C}$ does not contain $\tau_A$-periodic modules, that is, modules $X$ with $X = \tau_A^m X$ for some $m \geq 1$. Following [21], a subquiver $\mathcal{D}$ of $\Gamma_A$ is said to be *generalized standard* if $\mathrm{rad}_A^\infty(X,Y) = 0$ for all modules $X$ and $Y$ in $\mathcal{D}$. Recall that $\mathrm{rad}_A^\infty(X,Y)$ is the intersection of all finite powers $\mathrm{rad}_A^m(X,Y)$, for $m \geq 1$, of the radical $\mathrm{rad}_A(X,Y)$ of $\mathrm{Hom}_A(X,Y)$. Finally, the *annihilator* $r_A(\Sigma)$ of a subquiver $\Sigma$ of $\Gamma_A$ is the intersection of the (right) annihilators $r_A(X)$ of all modules $X$ in $\Sigma$. Clearly, $r_A(\Sigma)$ is a two-sided ideal of $A$. If $r_A(\Sigma) = 0$ then $\Sigma$ is said to be *faithful*. We refer to [2] for background on the Auslander–Reiten theory.

In this paper we are interested in the structure of selfinjective algebras whose Auslander–Reiten quiver contains nonperiodic generalized standard components. There has been work connecting tilting theory (see [8], [16], [17]) with selfinjective algebras via trivial extension algebras (see [1], [14], [19], [27]). Recently, a more general class of selfinjective algebras of tilted type has attracted much attention (see [6], [13], [17], [21]–[25]). Let $B$ be a tilted algebra of type $\Delta$ (see [8]) which is not a Dynkin quiver. Then the

Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ is of the form

$$\Gamma_{\widehat{B}} = \bigvee_{p\in\mathbb{Z}} (\mathcal{X}_p \vee \mathcal{R}_p)$$

where, for each $p \in \mathbb{Z}$, $\mathcal{X}_p$ is a component with the stable part of the form $\mathbb{Z}\Delta$, $\mathcal{R}_p$ is a family of components whose stable parts are tubes (if $\Delta$ is Euclidean) or of type $\mathbb{Z}\mathbb{A}_\infty$ (if $\Delta$ is wild), and $\nu_{\widehat{B}}(\mathcal{X}_p) = \mathcal{X}_{p+2}$ and $\nu_{\widehat{B}}(\mathcal{R}_p) = \mathcal{R}_{p+2}$ for the induced action of $\nu_{\widehat{B}}$ on $\Gamma_{\widehat{B}}$ (see [1], [6], [19]). Further, an automorphism $\varphi$ of $\widehat{B}$ is positive (respectively, strictly positive) if and only if there exists $q \geq 0$ (respectively, $q > 0$) such that $\varphi(\mathcal{X}_p) = \mathcal{X}_{p+q}$ and $\varphi(\mathcal{R}_p) = \mathcal{R}_{p+q}$ for all $p \in \mathbb{Z}$. In fact, it is known that any admissible group $G$ of automorphisms of $\widehat{B}$ is an infinite cyclic group generated by a strictly positive automorphism $g$ of $\widehat{B}$ (see [6, Lemma 3.6]). Further, the push-down functor $F^B_\lambda : \operatorname{mod}\widehat{B} \to \operatorname{mod}\widehat{B}/G$, associated to the Galois covering $F^B : \widehat{B} \to \widehat{B}/G$, is dense and preserves the Auslander–Reiten sequences (see [6, 3.7]). Therefore, if $g(\mathcal{X}_0) = \mathcal{X}_m$ with $m \geq 1$, then $\Gamma_{\widehat{B}/G}$ is obtained from $\Gamma_{\widehat{B}}$ by identifying, via $F^B_\lambda$, $\mathcal{X}_p$ with $\mathcal{X}_{p+m}$ and $\mathcal{R}_p$ with $\mathcal{R}_{p+m}$, for all $p \in \mathbb{Z}$. Thus $\Gamma_{\widehat{B}/G}$ is of the form

$$F^B_\lambda(\mathcal{X}_0) \vee F^B_\lambda(\mathcal{R}_0) \vee F^B_\lambda(\mathcal{X}_1) \vee F^B_\lambda(\mathcal{R}_1) \vee \ldots \vee F^B_\lambda(\mathcal{X}_{m-1}) \vee F^B_\lambda(\mathcal{R}_{m-1}).$$

Moreover, we have $m \geq 2$ (respectively, $m \geq 3$) if and only if $G$ is generated by an automorphism of the form $\varphi\nu_{\widehat{B}}$ for some positive (respectively, strictly positive) automorphism $\varphi$ of $\widehat{B}$ (see [6, Proposition 3.8 and Corollary 3.9]). Finally, for $m \geq 3$, all nonperiodic components $F^B_\lambda(\mathcal{X}_r)$, $0 \leq r \leq m-1$, are generalized standard (see [6, Corollaries 3.9 and 3.10]). We also mention that, for an arbitrary tilted algebra $B = \operatorname{End}_H(T)$ of type $\Delta$, there exist tilted algebras $B_1 = \operatorname{End}_H(T_1)$ and $B_2 = \operatorname{End}_H(T_2)$ of type $\Delta$, given by tilting $H$-modules $T_1$ and $T_2$ without nonzero preprojective or nonzero preinjective direct summands, respectively, such that $\widehat{B}_1 \cong \widehat{B} \cong \widehat{B}_2$ (see [1] and [13]).

THEOREM 1. *Let $A$ be a selfinjective algebra. Then the Auslander–Reiten quiver $\Gamma_A$ of $A$ admits at least three nonperiodic generalized standard components if and only if $A$ is isomorphic to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where $B$ is a tilted algebra not of Dynkin type and $\varphi$ is a strictly positive automorphism of $\widehat{B}$.*

*Proof.* The sufficiency part follows from the results stated above. For the necessity, assume that $\Gamma_A$ admits at least three nonperiodic generalized standard components. We know from [21, Theorem 2.3] that all such components have only finitely many $\tau_A$-orbits. Let $\mathcal{C}$ be a nonperiodic generalized standard component of $\Gamma_A$. Then it follows from the dual of [11, Theorem 3.4] that $\mathcal{C}$ admits a right stable full translation subquiver $\mathcal{D}$ of the form $(-\mathbb{N})\Delta$, for some (finite) valued quiver $\Delta$ without oriented cycles, which is closed under successors in $\mathcal{C}$. Let $M$ be the direct sum of modules lying on $\Delta$, $I$ the annihilator $r_A(\mathcal{D})$ of $\mathcal{D}$ in $A$, and $B = A/I$. Then $\mathcal{D}$ is a faithful full translation subquiver of the Auslander–Reiten quiver $\Gamma_B$ closed under successors (in $\Gamma_B$), and $M$ is a faithful $B$-module (see [20, Lemma 3]). Since $\mathcal{C}$ is a generalized standard component of $\Gamma_A$, $\mathcal{D}$ is a generalized standard subquiver of $\Gamma_B$, and we infer from [22, Proposition 5.3] that $M$ is a tilting $B$-module, $H = \mathrm{End}_B(M)$ is a hereditary algebra of type $\Delta$, and hence $B$ is a tilted algebra of the form $B \cong \mathrm{End}_H(T)$, for the tilting $H$-module $T = DM$ (see [20, Theorem 3]). In fact, $\mathcal{D}$ is a full translation subquiver of the connecting component $\mathcal{C}_T$ of $\Gamma_B$ determined by $T$, and hence $T$ has no nonzero preprojective direct summands (because $\mathcal{D}$ does not contain injective modules) (see [17]).

We may choose a complete set $\{e_i \mid 1 \le i \le n\}$ of primitive orthogonal idempotents of $A$ such that $1 = e_1 + \ldots + e_n$ and $\{e_i \mid 1 \le i \le m\}$, for some $m \le n$, is the set of all idempotents $e_i$ with $i \in \{1, \ldots, n\}$ which are not in $I$. Then the idempotent $e = e_1 + \ldots + e_m$ is uniquely determined by $I$ up to an inner automorphism, and is called a *residual identity* of $B = A/I$ (see [22]). We proved in [22, Theorem 5.1] that $IeI = 0$ and $Ie$ is an injective cogenerator in $\operatorname{mod} B$. Clearly, the ordinary quiver $Q_B$ of $B$ has no oriented cycles, since $B$ is a tilted algebra (see [8]). Therefore, applying [24, Theorem 4.1], we conclude that $A$ is socle equivalent to an algebra $\Lambda = \widehat{B}/(\psi\nu_{\widehat{B}})$ for a positive automorphism $\psi$ of $\widehat{B}$. Since the factor algebras $A/\operatorname{soc} A$ and $\Lambda/\operatorname{soc}\Lambda$ are isomorphic, the Auslander–Reiten quivers $\Gamma_{A/\operatorname{soc} A}$ and $\Gamma_{\Lambda/\operatorname{soc}\Lambda}$ are also isomorphic, and hence there is a canonical correspondence between the components of $\Gamma_A$ and $\Gamma_\Lambda$. In fact, it follows from [24, Proposition 5.1(ii)] that there is a bijection between the nonperiodic generalized standard components of $\Gamma_A$ and $\Gamma_\Lambda$.

Invoking now our assumption on $A$, we conclude that $\Gamma_\Lambda$ admits at least three nonperiodic generalized standard components, and consequently $\psi$ is a strictly positive automorphism of $\widehat{B}$. But then, for the Nakayama automorphism $\nu$ of $A$, we have $e_i \neq e_{\nu(i)}$ for all $i \in \{1, \ldots, n\}$ (see [25, Lemma 4.1]). Applying now [25, Proposition 2.5] we conclude that the canonical algebra epimorphism $eAe \to eAe/eIe$ splits. Finally, applying [24, Theorem 3.8], we infer that $A$ is isomorphic to an algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for a positive automorphism $\varphi$ of $\widehat{B}$, which is in fact strictly positive, because the number of nonperiodic generalized standard components in $\Gamma_A$ is at least three. This finishes the proof. ∎

In the above proof we have faced the situation when two selfinjective algebras of the forms $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$ and $\Lambda = \widehat{B}/(\psi\nu_{\widehat{B}})$, for a tilted algebra $B$ (not of Dynkin type) and strictly positive automorphisms $\varphi$ and $\psi$ of $\widehat{B}$, are socle equivalent. We point out that $A$ and $\Lambda$ can be nonisomorphic, as the following example shows.

EXAMPLE 2. Let $Q$ be a finite connected quiver with sink-source orientation, and $k$ a field. Fix a source $x$ in $Q$ and consider the quiver $\Delta$ obtained from $Q$ by adding two arrows $x \xrightarrow{\alpha} y$ and $x \xrightarrow{\beta} z$, where $y \neq z$ are not vertices of $Q$. Then $\Delta$ has again a sink-source orientation, and consequently the path algebra $B = K\Delta$ of $\Delta$ is a radical square zero hereditary algebra. Let $\sigma$ be the automorphism of the algebra $B$ induced by the automorphism of $\Delta$ exchanging the arrows $\alpha$ and $\beta$, and keeping the other arrows of $\Delta$ unchanged. Then $\sigma$ induces an automorphism $\varrho$ of $\widehat{B}$ whose restriction to each part $B_r$ is $\sigma$ [12, Lemma 3.1]. Fix $m \geq 1$, take the strictly positive automorphisms $\varphi_m = \nu_{\widehat{B}}^m$ and $\psi_m = \varrho\nu_{\widehat{B}}^m$ of $\widehat{B}$, and consider the algebras $A(m) = \widehat{B}/(\varphi_m \nu_{\widehat{B}})$ and $\Lambda(m) = \widehat{B}/(\psi_m \nu_{\widehat{B}})$. Then $A(m)$ is a selfinjective algebra whose Nakayama permutation (see [29] for definition) has order $m$, $\Lambda(m)$ is a selfinjective algebra whose Nakayama permutation has order $2m$, and consequently $A(m)$ and $\Lambda(m)$ are nonisomorphic. On the other hand, the socle factors $A(m)/\operatorname{soc} A(m)$ and $\Lambda(m)/\operatorname{soc} \Lambda(m)$ are radical square zero algebras having the same ordinary (Gabriel) quiver, and hence they are isomorphic. Therefore, $A(m)$ and $\Lambda(m)$ are socle equivalent.

Following [3, 8.2] an algebra $A$ is said to be *strictly wild* if there are modules $X$ and $Y$ in $\operatorname{mod} A$ whose endomorphism rings $\operatorname{End}_A(X)$ and $\operatorname{End}_A(Y)$ are division rings, $\operatorname{Hom}_A(X,Y) = 0 = \operatorname{Hom}_A(Y,X)$ and the inequality $\dim_{\operatorname{End}_A(Y)} \operatorname{Ext}^1_A(X,Y) \cdot \dim \operatorname{Ext}^1_A(X,Y)_{\operatorname{End}_A(X)} \geq 5$ holds. If $k$ is a field, then it follows from [15, 5.4] (see also [3, Lemma 8.2]) that $A$ is strictly wild if and only if there is a field extension $K$ of $k$ and a $K\langle x,y\rangle$-$A$-bimodule $M$ which is finitely generated projective over $K\langle x,y\rangle$ and such that the tensor product functor $-\otimes_{K\langle x,y\rangle} M : \operatorname{Mod} K\langle x,y\rangle \to \operatorname{Mod} A$ is fully faithful. Here, $K\langle x,y\rangle$ denotes the free associative $K$-algebra in two generators, and $\operatorname{Mod} K\langle x,y\rangle$ and $\operatorname{Mod} A$ the categories of all $K\langle x,y\rangle$-modules and all $A$-modules, respectively. Moreover, an algebra $A$ is said to be *wild* (see [3, 7.4]) if there exists a field extension $K$ of $k$ and a $K\langle x,y\rangle$-$A$-bimodule $M$, finitely generated and projective as $K\langle x,y\rangle$-module, such that the functor $-\otimes_{K\langle x,y\rangle} M : \operatorname{Mod} K\langle x,y\rangle \to \operatorname{Mod} A$ preserves indecomposability and isomorphism classes of modules. It is known that a wild hereditary algebra is strictly wild (see [3, Theorem 8.4]) but in general a wild algebra need not be strictly wild.

THEOREM 3. *Let $A$ be a selfinjective algebra. Then the following statements are equivalent*:

(i) *$A$ is not strictly wild and $\Gamma_A$ admits a nonperiodic generalized standard component.*

(ii) *$A$ is not wild and $\Gamma_A$ admits a nonperiodic generalized standard component.*

(iii) *$\Gamma_A$ admits a nonperiodic component and all nonperiodic components of $\Gamma_A$ are generalized standard.*

(iv) *$\Gamma_A$ admits a generalized standard component of Euclidean type.*

(v) *$A$ is isomorphic to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where $B$ is a tilted algebra of Euclidean type and $\varphi$ is a strictly positive automorphism of $\widehat{B}$.*

*Proof.* It follows from the proof of Theorem 1 that if $\Gamma_A$ admits a nonperiodic generalized standard component $\mathcal{C}$ then $A$ is socle equivalent to an algebra $\Lambda = \widehat{B}/(\psi\nu_{\widehat{B}})$, where $B$ is a tilted algebra $B = \mathrm{End}_H(T)$, for a hereditary algebra $H$ of type $\Delta$ not being a Dynkin quiver and a tilting $H$-module $T$ without nonzero preprojective (equivalently, preinjective) direct summands, and $\psi$ is a positive automorphism of $\widehat{B}$. Further, we know from [21, Corollary 3.3] and [18] that $\mathcal{C}$ is regular if and only if $\Delta$ is a wild quiver with at least three vertices and $T$ is a regular tilting $H$-module. Moreover, it follows from [6, Corollary 3.9] that if $\Gamma_\Lambda$ admits a nonregular nonperiodic generalized standard component then $\psi$ is strictly positive, and hence all nonperiodic components of $\Gamma_\Lambda$ are generalized standard. Finally, it follows from [3, Theorem 8.4] and [26, Theorem 7.5] (see also [10, Theorem 6.2]) that if $\Delta$ is wild then $B$ is strictly wild, and consequently $\Lambda$ is strictly wild.

Assume (i) holds. Then it follows from the above statements that $\Delta$ is a Euclidean quiver, and consequently all nonperiodic components of $\Gamma_\Lambda$ are nonregular. Since, by the assumption (i), $\Gamma_\Lambda$ admits at least one nonperiodic generalized standard component, we then infer that $\psi$ is strictly positive, and consequently $\Gamma_\Lambda$ admits at least three nonperiodic generalized standard components. Invoking now Theorem 1 we deduce that $A$ is isomorphic to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for a strictly positive automorphism $\varphi$ of $\widehat{B}$. Hence (v) holds. Further, the equivalence (iv)$\Leftrightarrow$(v) and the implication (v)$\Rightarrow$(iii) are also direct consequences of the above discussion. Moreover, the implication (ii)$\Rightarrow$(i) follows from the well known fact that every strictly wild algebra is wild (see [3, Lemma 8.2]).

Assume (iii) holds. It has been shown in [21, Theorem 2.3] that every generalized standard component of an Auslander–Reiten quiver contains at most finitely many nonperiodic orbits. On the other hand, if $\Delta$ is wild then $\Gamma_\Lambda$ admits nonperiodic components with stable parts of the form $\mathbb{Z}\mathbb{A}_\infty$, and consequently admits nonperiodic components which are not generalized standard. Therefore, $\Delta$ is a Euclidean quiver, and as above $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$ for a strictly positive automorphism $\varphi$ of $\widehat{B}$. Hence the implication (iii)$\Rightarrow$(v) holds. Moreover, it follows from the structure of the module categories of selfinjective algebras of Euclidean type (see [1], [4], [19]) that $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$

is not wild, and consequently the implication (iii)$\Rightarrow$(ii) also holds. This finishes the proof. ∎

In [24, Theorem 5.5] we proved that a selfinjective algebra $A$ is socle equivalent to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where $B$ is a tilted algebra not of Dynkin type and $\varphi$ is a positive automorphism of $\widehat{B}$, if and only if $\Gamma_A$ admits a nonperiodic generalized standard right stable (respectively, left stable) full translation subquiver which is closed under successors (respectively, predecessors) in $\Gamma_A$. We exhibit below a class of algebras showing that in general we cannot replace "socle equivalent" by "isomorphic" without assuming that $\varphi$ is strictly positive.

Let $K$ be a finite field extension of a field $k$ of characteristic 2 such that the Hochschild cohomology group $H^2(K,K)$ is nonzero, where $K$ is considered as a $k$-algebra (see [5], [22], [28] for existence of such extensions). Take a 2-cocycle $\alpha : K\times K \to K$ corresponding to a nonsplittable extension $0 \to K \to L \to K \to 0$. Now let $Q = (Q_0, Q_1)$ be a finite quiver without oriented cycles and double arrows and $H = KQ$ be the path algebra of $Q$ over $K$. For each vertex $i$ of $Q$, choose a primitive idempotent $e_i$ of $H$, and for each path from $i$ to $j$ in $Q$, choose an element $h_{ji} = e_j h_{ji} e_i$ of $H$. Then $DH = \mathrm{Hom}_K(KQ,K) \cong \mathrm{Hom}_k(KQ,k)$ has a dual basis $e_i^*$, $h_{ji}^*$ over $K$. Let $\widetilde{H} = H \oplus DH$ be the direct sum of $K$-spaces. Define multiplication on $\widetilde{H}$ in the following way:

$$(a,u)(b,v) = \Big(ab, av + ub + \sum_{i\in Q_0} \alpha(a_i,b_i)e_i^*\Big)$$

for $a,b\in H$, $u,v\in DH$, where $a_i$ and $b_i$ are elements of $K$ such that

$$a = \sum a_i e_i + \sum r_{ji}h_{ji}, \qquad b = \sum b_i e_i + \sum s_{ji}h_{ji},$$

for $r_{ji}, s_{ji} \in K$, are the basis presentations of $a$ and $b$. Thus we have an algebra extension

$$0 \to DH \to \widetilde{H} \xrightarrow{\varrho} H \to 0$$

with the canonical morphism $\varrho$ and the embedding $DH \to \widetilde{H}$. Moreover, $\widetilde{H}$ is selfinjective, and even weakly symmetric (see [28]). The elements $\widetilde{e}_i = (e_i - \alpha(1,1)e_i^*) \in \widetilde{H}$, $i \in Q_0$, form a complete set of primitive orthogonal idempotents of $\widetilde{H}$. We have proved in [22, Proposition 6.1] that for each $i \in Q_0$ there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{e}_i(DH)\widetilde{e}_i \arrow[r] \arrow[d, "\psi_1"] & e_i\widetilde{H}e_i \arrow[r, "e_i\varrho e_i"] \arrow[d, "\psi"] & e_iHe_i \arrow[r] \arrow[d, "\psi_0"] & 0 \\
0 \arrow[r] & K \arrow[r] & L \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

with $\psi_0, \psi_1, \psi$ being algebra isomorphisms. Since the elements $e_i^*$, $i \in Q_0$, belong to the socle of $\widetilde{H}$, we find that $\widetilde{H}$ is socle equivalent to the trivial extension $H \ltimes DH = \widehat{H}/(\nu_{\widehat{H}})$. On the other hand, we have the following fact.

PROPOSITION 4. *The algebra $\widetilde{H}$ is not isomorphic to an algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where $B$ is a $k$-algebra and $\varphi$ is a positive automorphism of $\widehat{B}$.*

The proposition follows directly from the above diagram of nonsplittable extensions and the following general fact.

PROPOSITION 5. *Let $k$ be a field, $\Lambda$ a $k$-algebra and $A$ a Hochschild extension of $\Lambda$ by $D\Lambda$. Assume that*:

(i) *$A$ is isomorphic to an algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where $B$ is a $k$-algebra and $\varphi$ is a positive automorphism of $\widehat{B}$.*

(ii) *There exists a primitive idempotent $e$ of $A$ such that $e\Lambda e$ is a simple $k$-algebra.*

*Then $eAe \cong e\Lambda e \ltimes D(e\Lambda e)$.*

*Proof.* Since $A$ is weakly symmetric [28], it follows from [12, Theorem 2.2] that $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}}) \cong B \ltimes (DB)_\sigma$, where $\sigma$ is an algebra automorphism of $B$ of the identity permutation type (see [12, Section 2] for definition). Let $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents of $A$ with $1 = e_1 + \ldots + e_n$. Let $f : A \to B \ltimes (DB)_\sigma$ be an algebra isomorphism, and let $f(e_i) = (e'_i, u_i)$, $e'_i \in B$, $u_i \in DB$, for $1 \le i \le n$. Then $\{f(e_i) \mid 1 \le i \le n\}$ and $\{(e'_i, 0) \mid 1 \le i \le n\}$ are two complete sets of orthogonal primitive idempotents of $B \ltimes (DB)_\sigma$. Since there is an algebra automorphism mapping $f(e_i)$ to $(e'_i, 0)$ for all $i$, renumbering the vertices if necessary, we may assume that $f(e_i) = (e'_i, 0)$ for all $i$. For simplicity of notation, we put $e_i = (e'_i, 0)$, $1 \le i \le n$. Since the permutation type of the automorphism $\sigma$ is identity, the restriction of $\sigma$ to $e_iBe_i$ defines an automorphism $\sigma_i$ of $e_iBe_i$, and we have an isomorphism $e_i(B \ltimes (DB)_\sigma)e_i \cong (e_iBe_i) \ltimes D(e_iBe_i)_{\sigma_i}$. Invoking [12, Theorem 2.2] again, we conclude that, for the primitive idempotent $e$ of $A$, there exist $i$ with $1 \le i \le n$ and an algebra isomorphism

$$eAe \cong e_iBe_i \ltimes D(e_iBe_i)_{\sigma_i} \cong \widehat{B}_i/(\varphi_i\nu_{\widehat{B}_i})$$

for $B_i = e_iBe_i$ and a positive automorphism $\varphi_i$ of $\widehat{B}_i$. Since $eAe$ is by our assumption a simple $k$-algebra, applying [12, Proposition 2.4], we conclude that $eAe \cong e\Lambda e \ltimes D(e\Lambda e)$. ∎

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