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ON LOCAL WEAK CROSSED PRODUCT ORDERS

BY

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Abstract. Let $\Lambda = (S/R, \alpha)$ be a local weak crossed product order in the crossed product algebra $A = (L/K, \alpha)$ with integral cocycle, and $H = \{\sigma \in \mathrm{Gal}(L/K) \mid \alpha(\sigma, \sigma^{-1}) \in S^*\}$ the inertial group of α, for S^* the group of units of S. We give a condition for the first ramification group of L/K to be a subgroup of H. Moreover we describe the Jacobson radical of Λ without restriction on the ramification of L/K.

1. Introduction. Let R be a Dedekind domain with quotient field K, let L be a finite Galois extension of K with Galois group G, and S be the integral closure of R in L.

For a ring T, T^* means the group of units and $T^{\#} := T \setminus \{0\}$. Let $\alpha : G \times G \to L^*$ be a normalized cocycle, that is, α satisfies the cocycle relation

$$\rho(\alpha(\sigma, \tau))\alpha(\rho, \sigma\tau) = \alpha(\sigma, \tau)\alpha(\rho\sigma, \tau) \quad \text{for all } \rho, \sigma, \tau \in G \tag{1.1}$$

and the relations

$$\alpha(\sigma, 1) = \alpha(1, \sigma) = 1 \quad \text{for all } \sigma \in G.$$

It is known that the cocycle α is cohomologous to a cocycle taking values in $S^{\#}$. So we assume in what follows that the cocycle α is normalized taking values in $S^{\#}$. Then we can define the crossed product K-algebra

$$A := (L/K, \alpha) := \bigoplus_{\sigma \in G} L u_\sigma$$

freely generated as an L-vector space by the symbols $\{u_\sigma \mid \sigma \in G\}$ and with multiplication given by the rule

$$x u_\sigma y u_\tau = x\sigma(y)\alpha(\sigma, \tau)u_{\sigma\tau}, \quad \forall x, y \in L, \forall \sigma, \tau \in G.$$

It is well known that A is a central simple K-algebra and L is a maximal commutative subalgebra of A consisting of all elements of A commuting with all elements of L. $(L/K, \alpha)$ is also called a *classical crossed product algebra.*

Let $\Lambda := (S/R, \alpha) := \bigoplus_{\sigma \in G} S u_\sigma$. Then Λ is an R-order in A called the *weak crossed product order* corresponding to A. If the cocycle α takes values

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in S^*, i.e. it is a factor set, then Λ is called the *classical crossed product order* corresponding to A. Let H be the inertial group of the cocycle α, that is, $H = \{\sigma \in G \mid u_\sigma \in \Lambda^*\} = \{\sigma \in G \mid \alpha(\sigma, \sigma^{-1}) \in S^*\}$.

In Section 2 we give all the preliminary concepts and the results we need in this article. In Section 3 we prove some properties of cocycles. The main result of Section 3 is Theorem 3.5 where we give a condition for G_1, the first ramification group of the extension L/K, to be a subgroup of H.

In case R is a complete discrete valuation ring, Kessler [20, Corollary 3.5] proves that H is the unique subgroup of G with index $m/(e, m)$, where m is the Schur index of A and e is the ramification index of the extension L/K in case Λ is optional.

In Section 4 we describe the Jacobson radical of Λ in case R is a complete discrete valuation ring without restriction on the ramification of L/K. Our result extends the relevant result of Haile [13] in case the extension L/K is unramified, and that of Wilson [32] in case the extension L/K is tamely ramified. In the case of a classical crossed product order, without any restriction on the ramification of the extension, the Jacobson radical of A has been described by Chalatsis and Theohari-Apostolidi [7] (see also [27]).

Results of a similar nature for classical orders and Cohen–Macaulay algebras are discussed in [2], [3], [11], [10], [17], [22, Chapter 13], [23]–[26] and [28]–[30].

We refer to [9] and [21] for the theory of orders and crossed product algebras, and to [14]–[15] and [20] for weak crossed product orders.

2. Preliminaries

2.1. Cocycles. Let E/F be a finite Galois field extension with Galois group

$$G = G(E/F).$$

Then we define the crossed product F-algebra

$$A := (E/F, \alpha) = \bigoplus_{\sigma \in G} E u_\sigma,$$

where $\alpha : G \times G \to E$ is a normalized cocycle taking values in E. We remark that some authors call cocycles taking values in E instead of E^* *almost invertible.* We refer to [15] and [13] for the theory of cocycles.

Let $Z^2(G, E)$ be the set of all cocycles of G on E. Then $Z^2(G, E)$ becomes a monoid with multiplication

$$\alpha \cdot \beta(\sigma, \tau) = \alpha(\sigma, \tau)\beta(\sigma, \tau)$$

for $\alpha, \beta \in Z^2(G, E)$ and $\sigma, \tau \in G$. A map $\delta : G \times G \to E^*$ is called a *principal* *cocycle* if there are elements $\delta_\sigma \in E^*$, for $\sigma \in G$, such that

$$\delta(\sigma,\tau) = \delta_\sigma \sigma(\delta_\tau)\delta_{\sigma\tau}^{-1}$$

for $\sigma, \tau \in G$. The set $B^2(G, E^*)$ of principal cocycles is a multiplicative group and submonoid of $Z^2(G, E)$.

Let $M^2(G,E) := Z^2(G,E)/B^2(G,E^*)$. Then $M^2(G,E)$ is a monoid, and two elements $\alpha, \beta \in Z^2(G,E)$ are called *cohomologous* if $\beta = \delta\alpha$ for some $\delta \in B^2(G,E^*)$. Moreover every cocycle is cohomologous to a normalized cocycle, that is, satisfying the cocycle relation and the relation $\alpha(\sigma,1) = 1 = \alpha(1,\sigma)$ for $\sigma \in G$. The invertible elements of $M^2(G,E)$ form the usual cohomology group $H^2(G,E^*)$. Each of the idempotents of the monoid $M^2(G,E)$ is represented by a unique idempotent cocycle $\varepsilon \in Z^2(G,E)$ such that

$$M^2(G,E) = \bigcup_{\varepsilon} M^2_\varepsilon(G,E),$$

where

$$M^2_\varepsilon(G,E) := \{[\alpha] \in M^2(G,E) \mid \exists \beta \in Z^2(G,E),\ [\alpha][\beta] = [\varepsilon]\}.$$

2.2. Local orders. For a ring T, $\operatorname{rad} T$ denotes the Jacobson radical of T and $\overline{T} = T/\operatorname{rad} T$.

Throughout this paper, R is a complete discrete valuation ring with quotient field K, L a finite Galois field extension of K of degree n with Galois group

$$G = \operatorname{Gal}(L/K), \tag{2.1}$$

S the integral closure of R in L, and πS (resp. $\pi_K R$) the unique maximal ideal of S (resp. R). We assume that the residue field $\overline{R}$ of R is finite. Let $\alpha : G \times G \to S^{\#}$ be a normalized cocycle of G on $S^{\#}$. Two elements $\alpha, \beta \in Z^2(G, S^{\#})$ are equivalent over S if there is a map $\delta : G \to S^*$ such that

$$\alpha(\sigma,\tau) = \frac{\delta(\sigma)\sigma(\delta(\tau))}{\delta(\sigma\tau)}\beta(\sigma,\tau)$$

for all $\sigma, \tau \in G$. Let $N^2(G,S)$ be the set of equivalence classes of elements of $Z^2(G,S^{\#})$. $N^2(G,S)$ is a monoid under pointwise multiplication. Then there is an epimorphism of monoids $N^2(G,S) \to H^2(G,K)$ and a canonical map $N^2(G,S) \to M^2(G,\overline{S})$. This canonical map is given by reducing the values of the cocycle modulo πS (see [13]). To a cocycle $\alpha \in Z^2(G,S^{\#})$ corresponds a crossed product order

$$\Lambda := (S/R, \alpha) = \bigoplus_{\sigma \in G} S u_\sigma. \tag{2.2}$$

The ring Λ is a free S-module with basis the symbols u_σ, $\sigma \in G$, and multiplication given by the relations

$$u_\sigma u_\tau = \alpha(\sigma,\tau) = u_{\sigma\tau} \quad \text{and} \quad u_\sigma s = \sigma(s) u_\sigma,$$

for $\sigma, \tau \in G$ and $s \in S$. Then Λ is an R-order in the crossed product K-algebra

$$A := (L/K, \alpha). \tag{2.3}$$

We recall from [13] the following definition:

Definition 2.4. Assume $R, S, G = G(L/K)$ and A are as above. The R-order Λ (2.2) is called a *weak crossed product order* in A (2.3).

The K-algebra A is a central simple K-algebra (see [21]). Let

$$A \cong M_r(D) \cong \operatorname{End}_D(V),$$

where D is a division ring with index, say m, and V is the unique simple left A-module which is an (A, D)-bimodule with $(V : L) = m$ and $(V : D) = r$. We remark that Λ is a G-graded but not a strongly G-graded R-algebra, since $\alpha(\sigma, \tau)$ is not a unit of S, that is, u_σ is not an element of Λ^* for all $\sigma, \tau \in G$.

We need some more notation. Let Δ be the unique maximal R-order in D with maximal ideal $\pi_D\Delta$ for a prime element π_D of Δ. Then the ramification index of D over K is m, i.e., $\pi_K\Delta = \pi_D^m\Delta$, and m is also the inertial degree of D over K, that is, $m = (\overline{\Delta} : \overline{R})$ for $\overline{\Delta} := \Delta/\pi_D\Delta$ (see [21, §14]).

Let e be the ramification index of L over K, that is, $\pi_K S = \pi^e S$, and f be the inertia degree of L over K, that is, $(\overline{S} : \overline{R}) = f$. Then

$$n = ef = mr.$$

One of our objects of interest in this paper is the subgroup of $G = \operatorname{Gal}(L/K)$ given by

$$H := \{\sigma \in G \mid u_\sigma \in \Lambda^*\} = \{\sigma \in G \mid \alpha(\sigma, \sigma^{-1}) \in S^*\}, \tag{2.5}$$

called the *inertial group of the cocycle* α (see [13]).

Let L^H be the field corresponding to the subgroup H, and S^H be the integral closure of R in L^H. Then for $\alpha_H := \alpha|_{H\times H}$, the crossed product $\Lambda_H := (S/S^H, \alpha_H)$ is an S^H-order in the crossed product L^H-algebra $(L/L^H, \alpha_H)$; of course Λ_H is a classical crossed product order since α_H takes values in S^*. Moreover $\Lambda = \Lambda_H \oplus I$, where $I := \bigoplus_{\sigma\notin H} S u_\sigma$.

In the study of the order Λ its overorders play a significant rôle. In [5] Benz and Zassenhaus define a chain of orders

$$\Lambda = \Lambda_0, \quad \Lambda_{i+1} := O_\ell(\operatorname{rad}\Lambda_i) := \{a \in A \mid a \operatorname{rad}\Lambda_i \subseteq \operatorname{rad}\Lambda_i\}.$$

This chain stops at a number χ and it turns out that Λ_χ is a hereditary order. In [8] Cliff and Weiss compute the number χ in the case of a factor set α, i.e., a cocycle taking values in S^*. In [20] Kessler computes the number χ in the case of a cocycle and classifies all local hereditary crossed product orders.

A *principal order* is a hereditary R-order Γ such that $\operatorname{rad}\Gamma = \pi_\Gamma\Gamma = \Gamma\pi_\Gamma$; each such element π_Γ is called a *prime element* of Γ. For a discussion of hereditary crossed product orders we refer also to [1], [16], [18], [31] and for principal orders to [6], [12]. Now we sum up some properties of principal orders from [4–6], [12] and [20, Theorem 1.5] that we need in this article. We follow the notation introduced earlier.

THEOREM 2.6. *Let $\Lambda := (S/R, \alpha)$ be a weak crossed product order (2.2) in $A := (L/K, \alpha)$ (2.3) for a cocycle $\alpha : G\times G \to S^\#$, where $G = \mathrm{Gal}(L/K)$. There exists exactly one hereditary order Γ containing Λ which is a principal order with the following properties:*

(i) *$S = \Gamma \cap L$ and $\pi S = S \cap \operatorname{rad}\Gamma$, where $\operatorname{rad}\Gamma = \pi_\Gamma\Gamma = \Gamma\pi_\Gamma$ for a prime element π_Γ of Γ.*
(ii) *There exists a number $k \in \mathbb{N}$ which divides r such that r/k is the block length of Γ and $(k, k, \ldots, k)$ (r/k times) are the invariants of Γ.*
(iii) *$\pi_K\Gamma = \pi_\Gamma^{mk}\Gamma$, that is, the ramification index of Γ over R is mk and $(\overline{\Gamma} : \overline{R}) = nrk$.*
(iv) *$\overline{\Gamma} \cong M_{r/k}(\overline{\Delta})^{(k)}$, where (k) means k copies.*
(v) *The ramification index of Γ over S is $d := m/(e, m)$, that is, $\pi\Gamma = \pi_\Gamma^d\Gamma$, and $k = de/m = e/(e, m)$ and $r/k = f/d$. Hence d divides f.*

Let K_0 be the inertia field of the extension L/K, so $(L : K_0) = e$ and $(K_0 : K) = f$. Moreover let K_d be the uniquely determined intermediate field $K \le K_d \le K_0$ with $(K_d : K) = d$, $G_0 := \mathrm{Gal}(L/K_0)$ and $G_d := \mathrm{Gal}(L/K_d)$. We denote by R_0 (resp. R_d) the integral closure of R in K_0 (resp. K_d), and by π_0 (resp. π_d) a prime element of R_0 (resp. R_d). Then $\overline{S} = \overline{R}_0$, $(\overline{S} : \overline{R}_d) = f/d$ and $(\overline{R}_d : \overline{R}) = d$.

It follows from [20, Corollary 3.5] that the inertial group of α is a subgroup of G_d.

3. Some properties of a cocycle. Let G be a group acting on a field E, and N a normal subgroup of G with fixed field E^N. For a cocycle $\alpha : G/N\times G/N \to E^N$, let $\hat\alpha : G\times G \to E$ be defined by $\hat\alpha(\sigma, \tau) = \alpha(\sigma N, \tau N)$ for $\sigma, \tau \in G$. Then $\hat\alpha$ is also a cocycle, called the *inflation* of α. Moreover for a cocycle $\beta : G \times G \to E$, the restriction $\beta|_{N\times N} : N \times N \to E$ is also a cocycle. In this section and the next we denote $\sigma_x := \sigma(x)$ for $\sigma \in G$ and $x \in E$.

We consider the inflation map

$$\mathrm{Inf}: M^2_{\varepsilon}(G/N, E^N) \to M^2_{\hat{\varepsilon}}(G, E), \qquad [\alpha] \mapsto [\hat{\alpha}],$$

and the restriction map

$$\mathrm{Res}: M^2_{\hat{\varepsilon}}(G, E) \to M^2_{\hat{\varepsilon}}(N, E), \qquad [\beta] \mapsto [\beta|_{N\times N}],$$

using the notation of Subsection 2.1.

LEMMA 3.1. *Let G be a group acting on a field E, and N be a normal subgroup of G such that $H^1(N, E^*) = 1$. Let $\alpha : G \times G \to E$ be a cocycle such that $\alpha|_{N\times N} \in B^2(N, E^*)$. Then α is cohomologous to a cocycle $\delta : G\times G \to E^N$ such that $\delta(\sigma, \tau) = \delta(\sigma n_1, \tau n_2)$ for all $\sigma, \tau \in G$ and $n_1, n_2 \in N$.*

Proof. Let N and α be as above. Since $\alpha|_{N\times N} \in B^2(N, E^*)$, there is a map $\mu : N \to E^*$, $\mu(n) = \mu_n$, such that

$$\alpha(n_1, n_2) = \mu_{n_1} {}^{n_1}\mu_{n_2} \mu^{-1}_{n_1 n_2} \tag{3.1}$$

for all $n_1, n_2 \in N$. Hence $\alpha(n_1, n_2) \neq 0$, and so N is a subgroup of the inertial group of α. We consider the elements $\varphi_\sigma \in E^*$ such that $\varphi_n = \mu_n$ for all $n \in N$, and $\varphi_\sigma = 1$ for all $\sigma \in G \setminus N$.

Then the map $\gamma : G \times G \to E$ defined by

$$\gamma(\sigma, \tau) = [\varphi_\sigma {}^{\sigma}\varphi_\tau]^{-1} \varphi_{\sigma\tau} \alpha(\sigma, \tau) \tag{3.2}$$

is cohomologous to α. Moreover from (3.1) and (3.2), and since $N \leq H$, we get

$$\gamma(n_1, n_2) = 1 \quad \text{and} \quad \gamma(\sigma, n) \neq 0, \tag{3.3}$$

for all $n_1, n_2, n \in N$ and $\sigma \in G$. Let now T be a complete set of representatives of left cosets of N in G such that $1 \in T$. Then if $\sigma \in G$, there exist unique elements $t_0 \in T$ and $n_0 \in N$ depending on σ such that

$$\sigma = t_0 n_0. \tag{3.4}$$

Using the relation (3.4), for $\sigma \in G$, we consider the elements $\lambda_\sigma \in E^*$ such that

$$\lambda_\sigma = \lambda_{t_0 n_0} = \gamma(t_0, n_0). \tag{3.5}$$

Then from (3.3) and (3.5) we have

$$\lambda_n = \gamma(1, n) = 1 \qquad \text{for all } n \in N. \tag{3.6}$$

Moreover applying the cocycle equation (1.1) for the cocycle γ and the elements $t \in T$ and $n, n_1 \in N$, we get

$${}^{t}\gamma(n, n_1)\gamma(t, nn_1) = \gamma(t, n)\gamma(tn, n_1),$$

which because of the relation (3.3) becomes

$$\gamma(t, nn_1) = \gamma(t, n)\gamma(tn, n_1). \tag{3.7}$$

In addition, for σ as in (3.4) and $n_1 \in N$, from (3.5) and (3.7) we get

$$\lambda_{\sigma n_1} = \lambda_{t_0 n_0 n_1} = \gamma(t_0, n_0 n_1) = \gamma(t_0, n_0)\gamma(t_0 n_0, n_1) = \lambda_\sigma \gamma(\sigma, n_1).$$

Hence

$$\lambda_{\sigma n_1} = \lambda_\sigma \gamma(\sigma, n_1) \quad \text{for all } \sigma \in G \text{ and } n_1 \in N. \tag{3.8}$$

Now we define a new weak cocycle by

$$\beta : G \times G \to E \quad \text{with} \quad \beta(\sigma, \tau) = \lambda_\sigma \, {}^\sigma\lambda_\tau \lambda_{\sigma\tau}^{-1} \gamma(\sigma, \tau),$$

which is obviously cohomologous to γ. From (3.6) and (3.8), and for $\sigma \in G$ and $n \in N$, we get

$$\beta(\sigma, n) = \lambda_\sigma \, {}^\sigma\lambda_n \lambda_{\sigma n}^{-1} \gamma(\sigma, n) = 1.$$

Hence

$$\beta(\sigma, n) = 1 \quad \text{for all } \sigma \in G \text{ and } n \in N. \tag{3.9}$$

Using the cocycle equation (1.1) for the weak cocycle β and the elements $\sigma, \tau \in G$ and $n \in n$, we get

$${}^\sigma\beta(\tau, n)\beta(\sigma, \tau n) = \beta(\sigma, \tau)\beta(\sigma\tau, n).$$

Then from (3.9) we have

$$\beta(\sigma, \tau n) = \beta(\sigma, \tau) \quad \text{for all } \sigma, \tau \in G \text{ and } n \in N. \tag{3.10}$$

Since $N \trianglelefteq G$, for $n \in N$ and $\sigma \in G$ we have $n\sigma = \sigma n'$ for some $n' \in N$. Using (3.10) we get

$$\beta(n_1, n\sigma) = \beta(n_1, \sigma n') = \beta(n_1, \sigma). \tag{3.11}$$

Hence the cocycle equation (1.1) for β and the elements $n_1, n \in N$ and $\sigma \in G$ becomes

$${}^{n_1}\beta(n, \sigma)\beta(n_1, n\sigma) = \beta(n_1, n)\beta(n_1 n, \sigma),$$

and from (3.11) and (3.9) the above equation becomes

$${}^{n_1}\beta(n, \sigma)\beta(n_1, \sigma) = \beta(n_1 n, \sigma). \tag{3.12}$$

We define the map $\zeta_\sigma : N \to E^*$, $n \mapsto \beta(n, \sigma)$, for $\sigma \in G$ and $n \in N$. From (3.12) we have

$$\zeta_\sigma(n_1 n_2) = {}^{n_1}\zeta_\sigma(n_2)\zeta_\sigma(n_1)$$

for $n_1, n_2 \in N$ and $\sigma \in G$. Hence ζ_σ is a 1-cocycle, and since $H^1(G, E^*) = 1$, there exists an element $k_\sigma \in E^*$ such that

$$\beta(n, \sigma) = \zeta_\sigma(n) = {}^n k_\sigma k_\sigma^{-1} \quad \text{for all } n \in N.$$

Hence

$$\beta(n, \sigma) = \zeta_\sigma(n) = {}^n k_\sigma k_\sigma^{-1}. \tag{3.13}$$

Now we consider the elements $\psi_\sigma = k_{t_0}$ for $\sigma = t_0 n_0$ as in (3.4), and we get

$$\psi : G \times G \to E^* \quad \text{with} \quad \psi(\sigma,\tau) = \psi_\sigma\, {}^\sigma\psi_\tau \psi_{\sigma\tau}^{-1}.$$

It is clear that $\psi \in B^2(G, E^*)$ and the following hold:

$$\psi_{\sigma n} = \psi_\sigma = \psi_{n\sigma} \quad \text{and} \quad \psi_n = 1, \quad \text{for } n \in N,\ \sigma \in G. \tag{3.14}$$

Let $\delta : G \times G \to E$ with $\delta(\sigma,\tau) = \psi_\sigma\, {}^\sigma\psi_\tau \psi_{\sigma\tau}^{-1} \beta(\sigma,\tau)$. Then δ is a cocycle cohomologous to β, and using (3.4), (3.10), (3.13) and (3.14) we obtain consecutively

$$\begin{aligned}
\beta(n,\sigma) &= \psi_n\, {}^n\psi_\sigma \psi_{n\sigma}^{-1} \delta(n,\sigma),\\
\beta(n,t_0n_0) &= \psi_n\, {}^n\psi_{t_0n_0} \psi_{t_0n_0}^{-1} \delta(n,\sigma),\\
\beta(n,t_0) &= {}^n k_{t_0} k_{t_0}^{-1} \delta(n,\sigma),\\
{}^n k_{t_0} k_{t_0}^{-1} &= {}^n k_{t_0} k_{t_0}^{-1} \delta(n,\sigma),
\end{aligned}$$

and finally

$$\delta(n,\sigma) = 1 \quad \text{for } n \in N \text{ and } \sigma \in G. \tag{3.15}$$

Moreover from the definition of ψ and δ, and the equations (3.9), (3.14) we get the following implications, for β and for $\sigma \in G$ and $n \in N$:

$$\beta(\sigma,n) = \psi_\sigma\, {}^\sigma\psi_n \psi_{\sigma n}^{-1} \delta(\sigma,n), \quad \text{so} \quad 1 = \psi_\sigma \psi_\sigma^{-1} \delta(\sigma,n),$$

that is,

$$\delta(\sigma,n) = 1 \quad \text{for } \sigma \in G \text{ and } n \in N. \tag{3.16}$$

Now applying the cocycle equation (1.1) for the cocycle δ and for the elements $\sigma,\tau \in G$ and $n \in N$, using (3.15) and (3.16) we get

$${}^\sigma\delta(\tau,n)\delta(\sigma,\tau n) = \delta(\sigma,\tau)\delta(\sigma\tau,n),$$

and hence

$$\delta(\sigma,\tau n) = \delta(\sigma,\tau), \tag{3.17}$$

for $\sigma,\tau \in G$ and $n \in N$.

Again from the cocycle equation (1.1) for the elements σ, n, τ and for the weak cocycle δ, using (3.15) and (3.16) for $\sigma,\tau \in G$ and $n \in N$, we obtain

$${}^\sigma\delta(n,\tau)\delta(\sigma,n\tau) = \delta(\sigma,n)\delta(\sigma n,\tau),$$

so that

$$\delta(\sigma,n\tau) = \delta(\sigma n,\tau). \tag{3.18}$$

Now for $\sigma,\tau \in G$ and $n_1, n_2 \in N$, and using (3.17) and (3.18), we get

$$\delta(\sigma n_1,\tau n_2) = \delta(\sigma n_1,\tau) = \delta(\sigma,n_1\tau) = \delta(\sigma,\tau n_1') = \delta(\sigma,\tau), \tag{3.19}$$

where $n_1\tau = \tau n_1'$ for some $n_1' \in N$.

In order to finish the proof of the lemma it is sufficient to prove that $\delta(\sigma, n) \in E^N$ for $\sigma, \tau \in G$. For this, let $n \in N$ and $\sigma, \tau \in G$. Then the cocycle equation (1.1) for the cocycle δ and for the elements n, σ, τ becomes

$$^n\delta(\sigma,\tau)\delta(n,\sigma\tau) = \delta(n,\sigma)\delta(n\sigma,\tau).$$

Now let $n\sigma = \sigma n'$ for some $n' \in N$. From (3.15) and (3.19), the above equation becomes

$$^n\delta(\sigma,\tau) = \delta(n\sigma,\tau) = \delta(\sigma n',\tau) = \delta(\sigma,\tau).$$

Hence $\delta(\sigma,\tau) \in E^N$ and the result follows. ∎

THEOREM 3.2. *Let G be a group acting on a field E, and N a normal subgroup of G with fixed field E^N such that $H^1(N, E^*) = 1$. Then the sequence*

$$1 \to M^2_\varepsilon(G/N, E^N) \xrightarrow{\text{Inf}} M^2_{\hat\varepsilon}(G,E) \xrightarrow{\text{Res}} H^2(N, E^*)$$

is exact. Moreover the equality $H^2(N, E^*) = 1$ yields

$$M^2_\varepsilon(G/N, E^N) \simeq M^2_{\hat\varepsilon}(G,E).$$

Proof. First we prove that $\text{Ker(Inf)} = \{[\varepsilon]\}$. Let $\alpha : G/N \times G/N \to E^N$ be a weak cocycle and $\hat\alpha : G \times G \to E$ be defined by $\hat\alpha(\sigma,\tau) = \alpha(\sigma N, \tau N)$. Then $\text{Inf}[\alpha] = [\hat\alpha]$. Let $[\alpha] \in \text{Ker(Inf)}$, so $[\hat\alpha] = [\hat\varepsilon]$. Then there exist elements $\mu_\sigma \in E^*$, for $\sigma \in G$, such that

$$\hat\alpha(\sigma,\tau) = \mu_\sigma\, {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}\hat\varepsilon(\sigma,\tau) \quad \text{ for } \sigma,\tau \in G.$$

We remark that for $n_1, n_2 \in N$,

$$\hat\alpha(n_1,n_2) = 1, \quad \text{so} \quad \mu_{n_1 n_2} = \mu_{n_1}\, {}^{n_1}\mu_{n_2},$$

hence for the map $\mu : G \to E^*$, $\mu(\sigma) = \mu_\sigma$, we see that $\mu|_N$ is a 1-cocycle and $[\mu|_N] \in H^1(N, E^*) = 1$, by assumption. Therefore, there exists an element $k \in E^*$ such that $\mu(n) = {}^nk \cdot k^{-1}$ for $n \in N$. Now we consider the map $\varphi : G \to E^*$ such that $\mu(\sigma) = {}^\sigma k k^{-1}\varphi(\sigma)$. Then

$$\varphi(n) = 1 \quad \text{ for } n \in N, \tag{3.20}$$

and

$$\begin{aligned}\hat\alpha(\sigma,\tau) &= {}^\sigma\mu_\tau \mu_\sigma \mu_{\sigma\tau}^{-1}\hat\varepsilon(\sigma,\tau)\\ &= {}^\sigma[{}^\tau k k^{-1}\varphi(\tau)]\, {}^\sigma k k^{-1}\varphi(\sigma)[{}^{\sigma\tau}k k^{-1}\varphi(\sigma\tau)]^{-1}\hat\varepsilon(\sigma,\tau)\\ &= {}^{\sigma\tau}k({}^\sigma k)^{-1}\, {}^\sigma\varphi(\tau)\, {}^\sigma k k^{-1}\varphi(\sigma)({}^{\sigma\tau}k)^{-1}k\varphi(\sigma\tau)^{-1}\hat\varepsilon(\sigma,\tau),\end{aligned}$$

and consequently

$$\hat\alpha(\sigma,\tau) = {}^\sigma\varphi(\tau)\varphi(\sigma)\varphi(\sigma\tau)^{-1}\hat\varepsilon(\sigma,\tau). \tag{3.21}$$

The above equation, for $\tau = n \in N$, yields

$$\hat\alpha(\sigma,n) = \varphi(\sigma)\, {}^\sigma\varphi(n)\varphi(\sigma n)^{-1}\hat\varepsilon(\sigma,n). \tag{3.22}$$

Since $\hat{\alpha}(\sigma, n) = \hat{\varepsilon}(\sigma, n) = 1$, using (3.20), the equation (3.22) gets the form

$$\varphi(\sigma n) = \varphi(\sigma) \quad \text{for } \sigma \in G,\ n \in N. \tag{3.23}$$

Moreover, the equation (3.21), for $\sigma = n \in N$, yields

$$\hat{\alpha}(n, \tau) = \varphi(n)\, {}^{n}\varphi(\tau)\varphi(n\tau)^{-1}\hat{\varepsilon}(n, \tau),$$

and since $\hat{\alpha}(n, \tau) = \hat{\varepsilon}(n, \tau) = 1$, in view of (3.20) we get

$$ {}^{n}\varphi(\sigma) = \varphi(n\sigma) \quad \text{for } \sigma \in G,\ n \in N. \tag{3.24}$$

We remark that since $N \trianglelefteq G$, we have $n\sigma = \sigma n'$ for some $n' \in N$. Hence (3.24) gets the form

$$ {}^{n}\varphi(\sigma) = \varphi(n\sigma) = \varphi(\sigma n') = \varphi(\sigma),$$

therefore we get

$$ {}^{n}\varphi(\sigma) = \varphi(\sigma) \quad \text{for } \sigma \in G,\ n \in N.$$

This means that $\varphi(\sigma) \in E^N$ for $\sigma \in G$. So there exists the map

$$\psi : G/N \to E^N, \qquad \psi(gN) = \varphi(g)$$

such that, for $\sigma, \tau \in G$, $\alpha(\sigma N, \tau N) = \psi(\sigma N)\, {}^{\sigma N}\psi(\tau N)\psi(\sigma\tau N)^{-1}\varepsilon(\sigma N, \tau N)$. In other words, the cocycle α is cohomologous to ε and so $\mathrm{Ker}(\mathrm{Inf}) = \{[\varepsilon]\}$.

To complete the proof we have to show that $\mathrm{Res} \circ \mathrm{Inf} = 1$. Let $[\alpha] \in M^2_{\varepsilon}(G/N, E^N)$ and $[\hat{\alpha}] = \mathrm{Inf}[\alpha]$. Then $\hat{\alpha}(n_1, n_2) = 1$ for $n_1, n_2 \in N$, and so $\mathrm{Res} \circ \mathrm{Inf}[\alpha] = [1]$. Hence $\mathrm{Im}(\mathrm{Inf}) \subseteq \mathrm{Ker}(\mathrm{Res})$. In order to prove that $\mathrm{Ker}(\mathrm{Res}) \subseteq \mathrm{Im}(\mathrm{Inf})$, let $\alpha : G \times G \to E$ be a cocycle such that $\alpha|_{N \times N} \in B^2(G, E)$. Then from Lemma 3.1, α is cohomologous to a cocycle $\beta : G \times G \to E$ such that $\beta(\sigma n_1, \tau n_2) = \beta(\sigma, \tau) \in E^N$ for $\sigma, \tau \in G$ and $n_1, n_2 \in N$. Therefore there exists a cocycle $\gamma : G/N \times G/N \to E^N$, $\gamma(\sigma N, \tau N) = \beta(\sigma, \tau)$, so that $\mathrm{Inf}[\gamma] = [\beta] = [\alpha]$, and this means that $\mathrm{Ker}(\mathrm{Res}) \subseteq \mathrm{Im}(\mathrm{Inf})$. ∎

Let now R be a complete discrete valuation ring, let $K, L, S, \pi_K, \pi S, G = \mathrm{Gal}(L/K), \overline{S}, \overline{R}, f$ be as in Section 2, and let G_1 denote the first ramification group of L/K, that is:

Definition 3.3.

$$G_1 = \{\sigma \in G \mid \sigma(a) \equiv a \ (\mathrm{mod}\, (\pi_K)^2) \text{ for all } a \in S\}.$$

The following result generalizes [7, Lemma 1.1], and implies the isomorphism $H^2(G/G_1, \overline{S}^*) \cong H^2(G, \overline{S})$.

Proposition 3.4. *Let $G = \mathrm{Gal}(L/K)$ be as in (2.1) and let $\varepsilon : G/G_1 \times G/G_1 \to \overline{S}$ be an idempotent cocycle, where G_1 is the first ramification group (3.3) of L/K. Then the inflation map $M^2_{\varepsilon}(G/G_1, \overline{S}) \xrightarrow{\mathrm{Inf}} M^2_{\hat{\varepsilon}}(G, \overline{S})$ is a group isomorphism.*

Proof. The first ramification group G_1 acts trivially on the field $\overline{S}$ and $(|G_1|, |\overline{S}^*|) = 1$, hence $H^1(G_1, \overline{S}^*) = 1$ [8, §39]. Also $H^2(G_1, \overline{S}^*) = 1$. Now from Theorem 3.2 we get the exact sequence

$$1 \to M^2_{\varepsilon}(G/G_1, \overline{S}^{G_1}) \xrightarrow{\text{Inf}} M^2_{\hat{\varepsilon}}(G, \overline{S}) \xrightarrow{\text{Res}} H^2(G_1, \overline{S}^*) = 1,$$

and the result follows. ∎

Now we are able to prove one of the main results of this paper.

THEOREM 3.5. *Let $G = G(L/K)$ be the group* (2.1), $G_1 = \mathrm{Ram}_1(L/K)$ *the first ramification group* (3.3), *and let $\hat{\varepsilon} : G \times G \to \overline{S}$ be an idempotent cocycle such that there exists an idempotent cocycle $\varepsilon : G/G_1 \times G/G_1 \to \overline{S}$ satisfying the relation $\hat{\varepsilon}(\sigma, \tau) = \varepsilon(\sigma G_1, \tau G_1)$. Then:*

- (i) *For every cocycle $\alpha : G \times G \to \overline{S}$ such that $[\alpha] \in M^2_{\hat{\varepsilon}}(G, \overline{S})$, there exists a cocycle $\hat{\beta} : G \times G \to \overline{S}$ such that $\hat{\beta}$ is cohomologous to α and $\hat{\beta}(\sigma, \tau) = 1$ if σ or τ belongs to G_1.*
- (ii) *For every cocycle $\alpha : G \times G \to S^{\#}$ such that $[\overline{\alpha}] \in M^2_{\hat{\varepsilon}}(G, \overline{S})$, where $\overline{\alpha}(\sigma, \tau) = \alpha(\sigma, \tau) \bmod \pi S$, there exists a cocycle $\beta : G \times G \to S^{\#}$ such that β is cohomologous to α and $\beta(\sigma, \tau) \in 1 + \pi S$ if $\sigma \in G_1$ or $\tau \in G_1$.*
- (iii) *The first ramification group G_1 is a subgroup of the inertial group H of the cocycle α.*

Proof. (i) We consider the inflation map

$$M^2_{\varepsilon}(G/G_1, \overline{S}) \xrightarrow{\text{Inf}} M^2_{\hat{\varepsilon}}(G, \overline{S}).$$

If $\alpha : G \times G \to \overline{S}$ is a cocycle such that $[\alpha] \in M^2_{\hat{\varepsilon}}(G, \overline{S})$ then, by Proposition 3.4, there exists $[\beta] \in M^2_{\varepsilon}(G/G_1, \overline{S})$ such that $\mathrm{Inf}[\beta] = [\hat{\beta}] = [\alpha]$. Then $\hat{\beta} : G \times G \to \overline{S}$ is a cocycle having the required properties.

(ii) Let $\alpha : G \times G \to S^{\#}$ be a cocycle. Then from (i) there exists a cocycle γ such that $[\gamma] = [\bar{\alpha}]$ and $\gamma(\sigma, \tau) = 1$ whenever $\sigma \in G_1$ or $\tau \in G_1$. Therefore there exist elements $\mu_\sigma \in \overline{S}^*$ for $\sigma \in G$ such that $\bar{\alpha}(\sigma, \tau) = \mu_\sigma {}^\sigma\mu_\tau \mu_{\sigma\tau}^{-1}$ for $\sigma \in G_1$ or $\tau \in G_1$. Let $\mu(\sigma) = \bar{s}_\sigma \in \overline{S}$ for some $s_\sigma \in S$. Then $\bar{\alpha}(\sigma, \tau) = \bar{s}_\sigma {}^\sigma\bar{s}_\tau \bar{s}_{\sigma\tau}^{-1}$, and hence $\overline{\alpha}(\sigma, \tau) = \overline{s_\sigma {}^\sigma s_\tau s_{\sigma\tau}^{-1}}$. So $\alpha(\sigma, \tau) - s_\sigma {}^\sigma s_\tau s_{\sigma\tau}^{-1} \in \pi S$ whenever $\sigma \in G_1$ or $\tau \in G_1$. We remark that the cocycle $\beta : G \times G \to S^{\#}$, $\beta(\sigma, \tau) = s_\sigma^{-1} {}^\sigma s_\tau^{-1} s_{\sigma\tau} \alpha(\sigma, \tau)$, has the required properties, and the result follows.

(iii) From (ii) we see that $\beta(\sigma, \tau) \in 1 + \pi S$ if $\sigma \in G_1$ or $\tau \in G_1$. Hence if $\sigma \in G_1$ or $\tau \in G_1$, then $\beta(\sigma, \tau) \in S^*$. Now from the definition of the inertial group H and the fact that the cocycle α is cohomologous to β, we conclude that G_1 is a subgroup of H. ∎

We remark that if $\alpha : G \times G \to S^*$ is a factor set, then there exists a factor set $\beta : G \times G \to S^*$ cohomologous to α such that $\beta(\sigma, \tau) \in 1 + \pi S$

whenever σ or τ belongs to G_1 (see [7, Lemma 1.3]). Theorem 3.5 gives a condition for an analogous result to hold in the case of a cocycle, and hence a condition for $G_1 \le H$.

4. The Jacobson radical of Λ. Throughout this section we assume that Λ is a weak crossed product order (2.2) in the algebra (2.3) for a cocycle $\alpha : G \times G \to S^{\#}$. In this section we study the Jacobson radical of Λ for any finite field extension L/K and a local field K. We denote by rad the Jacobson radical and follow the notation of the previous sections. We need the following result of Wilson (see [32, Lemmas 2.3 and 2.5]).

LEMMA 4.1. *Let $\alpha : G \times G \to S^{\#}$ be a weak cocycle. Then:*

(i) *For $\sigma \in G$ and $h \in H$, the elements $\alpha(\sigma, h)$ and $\alpha(h, \sigma)$ are both units of S.*
(ii) *If $\sigma, \tau \in G \setminus H$ and $\sigma\tau \in H$, then $\alpha(\sigma, \tau)$ is not a unit of S.*

PROPOSITION 4.2. $\operatorname{rad} \Lambda = \operatorname{rad} \Lambda_H \oplus I$, *where* $I = \bigoplus_{\sigma \in G-H} S u_\sigma$.

Proof. Since $\Lambda = \Lambda_H \oplus I$, we consider the map

$$\varphi : \Lambda_H \oplus I \to \Lambda_H/\operatorname{rad} \Lambda_H, \qquad \lambda_H + x \mapsto \lambda_H + \operatorname{rad} \Lambda_H,$$

for $x \in I$. It is clear that φ is an epimorphism of additive groups with kernel equal to $\operatorname{rad} \Lambda_H \oplus I$. We prove that φ preserves ring multiplication. Let $\lambda_H, \lambda'_H \in \Lambda_H$ and $x, x' \in I$. Then

$$(\lambda_H + x)(\lambda'_H + x') = \lambda_H \lambda'_H + \lambda_H x' + x \lambda'_H + x x'.$$

We remark that $\lambda_H \lambda'_H \in \Lambda_H$. Moreover $\lambda_H x', x\lambda'_H \in I$. Indeed, for $h \in H$ and $\sigma \in G - H$ we see that the elements

$$u_h u_\sigma = \alpha(h, \sigma) u_{h\sigma} \quad \text{and} \quad u_\sigma u_h = \alpha(\sigma, h) u_{\sigma h}$$

belong to I, and therefore $\lambda_H x'$ and $x \lambda'_H$ belong to I. For the element xx', let

$$x = \sum_{\sigma \in G-H} s_\sigma u_\sigma \quad \text{and} \quad x' = \sum_{\tau \in G-H} s_\tau u_\tau.$$

Then $xx' = \sum s_\sigma s_\tau^\sigma \alpha(\sigma, \tau) u_{\sigma\tau}$. If $\sigma\tau \notin H$ then $u_{\sigma\tau} \in I$, and so $xx' \in I$. If $\sigma\tau \in H$, then from Lemma 4.1(ii) we deduce that $\alpha(\sigma, \tau) \in \pi S$ and $s_\sigma s_\tau^\sigma \alpha(\sigma, \tau) u_{\sigma\tau} \in \pi \Lambda_H$. But $\pi \Lambda_H \subset \operatorname{rad} \Lambda_H$, and so $xx' \in \operatorname{rad} \Lambda_H$. Therefore in any case $xx' \in \operatorname{rad} \Lambda_H \oplus I$. Hence

$$\varphi[(\lambda_H + x)(\lambda'_H + x')] = \lambda_H \lambda'_H + \operatorname{rad} \Lambda_H = \varphi(\lambda_H + x)\varphi(\lambda'_H + x').$$

So we get

$$\Lambda_H \oplus I/(\operatorname{rad} \Lambda_H \oplus I) \cong \Lambda_H/\operatorname{rad} \Lambda_H,$$

and so $\operatorname{rad} \Lambda_H \oplus I \supset \operatorname{rad} \Lambda$, Λ_H being semisimple. It remains to prove that $\operatorname{rad} \Lambda \supset \operatorname{rad} \Lambda_H \oplus I$. For this we have to prove that there is a natural number

k such that $(\operatorname{rad} \Lambda_H \oplus I)^k \subset \pi\Lambda$, in other words $(\operatorname{rad} \Lambda_H \oplus I)/\pi\Lambda$ is a nilpotent ideal of the $\overline{R}$-algebra $\Lambda/\pi\Lambda$, $\operatorname{rad} \Lambda_H \oplus I$ being an ideal of Λ containing $\pi\Lambda$. In order to prove that $(\operatorname{rad} \Lambda_H \oplus I)/\pi\Lambda$ is a nilpotent ideal of $\Lambda/\pi\Lambda$ it is enough to show that it has an $\overline{R}$-basis consisting of nilpotent elements, by a theorem of Wedderburn (see [19, Ch. 11, Theorem 1.15]). Since, for $x \in \operatorname{rad} \Lambda_H$, the element $x + \pi\Lambda \in \Lambda/\pi\Lambda$ is nilpotent, it is enough to prove that an element $su_\sigma + \pi\Lambda$, for $s \in S$ and $\sigma \in G - H$, is nilpotent. For this let $\sigma \in G - H$ and k be the smallest natural number such that $\sigma^k \in H$. Then

$$\begin{aligned}(u_\sigma)^k &= \Big(\prod_{i=1}^{k-1} {}^{\sigma^{k-i-1}}\alpha(\sigma, \sigma^{k-1})\Big) u_{\sigma^k} \\ &= \Big(\prod_{i=1}^{k-2} {}^{\sigma^{k-i-1}}\alpha(\sigma, \sigma^{k-1})\Big)\, {}^{\sigma}\alpha(\sigma, \sigma^{k-1}) u_{\sigma^k}.\end{aligned}$$

Hence $(u_\sigma)^k \in \pi\Lambda$ because $\alpha(\sigma, \sigma^{k-1}) \in \pi S$, by Lemma 4.1(ii). Therefore $\operatorname{rad} \Lambda \supset \operatorname{rad} \Lambda_H \oplus I$, and we have proved that $\operatorname{rad} \Lambda = \operatorname{rad} \Lambda_H \oplus I$. ∎

For the next theorem we need some more notation. Let H_1 be the first ramification group of the field extension L/L^H with corresponding field L^{H_1}, and let α_1 be the restriction of the factor set α_H to $H_1 \times H_1$. Then

$$A_{H_1} := (L^{H_1}/L^H, \alpha_1) \cong \operatorname{End}_{D_1}(V_1) \cong M_{r_1}(D_1)$$

for a division ring D_1 centrally containing L^H with index, say, m_1. Moreover let Δ_1 be the unique maximal S^H-order in D_1 with maximal ideal $\Delta_1 \pi_{D_1}$. We remark that $\Lambda_{H_1} := (S^{H_1}/S^H, \alpha_1)$ is a hereditary S^H-order in A_{H_1} since the extension L^{H_1}/L^H is tamely ramified (see [31]).

THEOREM 4.3. *Let $\Lambda = (S/R, \alpha)$ be a weak crossed product order* (2.2) *in the crossed product K-algebra $A = (L/K, \alpha)$* (2.3), *and H be the inertial group of the cocycle α. Let H_1 be the first ramification group of the extension L/L^H, and X be a complete set of representatives of the left cosets of H_1 in H. Then*

(i)

$$\operatorname{rad} \Lambda = \bigoplus_{\sigma \in X} \pi S u_\sigma \oplus \Big(\bigoplus_{\substack{\sigma \in X \\ \rho \in H_1 - \{1\}}} S u_\sigma (u_\rho - u_1)\Big) \oplus \Big(\bigoplus_{\sigma \in G - H} S u_\sigma \Big).$$

(ii) $\Lambda/\operatorname{rad} \Lambda \cong \Lambda_H/\operatorname{rad} \Lambda_H \cong \Lambda_{H_1}/\operatorname{rad} \Lambda_{H_1} \cong M_{f_H}(\Delta_1/\Delta_1 \pi_{D_1})^{e_H/m_1}$, *where f_H is the inertial degree of the extension L/L^H, e_H is the tame ramification degree of the extension L/L^H, and m_1 is the index of D_1.*

Proof. (i) This follows from Proposition 4.2 and [7, Proposition 1.4].
(ii) The result follows from Proposition 4.2 and [7, Theorem 1.9]. ∎

The above result extends the relevant result of Haile [13] for unramified extensions and that of Wilson [32] for tamely ramified extensions.

5. Maximal orders containing Λ. We follow the notation of Subsection 2.2. Let Γ_0 be a maximal R-order in the crossed product algebra $A \cong \mathrm{End}_D(V)$ (2.3) containing the weak crossed product Λ (2.2). From the structure of maximal orders (see [21, §17]), there exists a unique up to isomorphism indecomposable Γ_0-lattice M full in V, i.e. $KM = V$, which is a (Γ_0, Δ)-lattice. Let $V = L\omega_1 \oplus \cdots \oplus L\omega_m$. Then we can choose $M = S\omega_1 \oplus \cdots \oplus S\omega_m$, and M is also an indecomposable left Λ-lattice and a (Λ, Δ)-bimodule. Of course M is a left Γ-lattice for the unique principal R-order Γ in A containing Λ (see Theorem 2.6). From the structure of hereditary orders we deduce that $\Gamma_i = \mathrm{End}_\Delta(\pi_\Gamma^i M)$, $0 \le i \le k-1$, are all the non-isomorphic maximal R-orders in A containing Γ, where $\Gamma = \bigcap_{i=0}^{k-1} \Gamma_i$ and $\pi_\Gamma^i M$, $0 \le i \le k-1$, are all the non-isomorphic indecomposable Γ-lattices. Therefore Γ_i, $0 \le i \le k-1$, are all the non-isomorphic maximal R-orders containing Λ. Moreover $\pi_\Gamma^i M$, $0 \le i \le k-1$, are also non-isomorphic indecomposable left Λ-lattices, full in V and (Λ, Δ)-bimodules. If N is another such left Λ-lattice, then $\mathrm{End}_\Delta(N)$ will be a maximal R-order in A, and hence one of Γ_i, $0 \le i \le k-1$. This means that N is isomorphic to one of $\pi_\Gamma^i M$, $0 \le i \le k-1$. So we conclude with the following:

Proposition 5.1. *Let A be a crossed product algebra* (2.3) *and let $V = L\omega_1 \oplus \cdots \oplus L\omega_m$ be the unique simple (A, D)-bimodule. Let $M = S\omega_1 \oplus \cdots \oplus S\omega_m$ and $\Gamma_0 := \mathrm{End}_\Delta(M)$. Then:*

(i) *$\Gamma_i := \pi_\Gamma^i \Gamma_0$, $0 \le i \le k-1$, are all the maximal R-orders in A containing the weak crossed product order Λ* (2.2).

(ii) *$\pi_\Gamma^i M$, $0 \le i \le k-1$, are all the non-isomorphic indecomposable Λ-lattices which are (Λ, Δ)-bimodules.*

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REFERENCES

[1] M. Auslander and D. S. Rim, *Ramification index and multiplicity*, Illinois J. Math. 7 (1963), 566–581.

[2] L. F. Barannyk, *Finite-dimensional twisted group algebras of semi-wild representation type*, Colloq. Math. 120 (2010), 277–298.

[3] L. F. Barannyk, *Finite groups of OTP projective representation type*, Colloq. Math. 126 (2012), 35–51.

[4] H. Benz, *Untersuchungen zur Arithmetik in lokalen einfachen Algebren, insbesondere über maximalen Teilkörpern*, J. Reine Angew. Math. 225 (1967), 30–75.

[5] H. Benz und H. Zassenhaus, *Über verschränkte Produktordnungen*, J. Number Theory 20 (1985), 282–298.

[6] C. Buschnell and A. Fröhlich, *Non-abelian congruence Gauss sums and p-adic simple algebras*, Proc. London Math. Soc. 50 (1985), 207–264.

[7] A. Chalatsis and Th. Theohari-Apostolidi, *Maximal orders containing local crossed-products*, J. Pure Appl. Algebra 50 (1988), 211–222.

[8] G. H. Cliff and A. R. Weiss, *Crossed products and hereditary orders*, Pacific J. Math. 122 (1986), 333–345.

[9] C. W. Curtis and I. Reiner, *Methods of Representation Theory, Vol. I, With Applications to Finite Groups and Orders*, Wiley, New York, 1981.

[10] Yu. A. Drozd, *Cohen–Macaulay modules and vector bundles*, in: Lecture Notes in Pure Appl. Math. 210, Dekker, 2000, 107–130.

[11] Yu. A. Drozd, A. G. Zavadskiĭ and V. V. Kirichenko, *Matrix problems and integral representations*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 291–293 (in Russian).

[12] A. Fröhlich, *Principal orders and embedding of local fields in algebras*, Proc. London Math. Soc. 54 (1987), 247–266.

[13] D. Haile, *Crossed product over discrete valuation rings*, J. Algebra 105 (1987), 116–148.

[14] D. Haile, *On crossed product algebras arising from weak cocycles*, J. Algebra 74 (1982), 270–279.

[15] D. Haile, R. Larson and M. Sweedler, *Almost invertible cohomology theory and the classification of idempotent cohomology classes and algebras by partially ordered sets with a Galois group action*, Amer. J. Math. 105 (1983), 689–814.

[16] M. Harada, *Some criteria for hereditarity of crossed products*, Osaka J. Math. 1 (1964), 69–80.

[17] Y. Hatzaras and Th. Theohari-Apostolidi, *Hermitian and quadratic forms over local classical crossed product orders*, Colloq. Math. 83 (2000), 43–53.

[18] G. Janusz, *Crossed product orders and the Schur index*, Comm. Algebra 8 (1980), 697–806.

[19] G. Karpilovsky, *Group Representations. Vol. 1*, North-Holland, Amsterdam, 1992.

[20] V. Kessler, *Crossed products, orders and non-commutative arithmetic*, J. Number Theory 46 (1994), 255–302.

[21] I. Reiner, *Maximal Orders*, London Math. Soc. Monogr. New Ser. 28, Oxford Univ. Press, 2003.

[22] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon & Breach Sci. Publ., New York, 1992.

[23] D. Simson, *Tame three-partite subamalgams of tiled orders of polynomial growth*, Colloq. Math. 8 (1999), 237–262.

[24] D. Simson, *A reduced Tits quadratic form and tameness of three-partite subamalgams of tiled orders*, Trans. Amer. Math. Soc. 352 (2000), 4843–4875.

[25] D. Simson, *Cohen–Macaulay modules over classical orders*, in: Lecture Notes in Pure Appl. Math. 210, Dekker, 2000, 345–382.

[26] O. Solberg, *Hypersurface singularities of finite Cohen–Macaulay type*, Proc. London Math. Soc. 58 (1989), 258–280.

[27] Th. Theohari-Apostolidi, *Local crossed-product orders of finite representation type*, J. Pure Appl. Algebra 41 (1986), 87–98.

[28] Th. Theohari-Apostolidi and H. Vavatsoulas, *On the separability of the restriction functor*, Algebra Discrete Math. 3 (2003), 95–101.

[29] Th. Theohari-Apostolidi and H. Vavatsoulas, *On strongly Gorenstein orders*, Algebra Discrete Math. 2 (2005), 80–89.

[30] Th. Theohari-Apostolidi and H. Vavatsoulas, *Induced modules of strongly group graded algebras*, Colloq. Math. 108 (2007), 93–104.

[31] S. Williamson, *Crossed product and hereditary orders*, Nagoya Math. J. 23 (1963), 103–120.

[32] C. J. Wilson, *Hereditary crossed product orders over discrete valuation rings*, J. Algebra 371 (2012), 329–349.

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