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ON CONTINUITY AT ZERO OF THE MAXIMAL OPERATOR FOR A SEMIFINITE MEASURE

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Abstract. For a sequence of linear maps defined on a Banach space with values in the space of measurable functions on a semifinite measure space, we examine the behavior of its maximal operator at zero.

1. Introduction and preliminaries. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Denote by $\mathcal{M}_0$ (resp. $\mathcal{M}$) the linear space of equivalence classes of all (respectively, a.e. finite) real or complex valued measurable functions on Ω. If X is a given set and $a_n : X \to \mathcal{M}$ is a sequence of maps, denote

$$a^\star(f)(\omega) = \sup_n |a_n(f)(\omega)|, \qquad f \in X,\ \omega \in \Omega.$$

The mapping $a^\star : X \to \mathcal{M}_0$ is called the *maximal operator* associated with the family $\{a_n\}$. For a fixed $f \in X$, the function $a^\star(f)$ is called a *maximal function* of the sequence $\{a_n(f)\}$.

If X is a Banach space, behavior of the maximal operator near $0 \in X$ plays an important role in the study of a.e. convergence of sequences $\{a_n(f)\}$, $f \in X$; see Theorem 1.2 below.

Let τ_μ be the *measure topology* on $\mathcal{M}$, that is, the topology defined by the following fundamental system of neighborhoods of $0 \in \mathcal{M}$:

$$\mathcal{N}(\epsilon, \delta) = \{f \in \mathcal{M} : \mu\{\omega : |f(\omega)| > \delta\} \le \epsilon\}, \qquad \epsilon, \delta > 0.$$

It is well-known and easy to see that $(\mathcal{M}, \tau_\mu)$ is a complete metrizable topological vector space [5].

REMARK 1.1. Note that if for some $f \in X$ the sequence $\{a_n(f)\}$ converges a.e., then clearly $a^\star(f) < \infty$ a.e., that is, $a^\star(f) \in \mathcal{M}$.

The classical Banach Principle can be stated as follows (see [3], and also [1]):

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Theorem 1.2. *Assume that* $\mu(\Omega) < \infty$*. Let* X *be a Banach space, and let* $a_n : X \to \mathcal{M}$ *be a sequence of* τ_μ*-continuous linear maps. Consider the following conditions:*

(A) $\{a_n(f)\}$ *converges a.e. for every* $f \in X$*;*
(B) $a^\star(f) \in \mathcal{M}$ *for every* $f \in X$*;*
(C) *the maximal operator* $a^\star : X \to \mathcal{M}$ *is* τ_μ*-continuous at* $0 \in X$*;*
(D) *the set* $\{f \in X : \{a_n(f)\}$ *converges a.e.*$\}$ *is closed in* X*.*

Then the implications (A)$\Rightarrow$(B)$\Rightarrow$(C)$\Rightarrow$(D) *hold. If, in addition, the sequence* $\{a_n(f)\}$ *converges a.e. for every* f *in a dense subset of* X*, then conditions* (A)–(D) *are equivalent.*

In particular, we have the following which is the crucial statement of Theorem 1.2.

Theorem 1.3. *Assume* $\mu(\Omega) < \infty$*. Let* X *be a Banach space, and let* $a_n : X \to \mathcal{M}$ *be a sequence of* τ_μ*-continuous linear maps. If* $\sup_n |a_n(f)| < \infty$ *a.e. for all* $f \in X$*, then the maximal operator* $a^\star : X \to \mathcal{M}$ *is* τ_μ*-continuous at* $0 \in X$*.*

One can ask what happens if the measure in question is not finite. In what follows we will show that Theorem 1.3 does not hold even when $\Omega = \mathbb{R}$ (with Lebesgue measure), but remains valid for a semifinite measure if the measure topology is replaced by the so-called local measure topology which is then weaker than the measure topology.

2. An example. The example below shows that if μ is not finite but is σ-finite, then Theorem 1.3 fails to hold.

Example 2.1. Let Ω be the set of real numbers endowed with Lebesgue measure μ. Take X to be the set of all continuous real valued functions on $\mathbb{R}$ that vanish outside the interval $(0, 1)$. Equipped with the norm $\|f\| = \max_{\omega\in\mathbb{R}} |f(\omega)|$, X is a Banach space. For every $n \in \mathbb{N}$ define a linear map $a_n : X \to \mathcal{M} = \mathcal{M}(\mathbb{R})$ by the formula

$$a_n(f)(\omega) = nf(n(\omega - n)), \qquad f \in X.$$

Then a_n is clearly continuous in $\|\cdot\|$, hence in τ_μ. Furthermore, since, for every $f \in X$, the function $a_n(f)$ vanishes outside the interval $(n, n + 1/n)$, we have $a_n(f)(\omega) \to 0$ for all $\omega \in \mathbb{R}$. In particular, given $f \in X$, the maximal function $a^\star(f)$ is finite everywhere. Therefore $a^\star(f) \in \mathcal{M}$ for every $f \in X$.

Next, fix $\gamma > 0$, and let $0 \neq f \in X$ be such that $\|f\| < \gamma$. Then, for some $\lambda > 0$,

$$\mu\{\omega : |f(\omega)| > \lambda\} = \Delta > 0.$$

Take $\delta > 0$. If $n \geq \delta/\lambda$, we have

$$\begin{aligned}\mu\{\omega : |a_n(f)(\omega)| > \delta\} &\geq \mu\{\omega : |f(n(\omega - n))| > \lambda\} \\ &= \mu\{\omega : |f(n\omega)| > \lambda\} = \Delta/n.\end{aligned}$$

Therefore, since the intervals $(n, n + 1/n)$ do not overlap, we can write

$$\mu\Big\{\omega : \sup_{n\geq\delta/\lambda} |a_n(f)(\omega)| > \delta\Big\} \geq \Delta\Big(\frac{1}{n} + \frac{1}{n+1} + \cdots\Big).$$

Thus, given $\gamma > 0$, for every $0 \neq f \in X$ with $\|f\| < \gamma$ and every $\delta > 0$, we have

$$\mu\{\omega : a^\star(f)(\omega) > \delta\} = \infty,$$

and we conclude that the maximal operator $a^\star : X \to \mathcal{M}$ is not τ_μ-continuous at $0 \in X$.

3. Theorem 1.3 for a semifinite measure. Assume now that μ is *semifinite*, that is, any subset of Ω of infinite measure contains a subset of non-zero finite measure.

The *local measure topology* t_μ on $\mathcal{M}$ can be defined by the following fundamental system of neighborhoods of $0 \in \mathcal{M}$:

$$\mathcal{N}(\epsilon, \delta, F) = \{f \in \mathcal{M} : \mu\{\omega \in F : |f(\omega)| > \delta\} \leq \epsilon\}, \quad \epsilon, \delta > 0,\ 0 < \mu(F) < \infty.$$

The t_μ-topology is strictly weaker than τ_μ, in general. However, if $\mu(\Omega) < \infty$, the distinction between these topologies disappears. $(\mathcal{M}, t_\mu)$ is a complete Hausdorff topological vector space that is not metrizable unless μ is σ-finite. For a detailed account on semifinite measure spaces and the local measure topology, see [2].

Let

$$\mathcal{L}^\infty = \Big\{f \in \mathcal{M} : \|f\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega} |f(\omega)| < \infty\Big\}.$$

One can verify the following.

Proposition 3.1. *For any $\epsilon, \delta > 0$, and F with $0 < \mu(F) < \infty$,*

$$\mathcal{N}(\epsilon, \delta, F) = \{f \in \mathcal{M} : \|f\chi_E\|_\infty \leq \delta \text{ for some } E \subset F \text{ with } \mu(F \setminus E) \leq \epsilon\}.$$

The proof of the next fact can be found in [1].

Lemma 3.2. *Let X be a topological space, and let $a_n : X \to \mathcal{M}$ be a sequence of t_μ-continuous maps. Then, given any $\epsilon > 0$ and $E \in \mathcal{A}$, the set*

$$X_L = \{f \in X : \|a^\star(f)\chi_G\|_\infty \leq L \text{ for some } G \subset E \text{ with } \mu(E \setminus G) \leq \epsilon\}$$

is closed in X for every $L > 0$.

Denote

$$\mathcal{A}_+^f = \{F \in \mathcal{A} : 0 < \mu(F) < \infty\}.$$

Proposition 3.3. *If $f \in \mathcal{M}_0$, then the following conditions are equivalent:*

(i) $f \in \mathcal{M}$;
(ii) *for every $F \in \mathcal{A}_+^f$ and $\epsilon > 0$ there exists a set $E \subset F$ with $\mu(F \setminus E) \leq \epsilon$ such that $y\chi_E \in \mathcal{L}^\infty$.*

Proof. (i)⇒(ii): We have $f_F = |f|\chi_F < \infty$ a.e. Therefore, if $F_N = \{\omega \in F : f_F(\omega) > N\}$, $N \in \mathbb{N}$, then $\lim_{N\to\infty} \mu(F_N) = 0$, hence $\mu(F_{N_0}) \leq \epsilon$ for some N_0. Setting $E = F \setminus F_{N_0}$, we obtain $\mu(F \setminus E) \leq \epsilon$ and $f_F(\omega) \leq N$ for every $\omega \in E$. Consequently, $f\chi_E = f_F\chi_E \in \mathcal{L}^\infty$.

The implication (ii)⇒(i) is obvious. ∎

Now we can extend Theorem 1.3 to the case of semifinite measure.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space, and let $a_n : X \to \mathcal{M}$ be a sequence of t_μ-continuous linear maps. If $\sup_n |a_n(f)| < \infty$ a.e. for all $f \in X$, then the maximal operator $a^\star : X \to \mathcal{M}$ is t_μ-continuous at $0 \in X$.*

Proof. Fix $\epsilon, \delta > 0$ and $F \in \mathcal{A}_+^f$. By Proposition 3.1, we need to show that there is $\gamma > 0$ such that $\|f\| < \gamma$ implies that

$$\|a^\star(f)\chi_E\|_\infty \leq \delta$$

for some $E \subset F$ with $\mu(F \setminus E) \leq \epsilon$.

For $L \in \mathbb{N}$ define

$$X_L = \{f \in X : \|a^\star(f)\chi_G\|_\infty \leq L \text{ for some } G \subset F \text{ with } \mu(F \setminus G) \leq \epsilon/2\}.$$

By Lemma 3.2, the sets X_L are closed, while $a^\star(f) = \sup_n |a_n(f)| < \infty$ a.e. for all $f \in X$ together with Proposition 3.3 implies that

$$X = \bigcup_L X_L.$$

By the Baire category theorem, there exists L_0 such that X_{L_0} contains a non-empty open set. Thus, there are $f_0 \in X$ and $\nu > 0$ such that, given $f \in X$ with $\|f - f_0\| < \nu$, one can present a set $G \subset F$ with $\mu(F \setminus G) \leq \epsilon/2$ satisfying

$$\|a^\star(f)\chi_G\|_\infty \leq L_0.$$

Consequently, if $\|f\| < \nu$, we can find E' and E'' such that $\mu(F \setminus E') \leq \epsilon/2$, $\mu(F \setminus E'') \leq \epsilon/2$, and

$$\|a^\star(f + f_0)\chi_{E'}\|_\infty \leq L_0, \quad \|a^\star(f_0)\chi_{E''}\|_\infty \leq L_0.$$

Defining $E = E' \cap E''$, we obtain $\mu(F \setminus E) \leq \epsilon$ and also

$$\|a^\star(f)\chi_E\|_\infty \leq \|a^\star(f + f_0)\chi_E\|_\infty + \|a^\star(f_0)\chi_E\|_\infty \leq 2L_0.$$

If $m > 0$ is chosen such that $2L_0/m \leq \delta$, then $\|f\| < \gamma = \nu/m$ implies $\|mf\| < \nu$, hence

$$\|a^\star(f)\chi_E\|_\infty \leq \delta. \ ∎$$

The following lemma can also be found in [1].

LEMMA 3.5. *Let* $(X, +)$ *be a semigroup, and let* $a_n : (X, +) \to (\mathcal{M}, +)$ *be a sequence of homomorphisms. Suppose that* $f \in X$ *is such that, given* $F \in \mathcal{A}_+^f$, *there exist a sequence* $\{f_k\} \subset X$ *and a set* $E \subset F$ *with* $\mu(E) > 0$ *satisfying*

(a) *the sequence* $\{a_n(f + f_k)\}$ *converges a.e. for each* k;
(b) $\|a^\star(f_k)\chi_E\|_\infty \to 0$ *as* $k \to \infty$.

Then the sequence $\{a_n(f)\}$ *also converges a.e.*

As an application of Theorem 3.4 we derive the following corollary.

COROLLARY 3.6. *Let* X *and* $\{a_n\}$ *be as in Theorem* 3.4. *If* $\sup_n |a_n(f)| < \infty$ *a.e., then the set*

$$C = \{f \in X : \{a_n(f)\} \text{ converges a.e.}\}$$

is closed in X.

Proof. Take any $F \in \mathcal{A}_+^f$ and fix $\epsilon > 0$ such that $\epsilon < \mu(F)$. By Theorem 3.4, given any $k \in \mathbb{N}$, there is $\gamma_k > 0$ for which $\|x\| < \gamma_k$ implies

$$\|a^\star(f)\chi_{E_k}\|_\infty \le 1/k$$

for some $E_k \subset F$ with $\mu(F \setminus E_k) \le \epsilon/2^k$.

Pick $f \in \overline{C}$. Then, given $k \in \mathbb{N}$, there is $g_k \in C$ such that $\|g_k - f\| < \gamma_k$. Therefore, denoting $f_k = g_k - f$ and letting $E = \bigcap_{k=1}^\infty E_k$, we obtain $\mu(F \setminus E) \le \epsilon$, hence $\mu(E) > 0$ since $\mu(F) > \epsilon$. Also, $f + f_k = g_k \in C$ for each k and

$$\|a^\star(f_k)\chi_E\|_\infty \le 1/k \to 0 \quad \text{as } k \to \infty.$$

By Lemma 3.5, $f \in C$, implying $\overline{C} = C$. ∎

REMARK 3.7. As was carried out in [4] for a finite measure, Theorem 3.4 and Corollary 3.6 can be proved in the case where X is a topological group of second Baire category.

REMARK 3.8. The main reason for deriving Corollary 3.6 from Theorem 3.4 here is to show that the t_μ-continuity, a weaker condition than τ_μ-continuity, of the maximal operator at zero is sufficient for the closedness of the set C. Alternatively, Corollary 3.6 can be derived directly from Theorem 1.2 as follows.

If we define

$$a_n^F(f) = a_n(f)\chi_F, \qquad F \in \mathcal{A}_+^f,$$

then, for a given $F \in \mathcal{A}_+^f$, $a_n^F : X \to \mathcal{M}(F)$ is a sequence of τ_μ-continuous linear maps such that $(a^F)^\star(f) \in \mathcal{M}(F)$ for every $f \in X$. By Theorem 1.2, the set

$$C_F = \{f \in X : \{a_n^F(f)\} \text{ converges a.e.}\}$$

is closed in X.

Clearly $C \subset \bigcap_{F \in \mathcal{A}_+^f} C_F$. If $f \in \bigcap_{F \in \mathcal{A}_+^f} C_F$ and

$$D = \{\omega \in \Omega : \{a_n(f)(\omega)\} \text{ does not converge}\},$$

then $D \in \mathcal{A}$. Suppose that $\mu(D) > 0$ and take $F \subset D$ such that $F \in \mathcal{A}_+^f$. Since $f \in C_F$, the sequence $\{a_n^F(f)\}$ converges a.e., contrary to the definition of D. Therefore $\mu(D) = 0$, hence $f \in C$, which implies that $C = \bigcap_{F \in \mathcal{A}_+^f} C_F$. Thus C is closed in X.

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