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COMPACTLY SUPPORTED COHOMOLOGY OF SYSTOLIC 3-PSEUDOMANIFOLDS

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Abstract. We show that the second group of cohomology with compact supports is nontrivial for three-dimensional systolic pseudomanifolds. It follows that groups acting geometrically on such spaces are not Poincaré duality groups.

1. Introduction. *Systolic complexes* are simply connected flag simplicial complexes satisfying some local combinatorial nonpositive-curvature-like conditions. They were introduced first by Chepoi [Che00] and then, independently (bringing them into the geometric group theory), by Januszkiewicz–Świątkowski [JŚ06] and Haglund [Hag03]. Many properties of systolic complexes resemble the ones of spaces of nonpositive curvature. Consequently, groups acting on such complexes *geometrically* (i.e., cocompactly and properly discontinuously by automorphisms), called *systolic groups*, behave similarly to CAT(0) groups. In contrast to CAT(0) complexes and groups, the systolic setting is purely combinatorial, which allows one to construct explicit examples of groups with various, often exotic, properties—see e.g. [JŚ06]. In particular, systolic groups are not fundamental groups of nonpositively curved manifolds of dimension greater than 2 (cf. [JŚ07, Osa07, Osa08, OŚ13]), although the dimension (cohomological or asymptotic) of systolic groups can be arbitrarily large. Moreover, it is conjectured that systolic groups do not contain subgroups isomorphic to fundamental groups of aspherical manifolds of dimension greater than 2. An attempt to establish such a result in a particular case is the main motivation for the current paper.

Main Theorem. *Let X be a locally finite systolic 3-pseudomanifold. Then the second group of cohomology with compact supports $H^2_c(X;\mathbb{Z})$ is nontrivial.*

From the Main Theorem we immediately obtain the following result establishing the aforementioned conjecture in the case of 3-pseudomanifolds (see e.g. [Dav08, Appendix F.5] for basics about Poincaré duality groups).

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Corollary. *Groups acting geometrically on systolic* 3*-pseudomanifolds are not Poincaré duality groups.*

Notice that systolic pseudomanifolds and, more generally, chamber complexes form the main source of examples of high-dimensional groups provided by "systolic" techniques (cf. [JŚ06]). We believe that methods similar to the ones used in our proof of the Main Theorem allow one to show that groups acting geometrically on any chamber complex of dimension greater than 2 are not Poincaré duality groups. Note that hyperbolic systolic groups with two-dimensional Gromov boundaries are not Poincaré duality groups since their boundaries are not spheres (cf. [Osa08, OŚ13]).

2. Preliminaries

2.1. Systolic complexes. In this section we follow the notation of [Osa07, JŚ06]. All simplicial complexes are finite-dimensional and locally finite. A simplicial complex X is *flag* if any finite set of vertices, which are pairwise connected by edges of X, *spans* a simplex of X (i.e., it is contained in some simplex of X). A subcomplex K of X is called *full* (in X) if any simplex of X spanned by a set of vertices in K is a simplex of K. Let σ and τ be two simplices of X. We denote by $\sigma * \tau$ the *join* of σ and τ, i.e., the minimal simplex of X (if it exists) containing both of them. The *link* of a simplex σ of X, denoted by X_σ, is a subcomplex of X consisting of all simplices that are disjoint from σ and which span a simplex of X together with σ. The *residue* of a simplex σ of X, denoted by $\mathrm{Res}(\sigma, X)$, is the minimal subcomplex of X containing all simplices that contain σ. A *cycle* in a simplicial complex X is a subcomplex γ of X isomorphic to some triangulation of $\mathbb{S}^1$. We denote by $|\gamma|$ the *length* of γ, i.e., the number of 1-simplices of γ. A *full cycle* in X is a cycle that is full as a subcomplex of X. We define the *systole* of X as $\mathrm{sys}(X) = \min\{|\gamma| : \gamma \text{ is a full cycle in } X\}$. In particular, we have $\mathrm{sys}(X) \geq 3$ for any simplicial complex X, and if there is no full cycle in X, then $\mathrm{sys}(X) = \infty$. If $k \geq 4$ is a natural number and X is a flag simplicial complex, then we say that X is *k-large* if $\mathrm{sys}(X) \geq k$; X is *locally k-large* if the residue of every simplex of X is k-large; and X is *k-systolic* if it is connected, simply connected and locally k-large. We abbreviate 6-systolic to *systolic* due to its importance in this theory. The following facts are immediate (see [JŚ06, Section 1]):

Fact 2.1.

(1) *A complex is locally k-large iff the link of its every nonempty simplex has the systole at least k.*
(2) *A* (*locally*) *k-large complex is* (*locally*) *m-large for $k \geq m$.*
(3) *A full subcomplex of a* (*locally*) *k-large complex is* (*locally*) *k-large.*
(4) *A simplicial complex is* 4*-large iff it is flag.*
(5) *For $k > 4$, X is k-large iff it is flag and* $\mathrm{sys}(X) \geq k$.

(6) *A k-large complex is locally k-large.*
(7) *The universal cover $\widetilde{X}$ of a connected, locally k-large complex X is k-systolic.*

Now let X be a systolic complex and let $\sigma \in X$ be a simplex. The *closed combinatorial ball* (resp. *sphere*) of radius i around σ in X, denoted $B_i(\sigma, X)$ (resp. $S_i(\sigma, X)$), is the full subcomplex of X spanned by vertices at combinatorial distance at most i (resp. exactly i) from σ. Notice that, by Fact 2.1(3), balls and spheres are 6-large. For subcomplexes Y, Z of X, we denote by $Y - Z$ the full subcomplex spanned by vertices of Y not in Z.

Closed combinatorial balls of small radii in k-large complexes are isomorphic to ones in the corresponding universal covers (i.e., in systolic complexes):

Lemma 2.2. *For a 6-large simplicial complex X and a simplex $\tau \in X$, let $\widetilde{X} \xrightarrow{p} X$ with $p(\tilde{\tau}) = \tau$ be the universal cover of X. Then $p|_{B_1(\tilde{\tau}, \widetilde{X})} : B_1(\tilde{\tau}, \widetilde{X}) \to B_1(\tau, X)$ is an isomorphism.*

The next lemma is fundamental for the understanding of this paper (see [JŚ06, Section 7] and [Osa07, Lemma 2.3] for the proof of the last part).

Lemma 2.3 (Projection Lemma). *For any $\tau \in S_i(\sigma, X)$, the intersection $\rho = S_{i-1}(\sigma, X) \cap X_\tau$ is a single (nonempty) simplex. Moreover, $X_\tau \cap B_i(\sigma, X) = B_1(\rho, X_\tau)$ and $X_\tau \cap S_i(\sigma, X) = S_1(\rho, X_\tau)$.*

In the rest of the paper we call the simplex ρ, as in the above lemma, the *projection* of τ on $B_{i-1}(\sigma, X)$.

Let X be a simplicial complex and let $\sigma \in X$ be a simplex. By the Projection Lemma we can define an elementary contraction

$$\pi_{B_i(\sigma,X)} : B_{i+1}(\sigma, X)' \to B_i(\sigma, X)'$$

between barycentric subdivisions of balls by putting

$$\pi_{B_i(\sigma,X)}(b_\nu) = \begin{cases} b_{\nu \cap B_i(\sigma,X)} & \text{if } \nu \cap B_i(\sigma, X) \neq \emptyset, \\ b_{X_\nu \cap B_i(\sigma,X)} & \text{if } \nu \cap B_i(\sigma, X) = \emptyset, \end{cases}$$

for $\nu \in B_i(\sigma, X)$, and then extending it simplicially. In [JŚ06, Section 8] it is shown that $\pi_{B_i(\sigma,X)}$ is a deformation retraction and $\pi_{B_i(\sigma,X)}(B_{i+1}(\sigma, X) - B_{i-1}(\sigma, X)) \subset S_i(\sigma, X)$. The combination of such maps gives a deformation retraction $h_{S_i(\sigma,X)} : X - B_{i-1}(\sigma, X) \to S_i(\sigma, X)$, implying the following:

Theorem 2.4 ([JŚ06, Theorem 4.1]). *Let X be a finite-dimensional systolic complex. Then X is contractible.*

Remark 2.5. It is well known that every closed surface can be triangulated. However, the sphere and projective plane do not admit 6-large triangulations (see [JŚ06, Example 1.8(5)]). Any other surface admits a 6-large triangulation (see [JŚ06, Example 1.8(3)]).

The fact that there is no k-large triangulation of the 2-sphere for $k \geq 6$ implies that no triangulation of a manifold of dimension greather than 2 is 6-large, since 2-spheres would occur as links of some simplices of such triangulation (see [JŚ06, Example 1.8(5)]).

2.2. Systolic pseudomanifolds. A simplicial complex X is called a *simplicial pseudomanifold* of dimension n (or shortly n-*pseudomanifold*) if it is a union of n-simplices such that every $(n-1)$-simplex is contained in exactly two n-simplices (cf. [JŚ06]). Let τ be a subsimplex of a maximal simplex σ of dimension n. We say that τ has *codimension* k if its dimension is $(n-k)$.

LEMMA 2.6. *Let X be an n-pseudomanifold and let σ be a k-simplex of X. Then X_σ is an $(n-k-1)$-pseudomanifold.*

Proof. First we show that X_σ is the union of $(n-k-1)$-simplices. By definition, $X_\sigma = \{\tau \in X \mid \tau \cap \sigma = \emptyset \text{ and } \sigma * \tau \text{ is a simplex of } X\}$. Take $\tau \in X_\sigma$. Since X is an n-pseudomanifold, $\sigma * \tau \subseteq \rho$, where ρ is some n-simplex of X. Let $\rho = \sigma * \tau'$, where $\tau' \cap \sigma = \emptyset$. Since $n = \dim \rho = \dim(\sigma * \tau') = \dim \sigma + \dim \tau' + 1 = k + \dim \tau' + 1$, it follows that $\dim \tau' = n - k - 1$. Therefore every simplex $\tau \in X_\sigma$ is contained in an $(n-k-1)$-simplex, and thus X_σ is the union of $(n-k-1)$-simplices.

Now we show that every codimension 1 simplex of X_σ is contained in exactly two $(n-k-1)$-simplices. Let ω be a codimension 1 simplex of X_σ. Therefore $\dim \omega = n-k-2$ and $\dim(\omega * \sigma) = \dim \omega + \dim \sigma + 1 = n-k-2+k+1 = n-1$. Since X is an n-pseudomanifold, there exist exactly two n-simplices α, β that contain $\omega * \sigma$. Now we take α' and β' such that $\alpha' \cap (\omega * \sigma) = \emptyset$, $\beta' \cap (\omega * \sigma) = \emptyset$ and $\alpha = (\omega * \sigma) * \alpha'$, $\beta = (\omega * \sigma) * \beta'$. Note that $\alpha' * \omega$ and $\beta' * \omega$ both belong to X_σ. Note also that $n = \dim \alpha = \dim(\sigma * \alpha' * \omega) = k + \dim(\alpha' * \omega) + 1$, and therefore $\dim(\alpha' * \omega) = n-k-1$. Similarly, $\dim(\beta' * \omega) = n-k-1$.

Finally, it remains to show that these two simplices are the only two $(n-k-1)$-simplices of X_σ. If there existed an $(n-k-1)$-simplex $\gamma \in X_\sigma$ containing ω such that $\gamma \neq \alpha' * \omega$ and $\gamma \neq \beta' * \omega$, then $\gamma * \sigma, \alpha$ and β would be three n-simplices of X containing $\omega * \sigma$, a contradiction. ∎

Spheres in n-pseudomanifolds are $(n-1)$-pseudomanifolds (compare e.g. [Osa07, Lemma 4.1]). For simplicity in the following two lemmas we prove it only for dimensions 2 and 3.

LEMMA 2.7. *Let X be a systolic pseudomanifold of dimension 2. Then $S_k(\sigma, X)$ is a one-dimensional pseudomanifold for all $k \geq 1$.*

Proof. First we show that $S_k(\sigma, X)$ is at most one-dimensional. Suppose there exists a 2-simplex $\tau \in S_k(\sigma, X)$. By the Projection Lemma, the projection of τ on $B_{k-1}(\sigma, X)$ is a nonempty simplex $\rho = S_{k-1}(\sigma, X) \cap X_\tau$. It means that $\rho \subset X_\tau$, and by definition of the link, $\tau * \rho$ must be a simplex of X. But $\dim(\tau * \rho) = 2 + 1 + \dim(\rho) > 2 = \dim(X)$, a contradiction.

Now we show that every simplex τ of $S_k(\sigma, X)$ is contained in some 1-simplex. We have seen that τ can just have dimensions 0 or 1. If it has dimension 1, we are done. Suppose it has dimension 0. By the Projection Lemma, $X_\tau \cap S_k(\sigma, X) = S_1(\rho, X_\tau)$, where $\rho = S_{k-1}(\sigma, X) \cap X_\tau$. Since X_τ is a 6-large (by Fact 2.1(3)) 1-pseudomanifold (by Lemma 2.6), it is a union of cycles. Therefore $S_1(\rho, X_\tau)$ consists of two vertices, say v and w. Thus $\tau * v$ is a 1-simplex of $S_k(\sigma, X)$ containing τ.

Moreover, $\tau * v$ and $\tau * w$ are the only two maximal simplices of $S_k(\sigma, X)$ that contain the codimension 1 simplex τ of $S_k(\sigma, X)$. ■

LEMMA 2.8. *Let X be a systolic pseudomanifold of dimension 3. Then $S_k(\sigma, X)$ is a two-dimensional pseudomanifold for all $k \geq 1$.*

Proof. To see that $S_k(\sigma, X)$ is at most two-dimensional, the proof is analogous to that in the previous lemma.

We show that every simplex τ of $S_k(\sigma, X)$ is contained in some 2-simplex. Now τ can have dimension $0, 1$ and 2. If it has dimension 2, we are done. Suppose it has dimension 0. By the Projection Lemma, $X_\tau \cap S_k(\sigma, X) = S_1(\rho, X_\tau)$, where $\rho = S_{k-1}(\sigma, X) \cap X_\tau$. Now X_τ is a 6-large (by Fact 2.1(1)) 2-pseudomanifold (by Lemma 2.6). By Fact 2.1(7) its universal cover $\widetilde{X}_\tau$ is systolic, and by Lemma 2.2 $S_1(\rho, X_\tau)$ is isomorphic to $S_1(\rho, \widetilde{X}_\tau)$. Therefore by Lemma 2.7 $S_1(\rho, X_\tau)$ is a 1-pseudomanifold. The span of one of its 1-simplices, call it τ', with τ is a 2-simplex of $S_k(\sigma, X)$.

The proof for dim $\tau = 1$ is exactly the same as the part of the proof of Lemma 2.7 for dim $\tau = 0$. ■

2.3. Cohomology with compact supports. In this paper we consider only simplicial cohomology. For cohomology with compact supports we follow the approach of [Dav08, Appendix G.2]. Finite subcomplexes of X form a directed set under inclusion. To each finite subcomplex $K \subset X$ we associate the group $H^i(X, X - K; \mathbb{Z})$, with a fixed i and a coefficient group $\mathbb{Z}$. For each inclusion $K \subset L$ of finite subcomplexes, we have the inclusion $X - L \overset{i}{\hookrightarrow} X - K$ and the associated natural homomorphism $H^i(X, X - K; \mathbb{Z}) \xrightarrow{i^*} H^i(X, X - L; \mathbb{Z})$. The cohomology group $H^i_{\mathrm{c}}(X; \mathbb{Z})$ equals, by definition, the resulting limit group $\varinjlim H^i(X, X - K; \mathbb{Z})$. Each element of this limit group is represented by a cocycle in $C^i(X, X - K; \mathbb{Z})$ for some finite subcomplex K; such a cocycle is zero in $\varinjlim H^i(X, X - K; \mathbb{Z})$ iff it is zero in $C^i(X, X - L; \mathbb{Z})$ for some finite subcomplex $L \supset K$, which means it is the coboundary of some cochain in $C^{i-1}(X, X - L; \mathbb{Z})$.

3. Proof of the Main Theorem. We now come to the main goal of this paper. Before that, we prove an important lemma:

Lemma 3.1. *If Y is a 6-large 2-pseudomanifold, then $H^1(Y;\mathbb{Z}) \neq \{0\}$.*

Proof. Let S be the disjoint union of all 2-simplices of Y. Consider the map $f : S \to Y$ such that $f(x) = x$. Now let $\bar{S} = S/\sim$ be the quotient space where we identify 1-simplices τ and τ' of S in the following way: for $x \in \tau \subseteq \sigma^{(1)}$ and $x' \in \tau' \subseteq \sigma'^{(1)}$ $(\sigma, \sigma' \subseteq S)$, $x \sim x'$ if $f(x) = f(x')$ and $f[\tau] = f[\tau']$. Observe that $\sim$ is an equivalence relation since each edge of Y belongs to exactly two triangles. Consider the map $i : \bar{S} \to Y$ such that $i([x]_\sim) = x$ for $x \in \sigma$. This map is of course well defined.

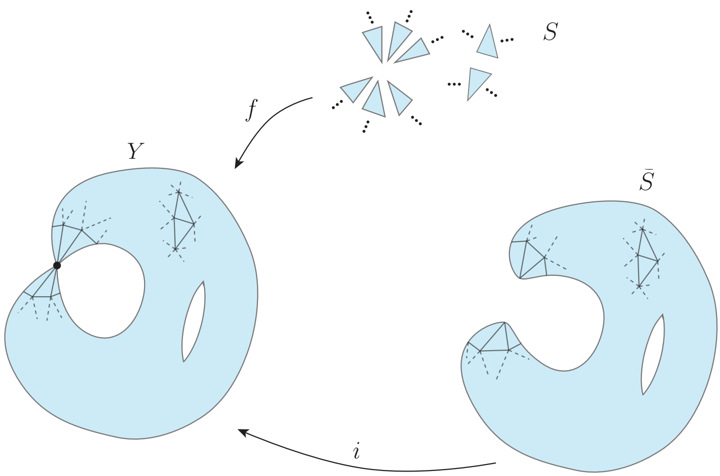

Fig. 1. Diagram of a 6-large 2-pseudomanifold Y and the spaces S and $\bar{S}$ formed from it, together with the maps between them

We now show that $\bar{S}$ is a surface. The space $\bar{S}$ is a simplicial complex being a union of 2-simplices, and by the construction, each 1-simplex is contained in exactly two 2-simplices. Therefore it is a pseudomanifold. Since $\bar{S}$ is finite, the link of every vertex, being a one-dimensional manifold (by Lemma 2.6), can only be a union of circles. It is easy to observe that such a link has to be connected and therefore it is just one circle. Thus we conclude that $\bar{S}$ is a surface. One can easily see that $\bar{S}$ is 6-large and that the map $i : \bar{S} \to Y$ is locally injective.

Given a cocycle $\bar{\varphi} \in Z^1(\bar{S};\mathbb{Z})$, suppose $[\bar{\varphi}] \in H^1(\bar{S};\mathbb{Z})$ is nonzero. We define a cochain φ in $C^1(Y;\mathbb{Z})$ as

$$\varphi(i([\tau]_\sim)) := \bar{\varphi}([\tau]_\sim) \quad \text{for an edge } [\tau]_\sim \in \bar{S}.$$

This is well defined since i is a bijection on the set of edges. We show that

φ is a cocycle. Consider a triangle $\sigma \in Y$. Let $[\sigma]_\sim$ be a triangle in $\bar{S}$ with $i([\sigma]_\sim) = \sigma$, $\partial[\sigma]_\sim = [\tau_1]_\sim + [\tau_2]_\sim + [\tau_3]_\sim$ and $i([\tau_i]_\sim) = \tau_i$ for $i \in \{1, 2, 3\}$. Then we have

$$\begin{aligned}\delta\varphi(\sigma) &= \varphi(\partial\sigma) = \varphi(\tau_1 + \tau_2 + \tau_3) = \varphi(\tau_1) + \varphi(\tau_2) + \varphi(\tau_3)\\ &= \bar{\varphi}([\tau_1]_\sim) + \bar{\varphi}([\tau_2]_\sim) + \bar{\varphi}([\tau_3]_\sim)\\ &= \bar{\varphi}([\tau_1]_\sim + [\tau_2]_\sim + [\tau_3]_\sim) = \bar{\varphi}(\partial[\sigma]_\sim) = \delta\bar{\varphi}([\sigma]_\sim) = 0.\end{aligned}$$

Finally we show that $[\varphi] \in H^1(Y;\mathbb{Z})$ is nontrivial. Suppose that $\varphi = \delta\alpha$ for some $\alpha \in C^0(Y;\mathbb{Z})$. We define an $\bar{\alpha} \in C^0(\bar{S};\mathbb{Z})$ as $\bar{\alpha}([v]_\sim) := \alpha(i([v]_\sim))$. Then $\bar{\varphi}([\tau]_\sim) = \varphi(i([\tau]_\sim)) = \delta\alpha(i([\tau]_\sim)) = \alpha(\partial i([\tau]_\sim)) = \alpha(i([v_1]_\sim)) - \alpha(i([v_2]_\sim)) = \bar{\alpha}([v_1]_\sim) - \bar{\alpha}([v_2]_\sim)$, where $[\tau]_\sim = [v_1]_\sim[v_2]_\sim$ and $i([v_i]_\sim) = v_i$. This is a contradiction since $\bar{\varphi}$ is not a coboundary.

Since $\bar{S}$ is 6-large, by Remark 2.5, it can only be a connected sum of tori (a single torus included) or a connected sum of $\mathbb{RP}^2$ (at least two projective planes). In this case it is known that $\bar{S}$ has nontrivial first cohomology group. So there exists $[\bar{\varphi}] \neq 0$ in $H^1(\bar{S};\mathbb{Z})$, thus $[\varphi] \neq 0$ in $H^1(Y;\mathbb{Z})$. ∎

Finally, we have all the necessary tools to prove the main theorem of this paper.

THEOREM 3.2. *Let X be a locally finite systolic 3-pseudomanifold. Then $H^2_{\mathrm{c}}(X;\mathbb{Z}) \neq \{0\}$.*

Proof. To simplify the notation, we denote the ball and sphere of radius k around a fixed simplex $\sigma \in X$ by B_k and S_k respectively.

Given the long exact sequence of cohomology groups

$$\cdots \to H^1(X;\mathbb{Z}) \xrightarrow{i^*} H^1(X - B_k;\mathbb{Z}) \xrightarrow{\delta} H^2(X, X - B_k;\mathbb{Z}) \xrightarrow{j^*} H^2(X;\mathbb{Z}) \to \cdots$$

we know that $H^1(X;\mathbb{Z}) = \{0\}$ and $H^2(X;\mathbb{Z}) = \{0\}$ since X is contractible (by Theorem 2.4). Therefore we have the sequence

$$\cdots \to 0 \xrightarrow{i^*} H^1(X - B_k;\mathbb{Z}) \xrightarrow{\delta} H^2(X, X - B_k;\mathbb{Z}) \xrightarrow{j^*} 0 \to \cdots.$$

In this situation, δ is an isomorphism, which means that $H^1(X - B_k;\mathbb{Z}) \simeq H^2(X, X - B_k;\mathbb{Z})$.

Now, by the Projection Lemma and by the discussion following it, for $\pi_{S_k} := \pi_{B_k}|_{S'_{k+1}}$, we have the following commutative diagram:

$$\begin{tikzcd} S_{k+2} \arrow[d, "\pi_{S_{k+1}}"'] & X - B_{k+1} \arrow[l, "h_{S_{k+2}}"'] \arrow[d, hook] \\ S_{k+1} & X - B_k \arrow[l, "h_{S_{k+1}}"'] \end{tikzcd}$$

Therefore, the induced diagram of cohomology maps is commutative with the maps $h^*_{S_k} : H^1(S_k;\mathbb{Z}) \to H^1(X - B_{k-1};\mathbb{Z})$ being isomorphisms. It follows that

$$H^2_{\mathrm{c}}(X;\mathbb{Z}) = \varinjlim H^2(X, X - B_k;\mathbb{Z}) = \varinjlim H^1(S_k;\mathbb{Z}),$$

where the last limit is over the directed set defined by π_{S_k}.

So our problem reduces to work on the cohomology group of pseudosurfaces S_k. By Lemma 2.8 each S_k is a pseudosurface and, by Lemma 3.1, $H^1(S_k;\mathbb{Z})$ is nontrivial. Thus all we need to prove is that given a cocycle in S_k different from zero, it can be mapped to S_{k+1} by the induced (by projection) homomorphism of cohomology groups so that its image is nontrivial too. For that we use the contraction $\pi_{S_k} : S'_{k+1} \to S'_k$ between barycentric subdivisions of spheres. For the induced map

$$\pi^*_{S_k} : H^1(S'_k;\mathbb{Z}) \to H^1(S'_{k+1};\mathbb{Z})$$

we want to prove that a cocycle $\tilde{\varphi} \in Z^1(S'_{k+1};\mathbb{Z})$ defined as $\tilde{\varphi} := \pi^*_{S_k}(\varphi)$ is nontrivial, if the cocycle $\varphi \in Z^1(S'_k;\mathbb{Z})$ is nontrivial.

Suppose that $\tilde{\varphi} = \delta\tilde{\alpha}$ for some $\tilde{\alpha} \in C^0(S'_{k+1};\mathbb{Z})$. We show that in this case we can construct an $\alpha \in C^0(S'_k;\mathbb{Z})$ such that $\varphi = \delta\alpha$, reaching a contradiction. We define such an α in S'_k in three steps:

(i) We define α on all the vertices of S'_k corresponding to barycenters of triangles in S_k. Consider a triangle τ in S_k. Since X is a 3-pseudomanifold, the link X_τ consists of two vertices $\tilde{v}_1$ and $\tilde{v}_2$. By the Projection Lemma, $S_{k-1} \cap X_\tau$ is a single (nonempty) simplex. Thus one of the vertices of the link, say $\tilde{v}_2$, belongs to S_{k-1}. Therefore the other vertex $\tilde{v}_1$ must belong to S_{k+1}. We can see this situation in Figure 2, where the barycenter of the triangle is named v. We define $\alpha(v) := \tilde{\alpha}(\tilde{v}_1)$.

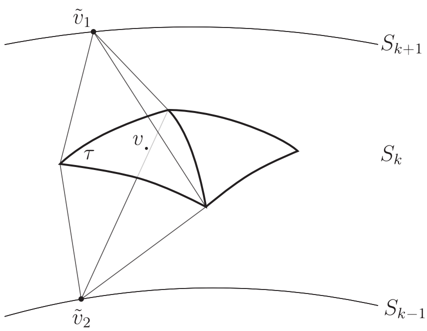

Fig. 2. Link of a triangle $\tau \in S_k$

(ii) We define α on all the vertices of S'_k corresponding to barycenters of edges in S_k. Consider an edge e. Since S_k is a pseudosurface, e belongs to two triangles τ_1 and τ_2. Let w be the barycenter of e. By Lemma 2.6, the link of this edge is a finite 1-pseudomanifold, thus a disjoint union of circles. By the Projection Lemma, only one of those circles intersects B_k, resulting in an interval. The vertices of the circle outside this interval belong to S_{k+1} and span (in X_e) a path $L = (\tilde{v}_1, \ldots, \tilde{v}_m)$, where $\tilde{v}_1$ and $\tilde{v}_m$ span a simplex with τ_1 and τ_2 respectively. Note that $m \geq 2$ due to the fact that X is systolic. Let $L' = (\tilde{v}_1, \tilde{w}_1, \tilde{v}_2, \tilde{w}_2, \ldots, \tilde{w}_{m-1}, \tilde{v}_m) \subseteq S'_{k+1}$ be the barycentric subdivision of L (see Figure 3).

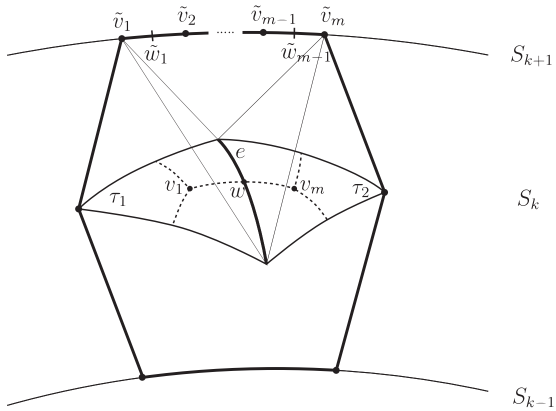

Fig. 3. Link of an edge $e \in S_k$

We define the value of α in w as $\alpha(w) := \tilde{\alpha}(\tilde{w}_1)$. We have to check that $\alpha(w)$ is well defined. For this it is enough to show that $\tilde{\alpha}(\tilde{w}_1) = \tilde{\alpha}(\tilde{w}_{m-1})$. Observe that π_{S_k} projects all the vertices of L' different from $\tilde{v}_1$ and $\tilde{v}_m$ to w. That gives $\tilde{\varphi}(\tilde{w}_1\tilde{v}_2) = \cdots = \tilde{\varphi}(\tilde{v}_{m-1}\tilde{w}_{m-1}) = 0$. This means that $0 = \tilde{\varphi}(\tilde{w}_1\tilde{v}_2) = \delta\tilde{\alpha}(\tilde{w}_1\tilde{v}_2) = \tilde{\alpha}(\tilde{v}_2) - \tilde{\alpha}(\tilde{w}_1)$, thus $\tilde{\alpha}(\tilde{v}_2) = \tilde{\alpha}(\tilde{w}_1)$. Applying the same argument to the rest of the edges yields $\tilde{\alpha}(\tilde{w}_1) = \tilde{\alpha}(\tilde{v}_2) = \cdots = \tilde{\alpha}(\tilde{v}_{m-1}) = \tilde{\alpha}(\tilde{w}_{m-1})$. Thus $\alpha(w)$ is well defined.

Furthermore, we use the fact that π_{S_k} projects the edges $\tilde{v}_1\tilde{w}_1$ and $\tilde{w}_{m-1}\tilde{v}_m$ onto the edges v_1w and wv_m respectively, to obtain

$$\varphi(v_1w) = \tilde{\varphi}(\tilde{v}_1\tilde{w}_1) = \delta\tilde{\alpha}(\tilde{v}_1\tilde{w}_1) = \tilde{\alpha}(\tilde{w}_1) - \tilde{\alpha}(\tilde{v}_1) = \alpha(w) - \alpha(v_1). \quad (1)$$

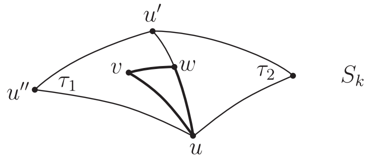

Fig. 4. Triangle $\triangle uvw \in S'_k$

(iii) We define α on all the vertices u of S_k. Consider a triangle $\triangle uu'u''$, and let w be the barycenter of uu' and v the barycenter of the triangle (see Figure 4). We define the value of α in u as $\alpha(u) := \varphi(wu) + \alpha(w)$. To check that this is well defined, it is enough to verify that $\varphi(wu) + \alpha(w) = \varphi(vu) + \alpha(v)$

(see Figure 4). Since φ is a cocycle, we have $\delta\varphi = 0$, and thus by (1) we obtain

$$0 = \delta\varphi(uvw) = \varphi(wu) + \varphi(vw) + \varphi(uv) = \varphi(wu) + \alpha(w) - \alpha(v) - \varphi(vu).$$

Therefore α is well defined.

In the three steps above we defined the cochain α satisfying, by (1) and by the definition in (iii), the equation $\delta\alpha = \varphi$. Therefore the proof is completed. ■

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