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ON THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER BY CERTAIN QUADRATIC FORMS

BY

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Abstract. For natural numbers a, b and positive integer n, let $R(a, b; n)$ denote the number of representations of n in the form

$$\sum_{i=1}^{a}(x_i^2 + x_iy_i + y_i^2) + 2\sum_{j=1}^{b}(u_j^2 + u_jv_j + v_j^2).$$

Lomadze discovered a formula for $R(6, 0; n)$. Explicit formulas for $R(1, 5; n)$, $R(2, 4; n)$, $R(3, 3; n)$, $R(4, 2; n)$ and $R(5, 1; n)$ are determined in this paper by using the $(p; k)$-parametrization of theta functions due to Alaca, Alaca and Williams.

1. Introduction. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. Throughout this paper, $q \in \mathbb{C}$ is taken to satisfy $|q| < 1$. For $i, n \in \mathbb{N}$, let

$$\sigma_i(n) = \sum_{d|n} d^i, \tag{1.1}$$

where d runs through the positive divisors of n. If n is not a positive integer, set $\sigma_i(n) = 0$. As usual, we write $\sigma(n)$ for $\sigma_1(n)$. For $a, b \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $R(a, b; n)$ denote the number of representations of n in the form

$$\sum_{i=1}^{a}(x_i^2 + x_iy_i + y_i^2) + 2\sum_{j=1}^{b}(u_j^2 + u_jv_j + v_j^2).$$

In 1989, Lomadze [L] proved that for $n \in \mathbb{N}$,

$$\begin{aligned} R(6, 0; n) = {} & \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) \\ & + \frac{18}{13} \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \\ x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2=n}} (9x_1^4 - 9nx_1^2 + n^2). \end{aligned} \tag{1.2}$$

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Recently, Xia and Yao [YX] found that for $n \in \mathbb{N}$,

$$R(6,0;n) = \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\sigma_5(n/3) + \frac{216}{13}b(n), \tag{1.3}$$

where $b(n)$ is defined by

$$\sum_{n=1}^{\infty} b(n)q^n = q\prod_{i=1}^{\infty}(1-q^i)^6(1-q^{3i})^6. \tag{1.4}$$

Equating the two expressions for $R(6,0;n)$ in (1.2) and (1.3), we see that for $n \in \mathbb{N}$,

$$b(n) = \frac{1}{12} \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4 \\ x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2=n}} (9x_1^4 - 9nx_1^2 + n^2). \tag{1.5}$$

The formula (1.3) was first proved by Chan and Cooper [CC]. Furthermore, they established a general formula for $R(2a,0;n)$ for any positive integer a.

The aim of this paper is to propose explicit formulas for $R(1,5;n)$, $R(2,4;n)$, $R(3,3;n)$, $R(4,2;n)$ and $R(5,1;n)$. The main results can be stated as follows.

Theorem 1.1. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} R(1,5;n) = {} & \frac{6}{7}\sigma_5(n) - \frac{132}{7}\sigma_5(n/2) + \frac{162}{7}\sigma_5(n/3) - \frac{3564}{7}\sigma_5(n/6) \\ & + \frac{36}{7}b(n) - \frac{288}{7}b(n/2) + \frac{648}{7}c(n), \end{aligned} \tag{1.6}$$

$$\begin{aligned} R(2,4;n) = {} & \frac{12}{13}\sigma_5(n) + \frac{240}{13}\sigma_5(n/2) - \frac{324}{13}\sigma_5(n/3) - \frac{6480}{13}\sigma_5(n/6) \\ & + \frac{144}{13}b(n) + \frac{1008}{13}b(n/2), \end{aligned} \tag{1.7}$$

$$\begin{aligned} R(3,3;n) = {} & \frac{18}{7}\sigma_5(n) - \frac{144}{7}\sigma_5(n/2) + \frac{486}{7}\sigma_5(n/3) - \frac{3888}{7}\sigma_5(n/6) \\ & + \frac{108}{7}b(n) - \frac{864}{7}b(n/2) + \frac{1944}{7}c(n), \end{aligned} \tag{1.8}$$

$$\begin{aligned} R(4,2;n) = {} & \frac{60}{13}\sigma_5(n) + \frac{192}{13}\sigma_5(n/2) - \frac{1620}{13}\sigma_5(n/3) - \frac{5184}{13}\sigma_5(n/6) \\ & + \frac{252}{13}b(n) + \frac{2304}{13}b(n/2), \end{aligned} \tag{1.9}$$

$$\begin{aligned} R(5,1;n) = {} & \frac{66}{7}\sigma_5(n) - \frac{192}{7}\sigma_5(n/2) + \frac{1782}{7}\sigma_5(n/3) - \frac{5184}{7}\sigma_5(n/6) \\ & + \frac{144}{7}b(n) - \frac{1152}{7}b(n/2) + \frac{2592}{7}c(n), \end{aligned} \tag{1.10}$$

where $b(n)$ is defined by (1.4) and $c(n)$ is defined by

$$\sum_{n=1}^{\infty} c(n)q^n = q^2 \prod_{n=1}^{\infty} (1-q^n)(1-q^{2n})(1-q^{3n})^5(1-q^{6n})^5. \tag{1.11}$$

From (1.6) and (1.8), we can obtain the following corollary.

COROLLARY 1.2. *For $n \in \mathbb{N}_0$,*

$$3R(1,5;2n+1) = R(3,3;2n+1). \tag{1.12}$$

2. The (p,k)-parametrization of Eisenstein series. The objective of this section is to recall the (p,k)-parametrization of Eisenstein series due to Alaca, Alaca and Williams [AAW1, AAW2, AW].

In his second notebook [R], Ramanujan gave the definitions of the Eisenstein series; one of them is

$$N(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \tag{2.1}$$

It is easy to see that

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{2.2}$$

Alaca, Alaca and Williams [AAW1] defined

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{2.3}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \tag{2.4}$$

where $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n \in \mathbb{Z}} q^{n^2}. \tag{2.5}$$

Alaca and Williams [AW] derived representations of $N(q)$ in terms of p and k. Equation (3.73) in [AW] is

$$\begin{aligned} N(q) = (1 &- 246p - 5532p^2 - 38614p^3 - 135369p^4 \\ &- 276084p^5 - 348024p^6 - 276084p^7 - 135369p^8 \\ &- 38614p^9 - 5532p^{10} - 246p^{11} + p^{12})k^6. \end{aligned} \tag{2.6}$$

Applying the duplication and triplication principles successively to (2.6), Alaca, Alaca and Williams [AAW2] obtained representations of $N(q^2)$,

$N(q^3)$ and $N(q^6)$ in terms of p and k. Equations (3.23), (3.24) and (3.26) in [AAW2] are

$$(2.7)\qquad N(q^2) = \Big(1 + 6p - 114p^2 - 625p^3 - \frac{4059}{2}p^4 - 4302p^5 - 5556p^6 - 4032p^7 - \frac{4059}{2}p^8 - 625p^9 - 114p^{10} + 6p^{11} + p^{12}\Big)k^6,$$

$$(2.8)\qquad N(q^3) = (1 + 6p + 12p^2 - 58p^3 - 297p^4 - 396p^5 - 264p^6 - 396p^7 - 297p^8 - 58p^9 + 12p^{10} + 6p^{11} + p^{12})k^6,$$

$$(2.9)\qquad N(q^6) = \Big(1 + 6p + 12p^2 + 5p^3 - \frac{27}{2}p^4 - 18p^5 - 12p^6 - 18p^7 - \frac{27}{2}p^8 + 5p^9 + 12p^{10} + 6p^{11} + p^{12}\Big)k^6,$$

respectively. Alaca, Alaca and Williams [AAW3] also derived the representations of $q^{1/24}\prod_{j=1}^{\infty}(1-q^j)$, $q^{1/12}\prod_{j=1}^{\infty}(1-q^{2j})$, $q^{1/8}\prod_{j=1}^{\infty}(1-q^{3j})$ and $q^{1/4}\prod_{j=1}^{\infty}(1-q^{6j})$ in terms of p and k. Equations (2.10), (2.11), (2.12) and (2.14) in [AAW3] are

(2.10)
$$q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = 2^{-1/6}p^{1/24}(1-p)^{1/2}(1+p)^{1/6}(1+2p)^{1/8}(2+p)^{1/8}k^{1/2},$$

(2.11)
$$q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n}) = 2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/1}k^{1/2},$$

(2.12)
$$q^{1/8}\prod_{n=1}^{\infty}(1-q^{3n}) = 2^{-1/6}p^{1/8}(1-p)^{1/6}(1+p)^{1/2}(1+2p)^{1/24}(2+p)^{1/24}k^{1/2},$$

(2.13)
$$q^{1/4}\prod_{n=1}^{\infty}(1-q^{6n}) = 2^{-1/3}p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2},$$

respectively. Jonathan and Peter Borwein [BB] introduced three 2-dimensional theta functions; one of them is

$$(2.14)\qquad a(q) = \sum_{(i,j)\in\mathbb{Z}^2} q^{i^2+ij+j^2}, \qquad q\in\mathbb{C}, \qquad |q|<1.$$

Alaca, Alaca and Williams [AAW1] found representations of $a(q)$ and $a(q^2)$ in terms of p and k:

$$(2.15)\qquad a(q) = (1+4p+p^2)k$$

and

$$a(q^2) = (1 + p + p^2)k. \tag{2.16}$$

3. Some identities involving $a(q)$ and Eisenstein series. In this section, we establish five identities involving $a(q)$ and Eisenstein series, which are used to prove Theorem 1.1.

THEOREM 3.1. *We have*

(3.1)
$$a(q)a^5(q^2) = -\frac{1}{588}N(q) + \frac{11}{294}N(q^2) - \frac{9}{196}N(q^3) + \frac{99}{98}N(q^6) + \frac{36}{7}\sum_{n=0}^{\infty} b(n)q^n - \frac{288}{7}\sum_{n=0}^{\infty} b(n)q^{2n} + \frac{648}{7}\sum_{n=0}^{\infty} c(n)q^n,$$

(3.2)
$$a^2(q)a^4(q^2) = -\frac{1}{546}N(q) - \frac{10}{273}N(q^2) + \frac{9}{182}N(q^3) + \frac{90}{91}N(q^6) + \frac{144}{13}\sum_{n=0}^{\infty} b(n)q^n + \frac{1008}{13}\sum_{n=0}^{\infty} b(n)q^{2n},$$

(3.3)
$$a^3(q)a^3(q^2) = -\frac{1}{196}N(q) + \frac{2}{49}N(q^2) - \frac{27}{196}N(q^3) + \frac{54}{49}N(q^6) + \frac{108}{7}\sum_{n=0}^{\infty} b(n)q^n - \frac{864}{7}\sum_{n=0}^{\infty} b(n)q^{2n} + \frac{1944}{7}\sum_{n=0}^{\infty} c(n)q^n,$$

(3.4)
$$a^4(q)a^2(q^2) = -\frac{5}{546}N(q) - \frac{8}{273}N(q^2) + \frac{45}{182}N(q^3) + \frac{72}{91}N(q^6) + \frac{252}{13}\sum_{n=0}^{\infty} b(n)q^n + \frac{2304}{13}\sum_{n=0}^{\infty} b(n)q^{2n},$$

(3.5)
$$a^5(q)a(q^2) = -\frac{11}{588}N(q) + \frac{8}{147}N(q^2) - \frac{99}{196}N(q^3) + \frac{72}{49}N(q^6) + \frac{144}{7}\sum_{n=0}^{\infty} b(n)q^n - \frac{1152}{7}\sum_{n=0}^{\infty} b(n)q^{2n} + \frac{2592}{7}\sum_{n=0}^{\infty} c(n)q^n,$$

where $b(n)$ and $c(n)$ are defined by (1.4) *and* (1.11), *respectively.*

Proof. We just prove (3.1). The rest can be proved similarly. By (1.4), (1.11) and (2.10)–(2.13), we see that

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{4}p(1-p)^4(1+p)^4(1+2p)(2+p)k^6, \tag{3.6}$$

$$\sum_{n=0}^{\infty} b(n)q^{2n} = \frac{1}{16}p^2(1-p)^2(1+p)^2(1+2p)^2(2+p)^2k^6, \tag{3.7}$$

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{8}p^2(1-p)^2(1+p)^4(1+2p)(2+p)k^6. \tag{3.8}$$

Combining (2.6)–(2.9) and (3.6)–(3.8), we deduce that

$$\begin{aligned} & -\frac{1}{588}N(q) + \frac{11}{294}N(q^2) - \frac{9}{196}N(q^3) + \frac{99}{98}N(q^6) + \frac{36}{7}\sum_{n=0}^{\infty} b(n)q^n \\ & - \frac{288}{7}\sum_{n=0}^{\infty} b(n)q^{2n} + \frac{648}{7}\sum_{n=0}^{\infty} c(n)q^n = (1+4p+p^2)(1+p+p^2)^5k^6. \end{aligned} \tag{3.9}$$

On the other hand, in view of (2.15) and (2.16), it is easy to check that

$$a(q)a^5(q^2) = (1+4p+p^2)(1+p+p^2)^5k^6. \tag{3.10}$$

Identity (3.1) follows from (3.9) and (3.10). ■

4. Proof of Theorem 1.1. In this section, we present a proof of Theorem 1.1 by employing Theorem 3.1. We deduce (1.6) from (3.1). The rest can be proved similarly.

By the definition of $R(a, b; n)$ and (2.14), it is easy to see that the generating function of $R(1, 5; n)$ is

$$\sum_{n=1}^{\infty} R(1,5;n)q^n = a(q)a^5(q^2) - 1. \tag{4.1}$$

By applying (2.2), (3.1) and (4.1), we find that

$$\begin{aligned} \sum_{n=1}^{\infty} R(1,5;n)q^n &= -\frac{1}{588}N(q) + \frac{11}{294}N(q^2) - \frac{9}{196}N(q^3) + \frac{99}{98}N(q^6) \\ &\quad + \frac{36}{7}\sum_{n=0}^{\infty} b(n)q^n - \frac{288}{7}\sum_{n=0}^{\infty} b(n)q^{2n} + \frac{648}{7}\sum_{n=0}^{\infty} c(n)q^n - 1 \\ &= -\frac{1}{588}\Big(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\Big) + \frac{11}{294}\Big(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\Big) \\ &\quad - \frac{9}{196}\Big(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}\Big) + \frac{99}{98}\Big(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}\Big) \end{aligned} \tag{4.2}$$

$$+\frac{36}{7}\sum_{n=0}^{\infty}b(n)q^n-\frac{288}{7}\sum_{n=0}^{\infty}b(n)q^{2n}+\frac{648}{7}\sum_{n=0}^{\infty}c(n)q^n-1$$
$$=\frac{6}{7}\sum_{n=1}^{\infty}\sigma_5(n)q^n-\frac{132}{7}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}+\frac{162}{7}\sum_{n=1}^{\infty}\sigma_5(n)q^{3n}-\frac{3564}{7}\sum_{n=1}^{\infty}\sigma_5(n)q^{6n}$$
$$+\frac{36}{7}\sum_{n=0}^{\infty}b(n)q^n-\frac{288}{7}\sum_{n=0}^{\infty}b(n)q^{2n}+\frac{648}{7}\sum_{n=0}^{\infty}c(n)q^n.$$

Equating the coefficients of q^n on both sides of (4.2), we obtain (1.6). ∎

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