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INDECOMPOSABLE PRIMARILY COMULTIPLICATION MODULES OVER A PULLBACK OF TWO DEDEKIND DOMAINS

BY

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Abstract. We describe all those indecomposable primarily comultiplication modules with finite-dimensional top over pullback of two Dedekind domains. We extend the definition and results given by R. Ebrahimi Atani and S. Ebrahimi Atani [Algebra Discrete Math. 2009] to a more general primarily comultiplication modules case.

1. Introduction. The idea of investigating a mathematical structure via its representations in simpler structures is commonly used and often successful. The representation theory of finite-dimensional algebras has developed greatly in the recent years. It is an area which is very firmly based on the detailed understanding of examples, and there are many powerful techniques for investigating representations of particular algebras and for relating representations of different algebras to one another. Its basic problem is that of classification: given an algebra which is finite-dimensional over a field describe its representations (modules) and the relations between them. However, apart from some nicest classes of algebras, this is impossible, and so the aim in practice is to classify the finite-dimensional representations and the relations between them. Thus arises the project of classifying all representations or, more realistically, all representations of a certain significant type. A commonly adopted strategy is to prove a decomposition theorem which says that every representation of the sort we are considering may be built up from certain simpler ones, and then to develop a classification and structure theory for those simpler building blocks. An optimal structure theory for the blocks is one which provides us with a complete list and with presentations of the members of the list, which are explicit enough to allow answering many questions about the blocks with relatively little effort. The reader is referred to [2], [29, Chapters 1 and 14] and [31] for a detailed discussion of classification problems, their representation types (finite, tame, or wild), and useful computational reduction procedures.

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We know that every module is an elementary substructure of a pure-injective module. In fact, there is a minimal pure-injective elementary extension of each module M, denoted by $h(M)$, called the pure-injective hull of M, and it is unique up to isomorphism fixing M. The class of pure-injectives is closed under direct summands and finite direct sums, but an infinite direct sum of pure-injectives need not be pure-injective. Observe that any finite module is pure-injective. In a sense, then, pure-injective modules are model-theoretically typical: for example, classification of the complete theories of R-modules reduces to classifying the (complete theories of) pure-injectives. Also, for some rings, "small" (finite-dimensional, finitely generated, ...) modules are classified and in many cases this classification can be extended to give a classification of (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and families of finitely generated modules. Therefore, pure-injective modules are very important (see [32], [28] and [16]). One point of this paper is to introduce a subclass of pure-injective modules.

Modules over pullback rings have been studied by several authors (see for example, [25], [3], [17], [12], [18] and [33]). The important work of Levy [20] provides a classification of all finitely generated indecomposable modules over Dedekind-like rings. L. Klingler [18] extended this classification to lattices over certain non-commutative Dedekind-like rings, and Klingler and J. Haefner ([13], [14]) classified lattices over certain non-commutative pullback rings, which they called special quasi triads. Common to all these classifications is the reduction to a "matrix problem" over a division ring (see [4], [29, Section, 17.9], [27], and [30] for background on matrix problems and their applications).

In the present paper we introduce a new class of R-modules, called primarily comultiplication modules (see Definition 2.1), and we study it in detail from the classification point of view. We are mainly interested in the case where R is either a Dedekind domain or a pullback of two local Dedekind domains. First, we give a complete description of the primarily comultiplication modules over a local Dedekind domain. Next, let R be a pullback of two local Dedekind domains over a common factor field. The main purpose of this paper is to give a complete description of the indecomposable primarily comultiplication R-modules with finite-dimensional top over $R/\mathrm{rad}(R)$ (for any module M we define its top as $M/\mathrm{rad}(R)M$). In fact, we extend the definition and results given in [5] to a more general case, as the class of primarily comultiplication modules contains the class of weak comultiplication modules defined in [5]. The classification is divided into two stages: we give a list of all indecomposable separated primarily comultiplication R-modules and then, using this list, we show that non-separated indecomposable primarily comultiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable primarily comultiplication R-modules. Then we use the classification of separated indecomposable primarily comultiplication modules from Section 3, together with results of Levy [20], [21] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable primarily comultiplication modules M with finite-dimensional top (see Theorem 4.11). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable primarily comultiplication modules (where infinite length primarily comultiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these modules.

For the sake of completeness, we state some definitions and notations used throughout. In this paper all rings are commutative with identity and all modules unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinatewise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [19]).

Definition 1.1. An R-module S is defined to be *separated* if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$).

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [19, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [19, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module; indeed, every R-module has a "minimal" such representation: a *separated representation* of an R-module M is an epimorphism $\varphi : S \to M$ of R-modules with S separated such that if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [19, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [19, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [19, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [19, Theorem 2.6].

Definition 1.2.

(a) If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be a *primary submodule* (resp. *prime submodule*) if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some n (resp. $m \in N$ or $r \in (N : M)$), so $\mathrm{rad}(N : M) = P$ (resp. $(N : M) = P'$) is a prime ideal of R, and N is said to be a *P-primary* (resp. *P'-prime*) *submodule.* The set of all primary submodules (resp. prime submodules) in an R-module M is denoted $\mathrm{pSpec}(M)$ (resp. $\mathrm{Spec}(M)$).
(b) An R-module M is defined to be a *comultiplication module* if for each submodule N of M, $N = (0 :_M I)$ for some ideal I of R. In this case we can take $I = \mathrm{Ann}(N)$ (see [1]).
(c) An R-module M is defined to be a *weak comultiplication module* if either $\mathrm{Spec}(M) = \emptyset$ or for every prime submodule N of M, $N = (0 :_M I)$ for some ideal I of R (see [5]).
(d) A submodule N of an R-module M is called a *pure submodule* if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called *relatively divisible* (or an *RD-submodule*) in M if $rN = N \cap rM$ for all $r \in R$ (see [16], [32]).
(e) A module M is *pure-injective* if it has the injective property relative to all pure exact sequences (see [16], [32]).
(f) A non-zero R-module M is said to be *coprimary* if for each $r \in R$, the homothety $M \xrightarrow{r\cdot} M$ is either injective or nilpotent. So $\mathrm{rad}(0 : M)$ $= J$, the radical $(0 : M)$ is a prime ideal of R, and M is said to be *J-coprimary* (see [26]).

Remark 1.3.

(i) An R-module is pure-injective if and only if it is algebraically compact (see [16] and [32]).
(ii) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M (see [32]).
(iii) It is easy to see that an R-module M is coprimary if and only if whenever $rm = 0$ (for some $r \in R$ and $m \in M$), then either $m = 0$ or $r^n M = 0$ for some n. Moreover, it is clear that if N is a J-primary submodule of M, then M/N is a J-coprimary R-module.

2. Properties of primarily comultiplication modules. In this section we collect some basic properties concerning primarily comultiplication modules. We begin with the key definition of this paper.

DEFINITION 2.1. Let R be a commutative ring. An R-module M is defined to be a *primarily comultiplication module* if either $\mathrm{pSpec}(M) = \emptyset$, or for every primary submodule N of M, $N = (0 :_M I)$ for some ideal I of R.

One can easily show that if M is a primarily comultiplication module, then $N = (0 :_M \mathrm{ann}(N))$ for every primary submodule N of M. It is easy to see that the class of primarily comultiplication modules contains the class of weak comultiplication modules defined in [5]. We need the following lemma proved in [26, p. 101, Corollary, and p. 99, Corollary 1].

LEMMA 2.2.

(i) *Let $K \subseteq N$ be submodules of an R-module M. Then N is a primary submodule of M if and only if N/K is a primary submodule of M/K.*
(ii) *Let N be a P-primary submodule of the R-module M and suppose that I is an ideal of R and K a submodule of M. If $IK \subseteq N$, then either $I \subseteq P$ or $K \subseteq N$.*

PROPOSITION 2.3. *If R is a domain (which is not a field) and M is a primarily comultiplication R-module with torsion submodule $T(M) \neq M$, then* $\mathrm{pSpec}(M) = \{T(M)\}$.

Proof. Since $T(M)$ is a prime submodule of M with $(T(M) : M) = 0$ and R is a domain, we must have $\mathrm{rad}(T(M) : M) = \mathrm{rad}(0) = 0$. If N is a non-zero primary submodule of M, then $N = (0 :_M I)$ for some non-zero ideal I of R, so $N \subseteq T(M)$. Let $x \in T(M)$. Then $rx = 0 \in N$ for some $0 \neq r \in R$; hence $x \in N$ since $\mathrm{rad}(N : M) \subseteq \mathrm{rad}(T(M) : M) = 0$, and so $T(M) = N$. Note that if we assume additionally that 0 is a primary submodule of M, then since $0 \subseteq T(M)$, we have $\mathrm{rad}(0 : M) \subseteq \mathrm{rad}(T(M) : M) = 0$, which implies $0 \in \mathrm{pSpec}(M)$ and so $T(M) = 0$, as needed. ∎

LEMMA 2.4. *Let M be a primarily comultiplication module over a commutative ring R. Then the following hold:*

(i) *If N is a pure submodule of M, then M/N is a primarily comultiplication R-module.*
(ii) *Every direct summand of M is a primarily comultiplication submodule.*

Proof. (i) Let K/N be a primary submodule of M/N. Then by Lemma 2.2, K is a primary submodule of M, so $L = (0 :_M I)$ for some ideal I of R. An inspection will show that $L/N = (0 :_{M/N} I)$. (ii) follows from (i) since direct summands are pure. ∎

LEMMA 2.5. *Let M be an R-module, N a proper submodule of M, and $I \subseteq (0 : M)$. Then the following hold:*

(i) *N is a P-primary R-submodule M if and only if N is a P/I-primary submodule of M as an R/I-module.*

(ii) *M is a primarily comultiplication R-module if and only if M is a primarily comultiplication module as an R/I-module.*

Proof. The proof of (i) is straightforward. To see (ii), apply Lemma 2.2 and the fact that $(0 :_M J) = (0 :_M (J + I)/I)$ for every ideal J of R. ∎

PROPOSITION 2.6. *Let M be a module over a local ring R with a unique maximal ideal P. If $(0 : M) = P^n$ for some positive integer n, then every proper submodule of M is a P-primary submodule.*

Proof. Let N be a proper submodule of M. Then $\text{rad}(N : M) \neq R$ and $P^n M = 0 \in N$, so $P^n \subseteq (N : M) \subseteq P$ by Lemma 2.2; hence $\text{rad}(N : M) = P$. Let $rm \in N$ for some $r \in R$ and $m \in M$ such that $r \notin P$. Then R local gives $r^{-1}rm = m \in N$, as required. ∎

REMARK 2.7. (1) Let R be a local Dedekind domain with a unique maximal ideal $P = Rp$.

(a) Let $M = R$ (as R-modules). For a primary submodule $P^n M$ $(n \geq 2)$ of M we have $(0 :_M (0 :_R P^n M)) = R$. So $M = R$ is not a primarily comultiplication R-module.

(b) We show that $\text{pSpec}(E(R/P)) = \emptyset$, where $E = E(R/P)$ is the injective hull of R/P. By [8, Lemma 2.6], every non-zero proper submodule L of E is of the form $L = A_n = (0 :_E P^n)$ $(n \geq 1)$, $L = A_n = Ra_n$ and $\text{rad}(L : E) = 0$ since E is divisible and R is an integral domain. If L is a primary submodule of E, then for any positive integer m, we have $p^m \notin \text{rad}(L : E) = 0$ and $a_{n+m} \notin L$, but $p^m a_{n+m} = an \in L$ (see [8, Lemma 2.6]). Thus E is primarily comultiplication.

(2) Let R be an integral domain which is not a field, and $Q(R)$ the field of fractions of R. We show that $\text{pSpec}(Q(R)) = \{0\}$. By [23, Theorem 1], for every proper submodule N of $Q(R)$, we must have $(N : Q(R)) = 0$. Clearly, 0 is a 0-primary submodule of $Q(R)$. To show that 0 is the only primary submodule of $Q(R)$, assume the contrary and let K be a non-zero primary submodule of $Q(R)$. Then $\text{rad}(K : M) = \text{rad}(0) = 0$ since R is a domain. By an argument like that in [23, Theorem 1], we get a contradiction. As $0 = (0 :_{Q(R)} R)$, $Q(R)$ is primarily comultiplication.

THEOREM 2.8. *Let R be a discrete valuation domain with a unique maximal ideal $P = Rp$. Then the class of indecomposable primarily comultiplication modules over R consists of the following:*

(1) *R/P^n, $n \geq 1$, the indecomposable torsion modules;*

(2) $E(R/P)$, *the injective hull of* R/P;
(3) $Q(R)$, *the field of fractions of* R.

Proof. First we note that each of the listed modules is indecomposable (by [7, Proposition 1.3]) and primarily comultiplication. In the case of R/P^n this follows because R/P^n is a comultiplication module (see [6]). Moreover, $Q(R)$ and $E(R/P)$ are primarily comultiplication by Remark 2.7.

Now let M be an indecomposable primarily comultiplication module and choose any non-zero $a \in M$. Let $h(a) = \sup\{n : a \in P^n M\}$, so $h(a)$ is a non-negative integer or ∞. Also let $(0 : a) = \{r \in R : ra = 0\}$, which is an ideal of the form P^m or 0. Because $(0 : a) = P^{m+1}$ implies $p^m a \neq 0$ and $p \cdot p^m a = 0$, we can choose a so that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

Case 1: $\mathrm{pSpec}(M) = \emptyset$. Since $\mathrm{Spec}(M) \subseteq \mathrm{pSpec}(M)$, it follows from [24, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. We may assume that $(0 : a) = P$. By an argument like that in [8, Proposition 2.7], $M \cong E(R/P)$. So we may assume that $\mathrm{pSpec}(M) \neq \emptyset$.

Case 2: $h(a) = n$. Then $(0 : a) = P$. Indeed, suppose not. Then $(0 : a) = 0$. Say $a = p^n b$. Then $rb = 0$ implies $ra = 0$ and so $r = 0$. Thus $Rb \cong R$. Moreover, Rb is pure in M (see [10, Theorem 2.12, Case 1]). As M is a torsion-free R-module by [15, Theorem 10], Rb is a prime submodule of M (see [22, Result 2]) (so a primary submodule); hence $R \cong Rb = (0 :_M 0) = M$, which is a contradiction by Remark 2.7(1). So $(0 : a) = P$. Say $a = p^n b$. Then $Rb \cong R/P^{n+1}$. Since Rb is a pure submodule of bounded order of M, it is a direct summand of M by [15, Theorem 5]; hence $M = Rb \cong R/P^{n+1}$.

Case 3: $h(a) = \infty$, $(0 : a) = P$. By an argument like that in [6, Theorem 2.5, Case 2], we get $M \cong E(R/P)$; hence $\mathrm{pSpec}(M) = \emptyset$ by Remark 2.7, contrary to assumption.

Case 4: $h(a) = \infty$, $(0 : a) = 0$. By an argument like that in [10, Theorem 2.12, Case 3], we obtain $M \cong Q(R)$. ■

Let M be a primarily comultiplication module over a local Dedekind domain. Then every direct summand of M is primarily comultiplication. Then by [15, Theorems 7–9], either M is indecomposable or M has all indecomposable direct summands primarily comultiplication. Therefore, we have the following consequence of Lemma 2.4(ii), Theorem 2.8 and [7, Proposition 1.3].

Theorem 2.9. *Let M be a primarily comultiplication module over a discrete valuation domain with maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n $(n \geq 1)$ and K is* *a direct sum of copies of $E(R/P)$ and $Q(R)$. In particular, every primarily comultiplication R-module is pure-injective.*

3. The separated case. Throughout this section we shall assume, unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{3.1}$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2. Let P denote $P_1 \oplus P_2$. Then $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is, $P_1 \oplus 0$) and P_2 (that is, $0 \oplus P_2$).

REMARK 3.1. Let R be the pullback ring as in (1), and let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \twoheadrightarrow S_i$. Set

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\},$$
$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}.$$

Then for each $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi_1' = \pi_1|T : T \twoheadrightarrow T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus \text{Ker}(f_2) \cap T) \cong T/(T \cap P_2 S) \cong (T + P_2 S)/P_2 S \subseteq S/P_2 S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\text{Ker}(f_1) = P_1 S_1$ and $\text{Ker}(f_2) = P_2 S_2$).

LEMMA 3.2. *Let R be the pullback ring as in* (3.1), *and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module and let T be a proper submodule of S. Then the following hold:*

(i) *T is a $P_1 \oplus 0$-prime submodule S if and only if T is a $P_1 \oplus 0$-primary submodule of S.*
(ii) *T is a $0 \oplus P_2$-prime submodule S if and only if T is a $0 \oplus P_2$-primary submodule of S.*

Proof. (i) The necessity is clear. Conversely, suppose that T is a $P_1 \oplus 0$-primary submodule. Since $(0 \oplus P_2)(P_1 \oplus 0)S = 0 \in T$, we must have $(P_1 \oplus 0)S \subseteq T$ by Lemma 2.2. It follows that $P_1 \oplus 0 \subseteq (T : S) \subseteq \text{rad}(T : S) = P_1 \oplus 0$; hence $(T : S) = P_1 \oplus 0$. Now $(T : S) = P_1 \oplus 0$ and T primary implies T is a prime submodule of S. The proof of (ii) is similar. ∎

PROPOSITION 3.3. *Let R be the pullback ring as in* (3.1), *and let $S = (S/P_2 S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1 S)$ be any separated R-module. Then the following hold:*

(i) *If S has a P-primary R-submodule $T = (T_1 \to \bar{T} \leftarrow T_2)$, then T_1 is a P_1-primary submodule of S_1, and T_2 is a P_2-primary submodule of S_2.*

(ii) *If S has a $P_1 \oplus 0$-primary R-submodule T, then T_1 is a P_1-primary submodule of S_1, and T_2 is a 0-primary submodule of S_2.*

(iii) *If S has a $0 \oplus P_2$-primary R-submodule T, then T_1 is a 0-primary submodule of S_1, and T_2 is a P_2-primary submodule of S_2.*

Proof. (i) Let $r_1 s_1 \in T_1$ for some $r_1 \in R_1$ and $s_1 \in S_1$. Then $v_1(r_1) = v_2(r_2)$ and $f_1(s_1) = f_2(s_2)$ for some $r_2 \in R_2$, $s_2 \in S_2$, so there is a positive integer n such that $(r_1^n, r_2^n)(s_1, s_2) \in P^n S \subseteq T$ (note that every ideal in a commutative Noetherian ring contains a power of its radical). Therefore, T primary gives either $s_1 \in T_1$ or $r_1 \in P_1$. Thus T_1 is a P_1-primary submodule of S_1. Similarly, T_2 is a P_2-primary submodule of S_2.

(ii) Suppose that T is a $P_1 \oplus 0$-primary submodule of S. Since $(0 \oplus P_2)(P_1 \oplus 0)S = 0 \in T$ and T is $P_1 \oplus 0$-primary, we must have $(P_1 \oplus 0)S \subseteq T$ (see Lemma 2.2(ii)); hence $T/(P_1 \oplus 0)S$ is a $P_1 \oplus 0$-primary R-submodule of $S/(P_1 \oplus 0)S$. So T_2 is a 0-primary R_2-submodule of S_2 by Lemma 2.5. Finally, there is a positive integer s such that $(P_1 \oplus 0)^s S \subseteq T$ since R is Noetherian, so $P_1 \subseteq \mathrm{rad}(T_1 : S_1) \subseteq P_1$; hence $\mathrm{rad}(T_1 : S_1) = P_1$. Therefore, T_1 is a P_1-primary submodule of S_1 by Proposition 2.6. The proof of (iii) is similar. ∎

LEMMA 3.4. *Let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated module over the pullback ring as in* (3.1) *and m, n be positive integers. Then the following hold:*

(i) *If $(0 : S) = P_1^n \oplus P_2^m$, then every proper submodule of S is a P-primary submodule. In particular, if $n, m > 1$, then $(0 :_R (P_1 \oplus 0)S) = P_1^{n-1} \oplus P_2$ and $(0 :_R (0 \oplus P_2)S) = P_1 \oplus P_2^{m-1}$.*

(ii) *If $(0 : S) = P_1^n \oplus 0$ and $\bar{S} \neq 0$, then $(0 \oplus P_2)S$ is a P-primary submodule of S. In particular, $(0 : PS) = P_1^{n-1} \oplus 0$.*

(iii) *If $(0 : S) = 0 \oplus P_2^m$ and $\bar{S} \neq 0$, then $(P_1 \oplus 0)S$ is a P-primary submodule of S. In particular, $(0 : PS) = 0 \oplus P_2^{m-1}$.*

Proof. (i) Let T be a proper submodule of S. Then $(T : S) \neq R$ and $P_1^n \oplus P_2^m \subseteq (T : S)$. Since $(T : S) \nsubseteq P_1 \oplus 0$ and $(T : S) \nsubseteq 0 \oplus P_2$, and $\mathrm{Spec}(R) = \{P_1 \oplus 0, 0 \oplus P_2, P\}$, we must have $\mathrm{rad}(T : S) = P$. Now the assertion follows from Proposition 2.6. Finally, by assumption, $P_1 \oplus 0S \neq 0$ and $(0 \oplus P_2)S \neq 0$. Since $(P_1^{n-1} \oplus P_2)(P_1 \oplus 0)S = 0$, we must have $P_1^{n-1} \oplus P_2 \subseteq (0 :_R (P_1 \oplus 0)S)$. For the reverse inclusion, suppose that $(r_1, r_2) \in (0 :_R (P_1 \oplus 0)S) \subseteq P$. It follows that $r_2 \in P_2$ and $(r_1 p_1, 0)S \subseteq (r_1, r_2)(P_1 \oplus 0)S = 0$. So $r_1 p_1 \in P_1^n$. Then $r_1 = w_1 p_1^{n-1} \in P_1^{n-1}$ for some $w_1 \in R_1$; hence $(r_1, r_2) \in P_1^{n-1} \oplus P_2$. Similarly, $(0 :_R (0 \oplus P_2)S) = P_1 \oplus P_2^{m-1}$.

(ii) By assumption, $0 \subsetneq (0 \oplus P_2)S \subsetneq S$. Since $(P_1^n \oplus 0)S = 0 \subseteq (0 \oplus P_2)S$, we have $P_1^n \oplus 0 \subseteq ((0 \oplus P_2)S : S)$. Since $(0 \oplus P_2)S : S) \nsubseteq P_1 \oplus 0$ and $((0 \oplus P_2)S : S) \nsubseteq 0 \oplus P_2$, and $\mathrm{Spec}(R) = \{P_1 \oplus 0, 0 \oplus P_2, P\}$, we must have $\mathrm{rad}(T : S) = P$. Now the assertion follows from Proposition 2.6. Finally, since $PS \neq S$, PS is a P-primary submodule of S. Moreover, $(P_1^{n-1} \oplus 0)PS = 0$ gives $P_1^{n-1} \oplus 0 \subseteq (0 : PS)$. For the reverse inclusion, assume that $(r_1, r_2) \in (0 : PS)$. Then $(r_1p_1, r_2p_2)S = 0$, so $r_1p_1 \in P_1^n$ and $r_2p_2 = 0$. Then there exists $u_1 \in R_1$ such that $r_1 = u_1p_1^{n-1}$ and $r_2 = 0$ since R_2 is a domain. Hence $(r_1, r_2) \in P_1^{n-1} \oplus 0$, as needed. The proof of (iii) is similar. ∎

PROPOSITION 3.5. *Let S be any primarily comultiplication separated module over the pullback ring as in* (3.1). *If $(0 :_R S) = 0$, then $\bar{S} = 0$.*

Proof. Suppose $\bar{S} \neq 0$. Then PS is a P-primary submodule of S. Let $(r_1, r_2) \in (0 :_R PS)$. Then $(r_1, r_2)(p_1, p_2)S \subseteq (r_1, r_2)PS = 0$, so $r_1p_1 = 0$ and $r_2p_2 = 0$; hence $r_1 = 0$ and $r_2 = 0$. Therefore, $(0 :_R PS) = 0$. Then S primarily comultiplication gives $PS = (0 :_S (0 :_R PS)) = (0 :_S 0) = S$, which is a contradiction. ∎

PROPOSITION 3.6. *Let S be any separated module over the pullback ring as in* (3.1). *Then the following hold:*

(i) $(0 :_R S) = I \oplus J$ *if and only if* $(0 :_{R_1} S_1) = I$ *and* $(0 :_{R_2} S_2) = J$, *where $I \neq 0$ and $J \neq 0$.*
(ii) *If* $(0 :_R S) = P_1^n \oplus 0$ *for some positive integer n, then* $(0 :_{R_1} S_1) = P_1^n$ *and* $(0 :_{R_2} S_2) = 0$.
(iii) *If* $(0 :_R S) = 0 \oplus P_2^n$ *for some positive integer n, then* $(0 :_{R_1} S_1) = 0$ *and* $(0 :_{R_2} S_2) = P_2^n$.

Proof. (i) Assume that $(0 :_R S) = I \oplus J$ and let $s_1 \in S_1$. Then there exists $s_2 \in S_2$ such that $(s_1, s_2) \in S$, so $Is_1 = 0$ and hence $I \subseteq (0 :_{R_1} S_1)$. For the other containment, assume that $r_1 \in (0 :_{R_1} S_1) \subseteq P_1$. So $r_1S_1 = 0$. Let $(s_1, s_2) \in S$. Then $(r_1, 0)(s_1, s_2) = 0$, so $(r_1, 0)S = 0$. Hence $r_1 \in I$, and we have equality. Similarly, $(0 :_{R_2} S_2) = J$. Conversely, assume that $(s_1, s_2) \in S$. Then $Is_1 = 0$ and $Js_2 = 0$, so $(I \oplus J)(s_1, s_2) = 0$; hence $I \oplus J \subseteq (0 :_R S)$. For the reverse containment, suppose $(r_1, r_2) \in (0 :_R S)$. Let $t_1 \in S_1$. Then there is an element $t_2 \in S_2$ such that $(t_1, t_2) \in S$, so $(r_1, r_2)(t_1, t_2) = 0$; hence $r_1S_1 = 0$. Thus $r_1 \in I$. Similarly, $r_2 \in J$, and the proof is complete.

(ii) By (i), it suffices to show that $(0 :_{R_2} S_2) = 0$. Suppose not. Let $0 \neq r_2 \in (0 :_{R_2} S_2)$. Then there exist $u \in R_2$ and a positive integer t such that $r_2 = up_2^t$. Let $(s_1, s_2) \in S$. Then $(P_1^n \oplus uP_2^t)(s_1, s_2) = 0$, so $(P_1^n \oplus uP_2^t)S = 0$, which is a contradiction. The proof of (iii) is similar. ∎

PROPOSITION 3.7. *Let S be any primarily comultiplication separated module over the pullback ring as in* (3.1) *with $\bar{S} \neq 0$.*

(i) *If $(0:_R S) = P_1^n \oplus P_2^m$ for some positive integers n, m, then either $m = 1$ or $n = 1$.*
(ii) *If $(0:_R S) = P_1 \oplus P_2^m$ for some positive integer $m > 1$, then $(0:_R PS) = P_1 \oplus P_2^{m-1}$.*
(iii) *If $(0:_R S) = P_1^m \oplus P_2$ for some positive integer $m > 1$, then $(0:_R PS) = P_1^{m-1} \oplus P_2$.*

Proof. (i) Suppose not. We may assume that $n > 1$ and $m > 1$. Clearly, $0 \neq (P_1 \oplus 0)S \subseteq PS \neq S$, $0 \neq (0 \oplus P_2)S \subseteq PS \neq S$, and the submodules $(P_1 \oplus 0)S$ and $(0 \oplus P_2)S$ are P-primary submodules of S by Lemma 3.4. Since S is a primarily comultiplication R-module, we must have $(P_1 \oplus 0)S = (0:_S (0:_R P_1 \oplus 0)) = (0:_S P_1^{n-1} \oplus P_2)$ and $(0 \oplus P_2)S = (0:_S P_1 \oplus P_2^{m-1})$ by Lemma 3.4. Let $s_1 \in S_1$. There exists $s_2 \in S_2$ such that $(s_1, s_2) \in S$. It follows from Proposition 3.6 that $p_1^n s_1 = 0$ and $p_2^m s_2 = 0$. Therefore, $(p_1, p_2^{m-1})(p_1^{n-1}s_1, p_2 s_2) = 0$, so $(p_1^{n-1}s_1, p_2 s_2) \in (0:_S P_1 \oplus P_2^{m-1}) = (0 \oplus P_2)S$; hence $p_1^{n-1}s_1 = 0$. In a similar way, we get $p_1 s_1 = 0$. Therefore, $P_1 S_1 \cong (P_1 \oplus 0)S = 0$, which is a contradiction.

(ii) By Proposition 3.6, $(0:_{R_1} S_1) = P_1$ and $(0:_{R_2} S_2) = P_2^m$. Since $(P_1 \oplus P_2^{m-1})PS = 0$, we have $P_1 \oplus P_2^{m-1} \subseteq (0:_R PS)$. For the reverse inclusion, assume that $(r_1, r_2) \in (0:_R PS)$. Then $(r_1 p_1^2, r_2 p_2)S \subseteq (r_1, r_2)PS = 0$, so $r_1 p_1^2 \in P_1$ and $r_2 p_2 \in P_2^m$; hence $r_1 p_1 = u p_1^2$ and $r_2 p_2 \in w p_2^m$ for some $u \in R_1$ and $w \in R_2$. It follows that $r_1 \in P_1$ and $r_2 \in P_2^{m-1}$ since R_1 and R_2 are domains, and we have equality. The proof of (iii) is similar. ∎

PROPOSITION 3.8. *Let S be any primarily comultiplication separated module over the pullback ring as in* (3.1) *with $\bar{S} \neq 0$. Then $(0:_R S) \neq P_1^n \oplus 0$ and $(0:_R S) \neq 0 \oplus P_2^n$ for every positive integer n.*

Proof. Suppose $(0:_R S) = P_1^n \oplus 0$. If $(0 \oplus P_2)S = 0$, then $0 \oplus P_2 \subseteq P_1^n \oplus 0$, which is a contradiction. So $(0 \oplus P_2)S \neq 0$ and $(0:_R (0 \oplus P_2)S) \neq R$. Now we show that $(0:_R (0 \oplus P_2)S) = P_1 \oplus 0$. Since $(P_1 \oplus 0)(0 \oplus P_2)S = 0$, we have $P_1 \oplus 0 \subseteq (0:_R (0 \oplus P_2)S)$. For the reverse inclusion, assume that $(r_1, r_2) \in (0:_R (0 \oplus P_2)S)$. We may suppose that $(r_1, r_2) \in P$ since R is local. Then $r_1 \in P_1$ and $(0, r_2 p_2)S = 0$, so $r_2 p_2 = 0$; hence $r_2 = 0$ and $(r_1, r_2) \in P_1 \oplus 0$, and so we have equality. Moreover, by Lemma 3.4, $(0 \oplus P_2)S$ is a P-primary submodule of S, so $(0 \oplus P_2)S = (0:_S P_1 \oplus 0)$ since S is primarily comultiplication. We may assume that $n > 1$. Since $(P_1 \oplus 0)(P_1^{n-1} \oplus P_2)S = 0$, we must have $(P_1^{n-1} \oplus P_2)S \subseteq (0:_S P_1 \oplus 0) = (0 \oplus P_2)S$. Let $s_1 \in S_1$. Then there is an element $s_2 \in S_2$ such that $(s_1, s_2) \in S$. Hence $(p_1^{n-1}, p_2)(s_1, s_2) \in (0 \oplus P_2)S$; hence $P_1^{n-1}S_1 = 0$.

Therefore, $P_1^{n-1} \subseteq (0 :_{R_1} S_1) = P_1^n$ by Proposition 3.6, which is a contradiction. Thus $(0 :_R S) \neq P_1^n \oplus 0$ for every positive integer n. Similarly, $(0 :_R S) \neq 0 \oplus P_2^n$ for every positive integer n. ∎

REMARK 3.9. Let R be the pullback ring as in (3.1), and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Then $\mathrm{pSpec}(S) = \emptyset$ if and only if $\mathrm{pSpec}(S_i) = \emptyset$ for $i = 1, 2$.

Proof. For the necessity, assume that $\mathrm{pSpec}(S) = \emptyset$ and let π be the projection map of R onto R_i. Suppose $\mathrm{pSpec}(S_1) \neq \emptyset$ and let T_1 be a primary submodule of S_1, so T_1 is a primary R-submodule of $S_1 = S/(0 \oplus P_2)S$; hence $\mathrm{pSpec}(S) \neq \emptyset$, by Lemma 2.2, which is a contradiction. Similarly, $\mathrm{pSpec}(S_2) = \emptyset$. The sufficiency is clear by Proposition 3.3. ∎

THEOREM 3.10. *Let R be a pullback ring as in* (3.1), *and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module. Then S is a primarily comultiplication R-module if and only if each S_i is a primarily comultiplication R_i-module, $i = 1, 2$.*

Proof. Note that by Remark 3.9, $\mathrm{Spec}(S) = \emptyset$ if and only if $\mathrm{Spec}(S_i) = \emptyset$ for $i = 1, 2$. So we may assume that $\mathrm{Spec}(S) \neq \emptyset$. Assume that S is a separated primarily comultiplication R-module. If $\bar{S} = 0$, then by [7, Lemma 2.7], $S = S_1 \oplus S_2$; hence for each i, S_i is primarily comultiplication by Lemma 2.4. So we may assume that $\bar{S} \neq 0$.

Let L (resp. L') be a primary submodule of S_1 (resp S_2). Then there exists a separated submodule $T = (T/P_2S = T_1 \xrightarrow{g_1} \bar{T} = T/PT \xleftarrow{g_2} T_2 = T/P_1T)$ (resp. $T' = (T'/P_2T' = T'_1 \xrightarrow{g'_1} \bar{T}' = T'/PT' \xleftarrow{g'_2} T'_2 = T'/P_1S)$) of S, where g_i (resp. g'_i) is the restriction of f_i over T_i (resp. T'_i), $i = 1, 2$, such that $L = T_1$ (resp. $L' = T'_2$). Since T_1 (resp. T'_2) is a primary submodule of S_1 (resp. S_2), it follows that $T/(0 \oplus P_2)S$ (resp. $T'/(P_1 \oplus 0)S$) is a primary R-submodule of $S/(0 \oplus P_2)S$ (resp. $S/(P_1 \oplus 0)S$); hence T (resp. T') is a primary R-submodule of S by Lemma 2.2. We split the proof into two cases for $(0 : S)$ by Propositions 3.7 and 3.8.

CASE 1: $(0 : S) = P_1 \oplus P_2^m$ for some positive integer m. If $m = 1$, then by assumption, $T = (0 :_S P_1^k \oplus P_2^s)$ for some integers k, s; we show that $T_1 = (0 :_{S_1} P_1^k)$. Let $s_1 \in (0 :_{S_1} P_1^k)$. Then $P_1^k s_1 = 0$ and there exists $s_2 \in S_2$ such that $(s_1, s_2) \in S$, so $(P_1^k \oplus P_2^s)(s_1, s_2) = 0$; hence $(s_1, s_2) \in T$. Therefore, $(0 :_{S_1} P_1^n) \subseteq T_1$. Now suppose that $x \in T_1$. Then there is an element $y \in T_2$ such that $g_1(x) = g_2(y)$, so $(x, y) \in T$; hence $P_1^k x = 0$, and so we have equality. Similarly, S_2 is a primarily comultiplication R_2-module. So we may assume that $m > 1$. By Proposition 3.6, $(0 :_{R_1} S_1) = P_1$ and $(0 :_{R_2} S_2) = P_2^m$. Since $(P_1 \oplus 0)S \cong P_1 S_1 = 0$ and $(0 \oplus P_2)S \subseteq T$, we get

$PS \subseteq T \subseteq S$, so $(0 :_R S) \subseteq (0 :_R T) \subseteq (0 :_R PS)$; thus $P_1 \oplus P_2^m \subseteq (0 :_R T) \subseteq P_1 \oplus P_2^{m-1}$ by Proposition 3.7. Therefore, either $(0 :_R T) = P_1 \oplus P_2^m$ or $(0 :_R T) = P_1 \oplus P_2^{m-1}$. Since S is primarily comultiplication, we have either $T = (0 :_S P_1 \oplus P_2^m) = S$ or $T = (0 :_S P_1 \oplus P_2^{m-1}) = PS$; hence $T = PS$ and $T_1 = (PS)/PS = 0$. Then $L = T_1 = (0 :_{S_1} R_1)$ implies that S_1 is primarily comultiplication. Now we will prove S_2 is primarily comultiplication. By hypothesis, $T' = (0 :_S P_1^s \oplus P_2^t)$ for some positive integers s, t. We show that $T'_2 = (0 :_{S_2} P_2^m)$. Since the inclusion $T'_2 \subseteq (0 :_{S_2} P_2^m)$ is clear, we will prove the reverse inclusion. Let $s_2 \in (0 :_{S_2} P_2^m)$. Then $P_2^m s_2 = 0$ and there exists $s_1 \in S_1$ such that $(s_1, s_2) \in S$, so $(P_1^s \oplus P_2^t)(s_1, s_2) = 0$; hence $(s_1, s_2) \in T'$. Therefore, $s_2 \in T'_2$, and so we have equality.

Case 2: $(0 : S) = P_1^m \oplus P_2$ for some positive integer m. The proof is similar to that in Case 1.

Conversely, assume that S_1, S_2 are primarily comultiplication R_i-modules and let T be a non-zero primary submodule of S. If $(0 : S) = P_1 \oplus P_2^m$ for some positive integer m, then $(0 :_{R_1} S_1) = P_1$ and $(0 :_{R_2} S_2) = P_2^m)$, so there exist positive integers k, s such that $T_1 = (0 :_{S_1} P_1^k)$, $T_2 = (0 :_{S_2} P_2^s)$, and so $T = (0 :_S P_1^k \oplus P_2^s)$. Similarly we argue when $(0 :_R S) = P_1^m \oplus P_2$ for some positive integer m. ■

Lemma 3.11. *Let R be the pullback ring as in* (3.1). *The following separated R-modules are indecomposable and primarily comultiplication:*

(I) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2))$, *where $E(R_i/P_i)$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*
(II) $S = (Q(R_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow Q(R_2))$, *where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$;*
(III) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ *for all positive integers n, m.*

Proof. By [7, Lemma 2.8], these modules are indecomposable. Being primarily comultiplication follows from Theorems 2.8 and 3.10. ■

We refer to modules of type (3.1) in Lemma 3.2 as P_1-*Prüfer* and P_2-*Prüfer* respectively.

Theorem 3.12. *Let R be the pullback ring as in* (3.1), *and let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be an indecomposable separated primarily comultiplication R-module. Then S is isomorphic to one of the modules listed in Lemma* 3.11.

Proof. First suppose that $\mathrm{pSpec}(S) = \emptyset$. Then $\mathrm{pSpec}(S_i) = \emptyset$ by Remark 3.9, so $S_i = P_i S_i$ for each $i = 1, 2$ by Theorem 2.8; hence $S = PS = P_1 S_1 \oplus P_2 S_2 = S_1 \oplus S_2$. Therefore, $S = S_1$ or S_2 and so S is of type (I) in the list of Lemma 3.11 by Theorem 2.8. So we may assume that $\mathrm{pSpec}(S) \neq \emptyset$. Next suppose that $PS = S$. Then by [7, Lemma 2.7(i)], $S = S_1$ or S_2 and so S is an indecomposable primarily comultiplication R_i-module for some i, and since $PS = S$, it is type (II) by Theorem 2.8. So we may assume that $S/PS \neq 0$.

By Theorem 3.10, S_i is a primarily comultiplication R_i-module for each $i = 1, 2$ (note that for each i, S_i is torsion and it is not a divisible R_i-module by Theorem 2.8). Hence, by the structure of primarily comultiplication modules over a discrete valuation domain (see Theorem 2.9), $S_i = M_i \oplus N_i$ where N_i is a direct sum of copies of R_i/P_i^n ($n \geq 1$) and M_i is a direct sum of copies of $E(R_i/P_i)$ and $Q(R_i)$. Then $S = (N_1 \to \bar{S} \leftarrow N_2) \oplus (M_1 \to 0 \leftarrow 0)$ $\oplus\, (0 \to 0 \leftarrow M_2)$. Since S is indecomposable and $S/PS \neq 0$ it follows that $S = (N_1 \to \bar{S} \leftarrow N_2)$. We will see that each S_i $(= N_i)$ is indecomposable. Then there are positive integers m, n and k such that $P_1^m S_1 = 0$, $P_2^k S_2 = 0$ and $P^n S = 0$. For $s \in S$, let $o(s)$ denote the least positive integer m such that $P^m s = 0$. Now choose $s \in S_1 \cup S_2$ with $\bar{s} \neq 0$ and such that $o(t)$ is maximal. There exists an $s = (s_1, s_2)$ such that $o(s) = n$, $o(s_1) = m$ and $o(s_2) = k$. Then $R_i s_i$ is pure in S_i for $i = 1, 2$ (see [7, Theorem 2.9]). Therefore, $R_1 s_1 \cong R_1/P_1^m$ (resp. $R_2 s_2 \cong R_2/P_2^k$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i s_i$ is pure-injective. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{s}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 s_1 = M_1 \to \bar{M} \leftarrow M_2 = R_2 s_2)$. Then M is an R-submodule of S which is primarily comultiplication by Lemma 3.11 and is a direct summand of S; this implies that $S = M$, and S is as in (III) (see [7, Theorem 2.9]. ∎

Corollary 3.13. *Let R be the pullback ring as in* (3.1)*, and let S be a separated primarily comultiplication R-module. Then S is of the form $M \oplus N$, where M is a direct sum of copies of modules as in* (I)–(II) *and N is a direct sum of copies of modules as in* (III) *of Lemma* 3.11*. In particular, every separated primarily comultiplication R-module is pure-injective.*

Proof. Apply Theorem 3.12 and [7, Theorem 2.9]. ∎

4. The non-separated case. We continue to use the notation already established, so R is the pullback ring as in (3.1). In this section we find all indecomposable non-separated primarily comultiplication R-modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable primarily comultiplication modules.

Proposition 4.1. *Let R be a pullback ring as in* (3.1)*. Then $E(R/P)$ is a non-separated primarily comultiplication R-module.*

Proof. It suffices to show that $\mathrm{pSpec}(E(R/P)) = \emptyset$. Let L be any submodule of $E(R/P)$ described in [8, Proposition 3.1]. Since $E(R/P)$ is divisible, we must have $(L : E(R/P)) = 0$; hence $\mathrm{rad}(L : E(R/P)) = \mathrm{rad}(0) = (P_1 \oplus 0) \cap (0 \oplus P_2) \cap P = 0$. Set $P = R(p_1, p_2) = Rp$. Then no L, say $E_1 + A_n$, is a primary submodule of $E(R/P)$, for if m is any positive integer, then $p^m \notin \mathrm{rad}(L : E(R/P)) = 0$ and $x_1 + a_{n+m} \notin E_1 + A_n$ $(x_1 \in E_1)$, but $p^m(x_1 + a_{n+m}) = p_1^m x_1 + a_n \in E_1 + A_n$. Therefore, $E(R/P)$ is a non-separated primarily comultiplication R-module (see [7, p. 4053]). ∎

Proposition 4.2. *Let R be the pullback ring as in* (3.1)*, and let M be any primarily comultiplication R-module. Then the following hold:*

(i) *If M has a $P_1 \oplus 0$-primary submodule N, then M/N and M are separated.*
(ii) *If M has a $0 \oplus P_2$-primary submodule N, then M/N and M are separated.*

Proof. (i) First, we show that the $P_1 \oplus 0$-coprimary R-module M/N is separated. It is enough to show $(P_1 \oplus 0)(M/N) = 0$. As $(0, p_2)(p_1, 0)(m + N) = 0$ $(m \in M)$, we must have $(p_1, 0)m = 0$. Thus M/N is a separated R-module. Since M is primarily comultiplication, there are ideals I of R_1 and J of R_2 such that $N = (0 :_M I \oplus J)$, so $I \oplus J \subseteq (0 :_R N) \subseteq \mathrm{rad}(N : M) = P_1 \oplus 0$; hence $J = 0$ and $I = P_1^n$ for some n. It suffices to show that $(0 \oplus P_2)M = 0$. Suppose not. Clearly, $(0 \oplus P_2)M \subseteq N$. So by Lemma 2.2, N primary gives either $M \subseteq N$ or $0 \oplus P_2 \subseteq P_1 \oplus 0$, which is a contradiction. Therefore, M is separated. The proof of (ii) is similar. ∎

Lemma 4.3. *Let R be the pullback ring as in* (3.1) *and let M be any R-module. Let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then the following hold:*

(i) *For each positive integer n, $0 \to K \to P^n S \to P^n M \to 0$ is a separated representation of $P^n M$. In particular, $K \subseteq P^n S$.*
(ii) *If T is a primary submodule of S, then $K \subseteq T$.*

Proof. (i) Since $\varphi^{-1}(P^n M) = P^n S$, the results follows from [9, Lemma 3.1].

(ii) If $\mathrm{rad}(T : S) = P$, then (i) gives $K \subseteq P^n S \subseteq T$ since R is Noetherian. So suppose that $\mathrm{rad}(T : S) = P_1 \oplus 0$ and $K \nsubseteq T$; we show that $\mathrm{rad}(T : S) = \mathrm{rad}(T : K)$. Since the inclusion $\mathrm{rad}(T : S) \subseteq \mathrm{rad}(T : K)$ is clear, we will prove the reverse inclusion. Let $a \in \mathrm{rad}(T : K)$ and $x \in K - T$. Then $a^n x \in T$ for some n, so T primary gives $a \in \mathrm{rad}(T : S)$, and so we have equality. Since for each s, $P^s K = 0$ by [19, Proposition 2.4], we must have $P \subseteq \mathrm{rad}(T : K) = \mathrm{rad}(T : S) = P_1 \oplus 0$, which is a contradiction. Likewise, if $\mathrm{rad}(T : S) = 0 \oplus P_2$, then $K \subseteq T$. ∎

Proposition 4.4. *Let R be the pullback ring as in* (3.1) *and let M be any R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then* $\mathrm{pSpec}_R(S) = \emptyset$ *if and only if* $\mathrm{pSpec}_R(M) = \emptyset$.

Proof. First suppose that $\mathrm{pSpec}_R(S) = \emptyset$ and $\mathrm{pSpec}_R(M) \neq \emptyset$. So $M \cong S/K$ has a primary submodule, say T/K where T is a primary submodule of S by Lemma 2.2, which is a contradiction. Next suppose that $\mathrm{pSpec}_R(M) = \emptyset$ and $\mathrm{pSpec}_R(S) \neq \emptyset$. Let T be a primary submodule of S. Then by Lemma 4.3, $K \subseteq T$; hence T/K is a primary submodule of M, which is a contradiction. ∎

Lemma 4.5. *Let A be any ring, M and M' R-modules, and $f : M \to M'$ an A-homomorphism. Let N be a primary submodule of M' such that $f(M) \nsubseteq N$. Then $f^{-1}(N)$ is a primary submodule of M.*

Proof. The proof is straightforward. ∎

Theorem 4.6. *Let R be a pullback ring as in* (3.1) *and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is primarily comultiplication if and only if M is primarily comultiplication.*

Proof. By Proposition 4.4, we may assume that $\mathrm{Spec}(S) \neq \emptyset$. Suppose that M is a primarily comultiplication R-module and let T be a non-zero primary submodule of S. Then by Lemma 4.3, $K \subseteq T$, and so T/K is a primary submodule of S/K. By an argument like that in [6, Theorem 4.4], S is primarily comultiplication. Conversely, assume that S is a primarily comultiplication R-module and let N be a non-separated primary submodule of M. Then $\varphi^{-1}(N) = U$ is a primary submodule of S by Lemma 4.5, so $U = (0 :_S P_1^n \oplus P_2^m)$ for some integers m, n. By [9, Lemma 3.1], $U/K \cong N$ is a primary submodule of $S/K \cong M$, so an inspection will show that $N = U/K = (0 :_{S/K} P_1^n \oplus P_2^m)$, as required. ∎

Proposition 4.7. *Let R be a pullback ring as in* (3.1) *and let M be an indecomposable primarily comultiplication non-separated R-module with finite-dimensional top over $\bar{R}$. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pure-injective.*

Proof. By [7, Proposition 2.6(i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Theorem 4.6 and Corollary 3.13. ∎

Let R be a pullback ring as in (3.1) and let M be an indecomposable primarily comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 4.7, S is pure-injective. So in the proofs of [7, Lemma 3.1, Propositions 3.2 and 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [7, Proposition 2.6(ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable primarily comultiplication non-separated R-module", because the main keys to those results are the pure-injectivity of S, and the indecomposability and non-separability of M. So we have the following results:

Corollary 4.8. *Let R be a pullback ring as in* (3.1) *and let M be an indecomposable primarily comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then the quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summands of S.*

Corollary 4.9. *Let R be the pullback ring as in* (3.1) *and let M be an indecomposable primarily comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is a direct sum of finitely many indecomposable primarily comultiplication modules.*

Corollary 4.10. *Let R be the pullback ring as in* (3.1) *and let M be an indecomposable primarily comultiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then at most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Recall that every indecomposable R-module of finite length is primarily comultiplication (see Theorem 3.10 and Lemma 4.3). So by Corollary 4.10, the infinite length non-separated indecomposable primarily comultiplication modules are obtained in just the same way as the deleted cycle type indecomposable ones are, except that at least one of the two "end" modules must be a separated indecomposable primarily comultiplication of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one cannot have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules: we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer are direct summands of S then we will describe these modules as *doubly infinite*. Those where S has just one infinite length summand we will call *singly infinite* (the reader is referred to [7], [9] and [11] for more details). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable primarily comultiplication modules.

THEOREM 4.11. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the class of indecomposable non-separated primarily comultiplication modules with finite-dimensional top consists of the following:*

(i) *the indecomposable modules of finite length (except R/P which is separated),*
(ii) *the doubly infinite primarily comultiplication modules as described above,*
(iii) *the singly infinite primarily comultiplication modules as described above, except the two Prüfer modules* (I) *in Lemma* 3.11.

Proof. Let M be an indecomposable non-separated primarily comultiplication R-module with finite-dimensional top and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Clearly, M is a primarily comultiplication R-module. The indecomposability follows from [21, 1.9].

(ii) and (iii) (involving one or two Prüfer modules) M is primarily comultiplication (see Corollary 3.12 and Proposition 4.1). Finally, the indecomposability follows from [7, Theorem 3.5]. ■

COROLLARY 4.12. *Let R be the pullback ring as described in Theorem* 4.11. *Then every indecomposable primarily comultiplication R-module with finite-dimensional top is pure-injective.*

Proof. Apply [7, Theorem 3.5] and Theorem 4.11. ■

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