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ON EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF CAFFARELLI–KOHN–NIRENBERG TYPE EQUATIONS

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Abstract. We investigate the solvability of a singular equation of Caffarelli–Kohn–Nirenberg type having a *critical-like* nonlinearity with a sign-changing weight function. We shall examine how the properties of the Nehari manifold and the fibering maps affect the question of existence of positive solutions.

1. Introduction. In this paper we are concerned with the existence of *positive* solutions for a singular class of equations in $\mathbb{R}^N$,

$$(1.1)\qquad -\mathrm{div}(|x|^{-pa}|\nabla u|^{p-2}\nabla u) - \lambda h(x)|x|^{-p(1+a)}|u|^{p-2}u = Q(x)|x|^{-qb}|u|^{q-2}u,$$

where $\lambda > 0$ is a parameter, $1 < p < N$, $0 \le a < b < a+1 < N/p$ and $q = q(a,b,p) := Np/(N+p(b-a)-p)$. Here, $h \ge 0$ and Q are given functions on $\mathbb{R}^N$ with Q changing sign. Throughout this paper we always assume that $Q \in L^\infty(\mathbb{R}^N)$, and $\lim_{|x|\to\infty} Q(x) =: Q(\infty) < 0$. Further assumptions on h and Q will be formulated later. We note that the weight function $Q(x)$ on the right-hand side of (1.1) is assumed to change sign. In such a situation (and in the *subcritical case* for the Laplacian operator, i.e. $2 < q < 2^* := 2N/(N-2), N \ge 3$), the existence of two positive solutions for λ in a small right-neighborhood of the principal eigenvalue of ($-\Delta$, Dirichlet) was first proved by Alama and Tarantello in their pioneering paper [1] for the equation $-\Delta u - \lambda u = Q(x)|u|^{q-2}u$, in the case of a bounded domain $\Omega \subset \mathbb{R}^N$ under Dirichlet boundary condition. On the other hand, the case $\Omega = \mathbb{R}^N$ was considered in [11].

In our present problem (1.1), the exponent $q = q(a,b,p)$ defined above is a kind of *critical exponent*. In fact, when $p = 2 < N$ and $a = b = 0$ then $q = 2^* = 2N/(N-2)$, the well-known critical Sobolev exponent. We also note that, when $h(x) = 1$ and $a = 0$, the left-hand side of (1.1) is

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a perturbation of the p-Laplacian by the so-called L^p-Hardy potential (or the more common Hardy potential $\lambda/|x|^2$ in the case $p = 2$ of the usual Laplacian).

General problems like (1.1) are related to the interpolation inequalities proved by Caffarelli, Kohn and Nirenberg in [4] and have been studied by other authors, but mostly in the case of bounded domains or else when $a = 0$ (Hardy potential) or $p = 2$. In particular, we could mention the works [16, 14, 26, 5, 9, 25, 10, 15, 18] (for Ω bounded), [29, 13, 27] (when $\Omega = \mathbb{R}^N$), and their references. Regarding the Caffarelli–Kohn–Nirenberg inequalities *per se*, in addition to the original paper [4], we would refer the interested reader to the papers [6, 19].

Our main goal in the present work is to obtain existence of two positive solutions for (1.1), again when λ is in a suitable right-neighborhood of the principal eigenvalue of (1.1). In our approach we make use of the Nehari manifold and the fibering method for our equation combined with the concentration-compactness principle of P.-L. Lions [22]. To our knowledge, the Nehari/fibering approach was first applied by Drábek and Pohozaev in [12] (for more recent applications see e.g. [3, 8])

The range of the parameter λ in (1.1) will be determined by the principal eigenvalue of the nonlinear eigenvalue problem

$$-\mathrm{div}(|x|^{-pa}|\nabla u|^{p-2}\nabla u) - \lambda h(x)|x|^{-p(a+1)}|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

Given $r \in [1, \infty)$ and $c \geq 0$, we denote by $L^r_c(\mathbb{R}^N) := L^r(\mathbb{R}^N, |x|^{-rc}dx)$ the Banach space of measurable functions on $\mathbb{R}^N$ whose rth power is Lebesgue integrable with respect to the measure $|x|^{-rc}\,dx$, endowed with the norm

$$\|u\|_{L^r_c} := \Big(\int_{\mathbb{R}^N} |x|^{-rc}|u|^r\,dx\Big)^{1/r}.$$

Note that $L^r_c(\mathbb{R}^N)$ consists of those functions u such that $u/|x|^c \in L^r(\mathbb{R}^N)$.

We will need the *Caffarelli–Kohn–Nirenberg inequality* [4]

$$\hat{S}\Big(\int_{\mathbb{R}^N} |x|^{-qb}|u|^q\,dx\Big)^{p/q} \leq \int_{\mathbb{R}^N} |x|^{-pa}|\nabla u|^p\,dx, \tag{1.3}$$

which holds for $u \in C^\infty_c(\mathbb{R}^N)$ and where $1 < p < N$, $0 \leq a \leq b \leq a+1 < N/p$, $q = q(a, b, p) := Np/(N + p(b-a) - p)$ and $\hat{S} = \hat{S}(a, b, p) > 0$.

Let $D^{1,p}_a(\mathbb{R}^N)$ be the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,p}_a} := \Big(\int_{\mathbb{R}^N} |x|^{-pa}|\nabla u|^p\,dx\Big)^{1/p}$$

and let $L^p_b(\mathbb{R}^N)$ be the space defined above. In view of (1.3) the weighted Sobolev space $D^{1,p}_a(\mathbb{R}^N)$ is continuously embedded in the weighted Lebesgue space $L_b^q(\mathbb{R}^N)$. We make the following assumption on the coefficient $h(x)$ in (1.2), where we are denoting $p_0 = p_0(a,b,p) := p - p(b-a)$:

(H) $0 \not\equiv h \geq 0$ is such that $h \in L_{p_0}^{N/p_0}(\mathbb{R}^N) \cap L_{\mathrm{loc}}^{N/p_0+\theta}(\mathbb{R}^N \setminus \{0\})$ for some $\theta > 0$, i.e.,

$$\int_{\mathbb{R}^N} \frac{|h(x)|^{N/p_0}}{|x|^N}\,dx < \infty, \quad \int_B |h(x)|^{N/p_0+\theta}\,dx < \infty \ \forall \text{ ball } \overline{B} \subset \mathbb{R}^N \setminus \{0\}.$$

Remark. We note that the hypothesis (H) is satisfied if $0 \leq h \in L^{N/p_0}$ ($h \not\equiv 0$) is continuous and such that $h(x) = O(|x|^s)$ as $|x| \to 0$ for some $s > 0$.

Proposition 1.1. *Suppose* (H) *holds. Then the nonlinear eigenvalue problem* (1.2) *has a principal eigenvalue* $\lambda_1(h) > 0$ *which is simple. Moreover, a corresponding eigenfunction* φ_1 *belongs to the space* $D_a^{1,p}(\mathbb{R}^N)$ *and can be taken to be positive in the sense that* $\varphi_1 > 0$ *a.e. in* $\mathbb{R}^N \setminus \{0\}$.

Proof. For simplicity of notation, from now on we will omit writing $\mathbb{R}^N$ in the pertinent spaces and integrals. For each fixed $u \in D_a^{1,p}$, consider the linear functional $K(u)$ defined by the formula

$$\langle K(u), \phi\rangle = \int h(x)|x|^{-p(a+1)}|u|^{p-2}u\phi\,dx \qquad \forall \phi \in D_a^{1,p}.$$

First of all, we must show that $K(u)$ is well-defined on $D_a^{1,p}$. Indeed, the continuous embedding of $D_a^{1,p}$ into L_b^q implies that

$$\beta_u(x) := |x|^{-b(p-1)}|u|^{p-2}u \in L^{q/(p-1)} \quad \text{and} \quad \gamma_\phi(x) := |x|^{-b}\phi \in L^q. \tag{1.4}$$

Therefore, writing the integrand of $\langle K(u), \phi\rangle$ as

$$h(x)|x|^{-p(a+1)}|u|^{p-2}u\phi = h(x)|x|^{p(b-a)-p}\beta_u(x)\gamma_\phi(x) := \alpha_h(x)\beta_u(x)\gamma_\phi(x)$$

and noticing that $\beta_u\gamma_\phi \in L^{q/p}$, we conclude by the Hölder inequality that $K(u)$ is well-defined on $D_a^{1,p}$ provided

$$\alpha_h(x) := h(x)|x|^{p(b-a)-p} \in L^r \quad \text{with} \quad r = \frac{q}{q-p} = \frac{N}{p-p(b-a)} = \frac{N}{p_0}. \tag{1.5}$$

But this holds true in view of the first integrability condition in (H), and we obtain the following estimate, for some $\widehat{C} = \widehat{C}(a,b,p) > 0$:

$$|\langle K(u), \phi\rangle| \leq C\|u\|_{L_b^q}\|\phi\|_{L_b^q} \leq \widehat{C}\|u\|_{D_a^{1,p}}\|\phi\|_{D_a^{1,p}} \quad \forall u, \phi \in D_a^{1,p}. \tag{1.6}$$

Next, we show that the mapping $u \mapsto K(u)$ from $D_a^{1,p}$ into $(D_a^{1,p})^*$ is compact. For that, let (u_m) be a weakly convergent sequence in $D_a^{1,p}$, say $u_m \rightharpoonup \hat{u} \in D_a^{1,p}$. Passing to a subsequence if necessary, we must show that

$$\langle K(u_m), \phi\rangle \to \langle K(\hat{u}), \phi\rangle$$

uniformly for $\|\phi\|_{D_a^{1,p}}$ bounded, say $\|\phi\|_{D_a^{1,p}} \leq 1$. Let us write

$$|\langle K(u_m), \phi\rangle - \langle K(\hat{u}), \phi\rangle| \leq \Big(\int\limits_{|x|<\delta} + \int\limits_{\delta\leq|x|\leq R} + \int\limits_{|x|>R} \Big) |\alpha_h(x)|\,|\beta_{u_m}(x) - \beta_{\hat{u}}(x)|\,|\gamma_\phi(x)| := [I] + [II] + [III],$$

where some large $R > 0$ and small $\delta > 0$ are chosen so that, for given $\varepsilon > 0$,

$$\int\limits_{|x|>R} \frac{|h(x)|^{N/p_0}}{|x|^N}\, dx \leq \frac{\varepsilon}{6\widehat{C}(\sup \|u_m\|_{D_a^{1,p}})}$$

and

$$\int\limits_{|x|<\delta} \frac{|h(x)|^{N/p_0}}{|x|^N}\, dx \leq \frac{\varepsilon}{6\widehat{C}(\sup \|u_m\|_{D_a^{1,p}})}$$

and, hence,

$$[III] \leq \varepsilon/3 \quad \text{and} \quad [I] \leq \varepsilon/3. \tag{1.7}$$

Next, note that one has the continuous embeddings

$$D_a^{1,p}(\mathbb{R}^N) \subset W_a^{1,p}(B_R \setminus B_\delta) \subset L^{q_1}(B_R \setminus B_\delta)$$

for all $2 \leq q_1 \leq p^* := Np/(N-p)$, with the last inclusion being compact if $q_1 < p^*$. Then, since $u_m \rightharpoonup \hat{u}$ weakly in $D_a^{1,p}$, we have (passing to a subsequence if necessary)

$$u_m \to \hat{u} \quad \text{a.e. in } \mathbb{R}^N,$$
$$u_m \to \hat{u} \quad \text{strongly in } L^{q_1}(B_R \setminus B_\delta),$$

if $2 \leq q_1 < p^*$. And since we have assumed that $h \in L_{\text{loc}}^{N/p_0+\theta}(\mathbb{R}^N \setminus \{0\})$ for some $\theta > 0$, it follows that $\alpha_h \in L^{r_1}(B_R \setminus B_\delta)$ with $r_1 > r = N/p_0$ (see (1.5)). Therefore, if we define $q_1 := pr_1/(r_1 - 1)$ (note that $q_1 < q$), recall the definitions of β_{u_m} and β_u in (1.4), and use Hölder's inequality as before, we infer that

$$\beta_{u_m} \to \beta_u \quad \text{strongly in } L^{q_1/(p-1)}(B_R \setminus B_\delta)$$

with $q_1 < q \leq p^*$, and hence

$$[II] = \int\limits_{\delta\leq|x|\leq R} |\alpha(x)|\,|\beta_{u_m}(x) - \beta_{\hat{u}}(x)|\,|\gamma_\phi(x)|\, dx \leq \varepsilon/3$$

for all m large, uniformly for $\|\phi\|_{D_a^{1,p}} \leq 1$. Combining the above estimate with the ones in (1.7) we conclude that

$$\langle K(u_m), \phi\rangle \to \langle K(\hat{u}), \phi\rangle$$

uniformly for $\|\phi\|_{D_a^{1,p}} \leq 1$, in other words, the mapping $K : D_a^{1,p} \to (D_a^{1,p})^*$

is compact. In particular, the function

$$\langle K(u), u\rangle = \int_{\mathbb{R}^N} h(x)|x|^{-p(a+1)}|u|^p\,dx, \qquad u \in D_a^{1,p},$$

is completely continuous and the principal eigenvalue $\lambda_1(h) > 0$ is defined by the formula

$$\frac{1}{\lambda_1(h)} = \sup_{u\in D_a^{1,p}} \frac{\int |x|^{-p(a+1)}h(x)|u|^p\,dx}{\int |x|^{-pa}|\nabla u|^p\,dx}.$$

Next, we will show as a consequence of the work in [17] (see also [28, 24]) that $\lambda_1(h)$ is simple and possesses a corresponding eigenfunction $\varphi_1(x)$, with $\|\varphi_1\|_{D_a^{1,p}} = 1$, and such that $\varphi_1 > 0$ a.e. in $\mathbb{R}^N\setminus\{0\}$. We point out that, when $a = b = 0$, $|x|^{-p}h(x) := w(x)$ is a bounded function and one is dealing with a bounded domain, the simplicity of the principal eigenvalue and constant sign of a corresponding eigenfunction are well-known facts dating back to Anane [2] and Lindqvist [20, 21]. We refer the interested reader to the already cited work [17] of Kawohl–Lucia–Prashanth (and references therein), where a comprehensive study is done on simplicity of the principal eigenvalue for a large class of quasilinear problems.

In our present case, where the positive weight $|x|^{-pa}$ is degenerate and unbounded on $\mathbb{R}^N\setminus\{0\}$, we will make an adaptation of the results in [17]. To start, note that we may assume that any $\lambda_1(h)$-eigenfunction φ_1 is nonnegative by replacing φ_1 with $|\varphi_1|$. Then we use the maximum principle given by Proposition 3.2 in [17] for the differential inequality

$$(1.8)\qquad -\mathrm{div}(a(x,\nabla u)) + V(x)|u|^{q-2}u \ge 0, \quad u \in W^{1,p}_{\mathrm{loc}}(\Omega),$$

where $V \in L^1_{\mathrm{loc}}(\Omega)$, $V \ge 0$, $q \ge p > 1$, and $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying (for some $\alpha > 1$)

$$\langle a(x,\eta),\eta\rangle \ge \frac{1}{\alpha}\,|\eta|^p \qquad \text{a.e. } x\in\Omega,\ \forall\eta\in\mathbb{R}^N,$$
$$|a(x,\eta)| \le \alpha|\eta|^{p-1} \quad \text{a.e. } x\in\Omega,\ \forall\eta\in\mathbb{R}^N.$$

For our situation, we let $\Omega = \Omega_R := \{x\in\mathbb{R}^N \mid 1/R < |x| < R\}$ (with $R > 1$ fixed), consider the differential inequality

$$(1.9)\quad -\mathrm{div}(|x|^{-pa}|\nabla\varphi_1|^{p-2}\nabla\varphi_1) = \lambda_1 h(x)|x|^{-p(a+1)}|\varphi_1|^{p-2}\varphi_1 \ge 0 \quad \text{in } \Omega_R,$$

and use Proposition 3.2 from [17] to conclude that the set $\mathcal{Z}$ of zeros of φ_1 has $W^{1,p}$-capacity zero.

Finally, the simplicity of $\lambda_1(h)$ (i.e., the fact that the solutions of (1.9) form a 1-dimensional space) follows from arguments in [23], exactly as in Section 6.2 of [17]. The proof of Proposition 1.1 is now complete, since a solution φ_1 of (1.2) is also a solution of (1.9) for any $R > 0$. ■

REMARK. Clearly, by the above proposition and through the Krasnosel'skiĭ *genus*, the nonlinear eigenvalue problem (1.2) has a sequence of eigenvalues $0 < \lambda_1(h) < \lambda_2(h) \le \cdots \to +\infty$ (and, if $h(x)$ changes sign, there also exists a corresponding sequence of negative eigenvalues). Since we are concerned with *positive* solutions, the parameter $\lambda > 0$ will not be interacting with eigenvalues higher than $\lambda_1(h)$.

2. The singular problem. We now consider our singular problem (1.1) mentioned in the Introduction:

$$-\operatorname{div}(|x|^{-pa}|\nabla u|^{p-2}\nabla u) - \lambda h(x)|x|^{-p(1+a)}|u|^{p-2}u = Q(x)|x|^{-qb}|u|^{q-2}u, \tag{2.1}$$

where $\lambda > 0$ is a parameter, $1 < p < N$, $0 \le a < b < a+1 < N/p$ and $q = q(a,b,p) := Np/(N + p(b-a) - p)$. As before, we assume that the weight function $h(x)$ satisfies condition (H) introduced in the previous section, namely

(H) $0 \not\equiv h \ge 0$ is such that $h \in L_{p_0}^{N/p_0}(\mathbb{R}^N) \cap L_{\mathrm{loc}}^{N/p_0+\theta}(\mathbb{R}^N \setminus \{0\})$ for some $\theta > 0$,

and we shall make the following assumption on the coefficient $Q(x)$:

(Q) $Q \in C(\mathbb{R}^N)$ changes sign, $Q(0) \le 0$, and $\lim_{|x|\to\infty} Q(x) =: Q(\infty) < 0$.

Under these hypotheses, solutions of problem (2.1) will be obtained as critical points of the functional

$$J_\lambda(u) = \frac{1}{p}\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx - \frac{1}{q}\int |x|^{-qb}Q(x)|u|^q\,dx,$$

which is of class C^1 on $E := D_a^{1,p}$ and it is not bounded from below on E (we recall that we will be dropping $\mathbb{R}^N$ when writing the pertinent integrals and spaces). In addition, any solution $u \in E$ of (2.1) belongs to the so-called *Nehari manifold* (1)

$$S(\lambda) = \Big\{u \in E \,\Big|\, \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx = \int |x|^{-qb}Q(x)|u|^q\,dx\Big\}.$$

We shall follow some ideas from the papers [3, 8, 12]. With each $u \in E\setminus\{0\}$ we associate the *fibering map* $\varphi_u(t)$ defined by $\varphi_u(t) = J_\lambda(tu)$, $0 \le t < \infty$. The three results that follow are basic as they relate $S(\lambda)$ and critical points of J_λ. In particular, Lemma 2.3 says that "most" local minimizers of J_λ on $S(\lambda)$ are critical points of J_λ.

(1) In fact, it can be shown that $S(\lambda) \setminus S^\circ(\lambda)$ is indeed a C^1-submanifold of E of codimension 1, where $S^\circ(\lambda)$ is a "meager" subset of $S(\lambda)$ to be defined below.

LEMMA 2.1. *If $u \in E$ is a local minimizer of J_λ, then $\varphi_u(t)$ has a local minimum at $t = 1$. If $u \in E \setminus \{0\}$ and $tu \in S(\lambda)$ for some $t > 0$, then $\varphi_u'(t) = 0$.*

Therefore, elements in $S(\lambda)$ are stationary points of the maps $\varphi_u(t)$. This leads us to the decomposition of $S(\lambda)$ into three subsets:

$$S^+(\lambda) = \Big\{u \in S(\lambda) \,\Big|\, (p-1)\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx - (q-1)\int |x|^{-qb}Q(x)|u|^q\,dx > 0\Big\},$$

$$S^-(\lambda) = \Big\{u \in S(\lambda) \,\Big|\, (p-1)\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx - (q-1)\int |x|^{-qb}Q(x)|u|^q\,dx < 0\Big\},$$

$$S^\circ(\lambda) = \Big\{u \in S(\lambda) \,\Big|\, (p-1)\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx - (q-1)\int |x|^{-qb}Q(x)|u|^q\,dx = 0\Big\}.$$

This partition of $S(\lambda)$ corresponds to local minima, local maxima and inflection points of the fibering maps $\varphi_u(t)$. Therefore we have

LEMMA 2.2. *If $u \in S(\lambda)$ then $\varphi_u'(1) = 0$. Moreover,*

(i) *if $\varphi_u''(1) > 0$, then $u \in S^+(\lambda)$,*
(ii) *if $\varphi_u''(1) < 0$, then $u \in S^-(\lambda)$,*
(iii) *if $\varphi_u''(1) = 0$, then $u \in S^\circ(\lambda)$.*

LEMMA 2.3. *If $u_\circ$ is a critical point of $J_\lambda|S(\lambda)$ (in particular, a local minimizer on $S(\lambda)$) such that $u_\circ \notin S^\circ(\lambda)$, then $J_\lambda'(u_\circ) = 0$.*

For the proof of these lemmas we refer to [3].

Now, as recalled earlier, the principal eigenvalue of (1.2) is given by

$$\frac{1}{\lambda_1(h)} = \sup_{u\in E} \frac{\int |x|^{-p(a+1)}h(x)|u|^p\,dx}{\int |x|^{-pa}|\nabla u|^p\,dx},$$

so that

$$\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx > 0$$

for every $u \in E \setminus \{0\} := D_a^{1,p} \setminus \{0\}$ and $0 < \lambda < \lambda_1(h)$. In fact, a standard argument shows that, for every $0 \le \lambda < \lambda_1(h)$, there exists $\delta(\lambda) > 0$ such that

$$\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx \ge \delta(\lambda)\int |x|^{-pa}|\nabla u|^p\,dx \tag{2.2}$$

for all $u \in E$. Next we observe that if $u \in S(\lambda)$, then

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{q}\right) \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\, dx$$
$$= \left(\frac{1}{p} - \frac{1}{q}\right) \int |x|^{-qb} Q(x)|u|^q\, dx.$$

We also derive the following characterizations of $S^+(\lambda)$, $S^-(\lambda)$ and $S^\circ(\lambda)$:

$$S^+(\lambda) = \left\{u \in S(\lambda) \,\middle|\, \int |x|^{-qb}Q(x)|u|^q\, dx < 0\right\},$$

$$S^-(\lambda) = \left\{u \in S(\lambda) \,\middle|\, \int |x|^{-qb}Q(x)|u|^q\, dx > 0\right\},$$

$$S^\circ(\lambda) = \left\{u \in S(\lambda) \,\middle|\, \int |x|^{-qb}Q(x)|u|^q\, dx = 0\right\}.$$

Now, if for any given $u \in E \setminus \{0\}$ we denote $B(u) := \int |x|^{-qb}Q(x)|u|^q\, dx$ and $A_\lambda(u) := \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\, dx$, then it is easy to see that $\varphi_u(t)$ has exactly one stationary point in $(0, \infty)$ given by

$$t(u) = \left(\frac{A_\lambda(u)}{B(u)}\right)^{1/(q-p)}$$

provided that $A_\lambda(u)B(u) > 0$. By contrast, $\varphi_u(t)$ has no stationary point in $(0, \infty)$ if $A_\lambda(u)B(u) < 0$.

We also need the following sets (cf. [3, 8]) in order to better characterize the stationary points of $\varphi_u(t)$:

$$L^+(\lambda) = \left\{u \in E \,\middle|\, \|u\|_E = 1, \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\, dx > 0\right\},$$

$$L^-(\lambda) = \left\{u \in E \,\middle|\, \|u\|_E = 1, \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\, dx < 0\right\},$$

$$L^\circ(\lambda) = \left\{u \in E \,\middle|\, \|u\|_E = 1, \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\, dx = 0\right\},$$

and

$$B^+ = \left\{u \in E \,\middle|\, \|u\|_E = 1, \int |x|^{-bq}Q(x)|u|^q\, dx > 0\right\},$$

$$B^- = \left\{u \in E \,\middle|\, \|u\|_E = 1, \int |x|^{-bq}Q(x)|u|^q\, dx < 0\right\},$$

$$B^\circ = \left\{u \in E \,\middle|\, \|u\|_E = 1, \int |x|^{-bq}Q(x)|u|^q\, dx = 0\right\}.$$

Then, by looking at the behavior of $\varphi_u(t)$ for small $t > 0$ and for $t \to \infty$ we get the following characterization of the stationary points of $\varphi_u(t)$ (where

$\mathbb{R}^+u := \{tu \mid t > 0\}$ denotes the positive ray through u):

(a) $S^-(\lambda) \cap \mathbb{R}^+u \neq \emptyset$ if and only if $u/\|u\|_E \in L^+(\lambda) \cap B^+$;
(b) $S^+(\lambda) \cap \mathbb{R}^+u \neq \emptyset$ if and only if $u/\|u\|_E \in L^-(\lambda) \cap B^-$;
(c) $S(\lambda) \cap \mathbb{R}^+u = \emptyset$ whenever $u/\|u\|_E \in L^+(\lambda) \cap B^-$ or $u/\|u\|_E \in L^-(\lambda) \cap B^+$.

Finally, we need the following version of the concentration-compactness principle (see [22, 7]):

Concentration-Compactness Principle. Let $1 < p < N$, $0 \le a < b < a+1 < N/p$ and let (u_m) be a sequence in $E := D_a^{1,p}$ such that

$$\begin{aligned} u_m(x) &\to u(x) && \text{a.e. in } \mathbb{R}^N, \\ u_m &\rightharpoonup u && \text{in } D_a^{1,p}, \\ |x|^{-a}|\nabla(u_m - u)|^p &\rightharpoonup \mu && \text{in } \mathcal{M}(\mathbb{R}^N), \\ |x|^{-b}|u_m - u|^q &\rightharpoonup \nu && \text{in } \mathcal{M}(\mathbb{R}^N), \end{aligned}$$

where $\mathcal{M}(\mathbb{R}^N)$ denotes the space of bounded measures in $\mathbb{R}^N$. Define the quantities (measuring loss of mass at infinity of weakly convergent sequences in E):

$$\mu_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int\limits_{|x|>R} |x|^{-pa}|\nabla u_m|^p\,dx, \tag{2.3}$$

$$\nu_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int\limits_{|x|>R} |x|^{-qb}|u_m|^q\,dx. \tag{2.4}$$

Then it follows (with $\hat{S} := \hat{S}(a,b,p)$ defined in (1.3)) that

$$\hat{S}\|\nu\|^{p/q} \le \|\mu\|, \qquad \hat{S}\nu_\infty^{p/q} \le \mu_\infty$$

and

$$\limsup_{m\to\infty} \|\, |x|^{-a}\nabla u_m\|_E^p = \|\, |x|^{-a}\nabla u\|_E^p + \|\mu\| + \mu_\infty, \tag{2.5}$$

$$\limsup_{m\to\infty} \|\, |x|^{-b}u_m\|_{L^q}^q = \|\, |x|^{-b}u\|_{L^q}^q + \|\nu\| + \nu_\infty \tag{2.6}$$

(see [7]). Since $a < b$ we have $q < p^*$ and the measures μ and ν are concentrated at 0. Therefore, (2.5) and (2.6) take the form

$$\limsup_{m\to\infty} \|\, |x|^{-a}\nabla u_m\|_E^p = \|\, |x|^{-a}\nabla u\|_E^p + \mu_0 + \mu_\infty, \tag{2.7}$$

$$\limsup_{m\to\infty} \|\, |x|^{-b}u_m\|_{L^q}^q = \|\, |x|^{-b}u\|_{L^q}^q + \nu_0 + \nu_\infty, \tag{2.8}$$

where $\mu_0 > 0$ and $\nu_0 > 0$ are constants satisfying

$$\hat{S}\nu_0^{p/q} \le \mu_0.$$

3. The case $0 < \lambda < \lambda_1(h)$. In this section we show the existence of a minimizer of J_λ on $S^-(\lambda)$. In this case inequality (2.2) implies that $L^-(\lambda)$ and $L^\circ(\lambda)$ are empty and hence $S^+(\lambda)$ is also empty and $S^\circ(\lambda) = \{0\}$.

PROPOSITION 3.1. *Assume* (H), (Q), *and* $0 < \lambda < \lambda_1(h)$. *Then*

(i) $\inf_{S^-(\lambda)} J_\lambda > 0$,
(ii) *there exists* $u \in S^-(\lambda)$ *such that* $J_\lambda(u) = \inf_{S^-(\lambda)} J_\lambda$.

Proof. Clearly $\inf_{S^-(\lambda)} J_\lambda \geq 0$. We claim that $\inf_{S^-(\lambda)} J_\lambda > 0$. Indeed, if $u \in S^-(\lambda)$, then $v = u/\|u\|_E \in L^+(\lambda) \cap B^+(\lambda)$ and $u = t(v)v$ with

$$t(v) = \left(\frac{\int (|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\, dx}{\int |x|^{-qb}Q(x)|v|^q\, dx} \right)^{1/(q-p)}.$$

We then have

$$\begin{aligned} J_\lambda(u) = J_\lambda(t(v)v) &= \left(\frac{1}{p} - \frac{1}{q}\right) t(v)^p \int (|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\, dx \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \frac{\left(\int (|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\, dx\right)^{q/(q-p)}}{\left(\int |x|^{-qb}Q(x)|v|^q\, dx\right)^{p/(q-p)}} \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \frac{\delta(\lambda)^{q/(q-p)}}{\left(\int |x|^{-qb}Q(x)|v|^q\, dx\right)^{p/(q-p)}}. \end{aligned}$$

On the other hand, in view of the (C-K-N) inequality (1.3) we estimate the integral appearing in the above denominator as

$$\begin{aligned} \int |x|^{-qb}Q(x)|v|^q\, dx &\leq \|Q\|_{L^\infty} \int |x|^{-qb}|v|^q\, dx \\ &\leq \hat{S}^{-q/p}\, \|Q\|_{L^\infty} \left(\int |x|^{-pa}|\nabla v|^p\, dx\right)^{q/p} \\ &= \hat{S}^{-q/p}\|Q\|_{L^\infty}\|v\|_E^q = \hat{S}^{-q/p}\|Q\|_{L^\infty}. \end{aligned}$$

Assertion (i) follows from the last two estimates.

Next, set $A = \inf_{S^-(\lambda)} J_\lambda$ and let $(u_m) \subset S^-(\lambda)$ be a minimizing sequence for A. Then (u_m) is bounded in E, so that we may assume that $u_m \rightharpoonup u$ in E. In addition, the (C-K-N) inequality (1.3) shows that the sequence of integrals $\int |x|^{-qb}Q(x)|u_m|^q\, dx$ is also bounded.

On the other hand, in view of the concentration-compactness principle, and since

$$\int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\, dx \quad \text{and} \quad \int |x|^{-qb}Q(x)|u_m|^q\, dx$$

converge to the same limit (as $u_m \in S(\lambda)$), we have

$$\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx + \mu_0 + \mu_\infty \leq \int |x|^{-qb}Q(x)|u|^q\,dx + Q(0)\nu_0 + Q(\infty)\nu_\infty.$$

If $u \equiv 0$ on $\mathbb{R}^N$ it follows that

$$\mu_0 + \mu_\infty \leq Q(0)\nu_0 + Q(\infty)\nu_\infty,$$

hence $\mu_0 = \mu_\infty = 0$ since $Q(0) \leq 0$ and $Q(\infty) < 0$ by (Q). It follows that $u_m \to 0$ in E, which is impossible. Therefore, we must have $u \not\equiv 0$ on $\mathbb{R}^N$.

We now claim that $\mu_\infty = 0$. Otherwise, we have

$$0 < \int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx < \int |x|^{-qb}Q(x)|u|^q\,dx,$$

hence

$$\int (|x|^{-pa}|\nabla (su)|^p - \lambda h(x)|x|^{-p(a+1)}|su|^p)\,dx = \int |x|^{-qb}Q(x)|su|^q\,dx$$

for some $0 < s < 1$. This implies that $su \in S^-(\lambda)$ and, since we can assume that $\int h(x)|x|^{-p(a+1)}|u_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|u|^p\,dx$, we deduce that

$$\begin{aligned}\int (|x|^{-pa}|\nabla u|^p &- \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx \\ &\leq \liminf_{m\to\infty} \int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx \\ &= \frac{pq}{q-p}A \leq \int (|x|^{-pa}|\nabla (su)|^p - \lambda h(x)|x|^{-p(a+1)}|su|^p)\,dx,\end{aligned}$$

which yields the contradiction $s \geq 1$. Thus $\mu_\infty = 0$ and a similar argument also shows that $\mu_0 = 0$. Consequently, we conclude that $u_m \to u$ in E and

$$J_\lambda(u) = A = \inf_{S^-(\lambda)} J_\lambda.$$

Since $\int |x|^{-qb}Q(x)|u|^q\,dx > 0$, it is clear that $u \notin S^\circ(\lambda)$ so that, by Lemma 2.3, u is a critical point of J_λ. Finally, since $J_\lambda(|u|) = J_\lambda(u)$, we may assume by the maximum principle that $u > 0$ on $\mathbb{R}^N$. ∎

Next, we examine the behavior of $\inf_{S^-(\lambda)} J_\lambda$ when $\lambda \to \lambda_1(h)^-$. In Proposition 3.2 below we assume that $\int |x|^{-qb}Q(x)\varphi_1^q\,dx > 0$. We note that the principal eigenfunction φ_1 automatically belongs to $L_b^q(\mathbb{R}^N)$ in view of (C-K-N) and the fact that $\varphi_1 \in D_a^{1,p}$ by Proposition 1.1. Since $Q(\infty) < 0$, this "positivity" condition on the above integral guarantees that "most" of the L_b^q-norm of φ_1 lies in the region $\{x \mid Q(x) > 0\}$.

PROPOSITION 3.2. *Assume* (H), (Q) *and let* $\int |x|^{-qb}Q(x)\varphi_1^q\,dx > 0$. *Then*

(i) $\lim_{\lambda\to\lambda_1(h)^-}(\inf_{S^-(\lambda)} J_\lambda) = 0$,
(ii) *if* $\lambda_m \to \lambda_1(h)^-$ *and* u_m *minimizes* J_{λ_m} *on* $S^-(\lambda)$ *then* $u_m \to 0$ *in* E.

Proof. (i) Since $0 < \lambda < \lambda_1(h)$, we have $\varphi_1 \in L^+(\lambda)\cap B^+$ and $J_\lambda(t(\varphi_1)\varphi_1) \to 0$ as $\lambda \to \lambda_1(h)^-$. Thus, assertion (i) follows.

(ii) First we show that (u_m) is bounded in E. Arguing by contradiction, we assume (up to a subsequence) that $\|u_m\| \to \infty$ and set $v_m = u_m/\|u_m\|_E$. Then, again up to a subsequence, we may assume that

$$v_m \to v \quad \text{a.e. in } \mathbb{R}^N,$$
$$v_m \rightharpoonup v \quad \text{in } E,$$
$$\int h(x)|x|^{-p(a+1)}|v_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|v|^p\,dx.$$

Since

$$\frac{J_\lambda(u_m)}{\|u_m\|_E^p} = \left(\frac{1}{p} - \frac{1}{q}\right)\int (|x|^{-pa}|\nabla v_m|^p - \lambda_m h(x)|x|^{-p(a+1)}|v_m|^p)\,dx$$
$$= \left(\frac{1}{p} - \frac{1}{q}\right)\|u_m\|_E^{q-p}\int |x|^{-qb}Q(x)|v_m|^q\,dx,$$

we see that

$$\lim_{m\to\infty}\int (|x|^{-pa}|\nabla v_m|^p - \lambda_m h(x)|x|^{-p(a+1)}|v_m|^p)\,dx$$
$$= \lim_{m\to\infty}\int |x|^{-qb}Q(x)|v_m|^q\,dx = 0.$$

Now, if $v_m \not\to v$ in E we obtain

$$\int (|x|^{-pa}|\nabla v|^p - \lambda_1(h)h(x)|x|^{-p(a+1)}|v|^p)\,dx < \lim_{m\to\infty}\int |x|^{-qb}Q(x)|v_m|^q\,dx = 0,$$

which is impossible. Therefore, it follows that $v_m \to v = k\varphi_1$ for some $k \in \mathbb{R}$. On the other hand, since $S(\lambda)\setminus S^\circ(\lambda)$ is a natural constraint for J_λ, we have $J'_{\lambda_m}(u_m) = 0$. Therefore,

$$\int (|x|^{-pa}|\nabla v_m|^{p-2}\nabla v_m\cdot\nabla\phi - \lambda_m h(x)|x|^{-p(a+1)}|v_m|^{p-2}v_m\phi)\,dx$$
$$= \|u_m\|_E^{q-p}\int |x|^{-qb}Q(x)|v_m|^{q-2}v_m\phi\,dx$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Letting $m \to \infty$ we get

$$|k|^{q-2}k\int |x|^{-qb}Q(x)\varphi_1^{q-2}\varphi_1\phi\,dx = 0$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. If $k \neq 0$ then $\varphi_1 = 0$ on the set $\{x \mid Q(x) > 0\} \cup \{x \mid Q(x) < 0\}$, which is impossible since $\varphi_1 > 0$ on $\mathbb{R}^N \setminus \{0\}$. Therefore, we conclude that $v_m \to 0$ in E, which contradicts the fact that $\|v_m\|_E = 1$ for all $m \geq 1$.

Consequently, (u_m) is bounded in E and we may assume that

$$\begin{gathered} u_m \to u \quad \text{a.e. in } \mathbb{R}^N, \\ u_m \rightharpoonup u \quad \text{in } E, \\ \int h(x)|x|^{-p(a+1)}|u_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|u|^p\,dx. \end{gathered}$$

If

$$\int |x|^{-pa}|\nabla u|^p\,dx < \lim_{m\to\infty} \int |x|^{-pa}|\nabla u_m|^p\,dx$$

then

$$\begin{aligned} &\int (|x|^{-pa}|\nabla u|^p - \lambda_1(h)h(x)|x|^{-p(a+1)}|u|^p)\,dx \\ &\qquad < \lim_{m\to\infty} \int (|x|^{-pa}|\nabla u_m|^p - \lambda_m h(x)|x|^{-p(a+1)}|u_m|^p)\,dx = 0, \end{aligned}$$

which is impossible. Therefore, for some $k \in \mathbb{R}$, we have $u_m \to u = k\varphi_1$ in E. As in the previous part of the above proof, we show that $k = 0$. So, $u_m \to 0$ in E and the result stated in (ii) follows. The proof is complete. ∎

4. The case $\lambda > \lambda_1(h)$. If $\lambda > \lambda_1(h)$ then the principal eigenfunction $\varphi_1 > 0$ satisfies

$$\begin{aligned} \int (|x|^{-pa}|\nabla \varphi_1|^p - \lambda h(x)|x|^{-p(a+1)}\varphi_1^p)\,dx &= (\lambda_1(h) - \lambda) \int h(x)|x|^{-p(a+1)}\varphi_1^p\,dx \\ &< 0, \end{aligned}$$

so $\varphi_1 \in L^-(\lambda)$. If we assume $\int |x|^{-qb}Q(x)\varphi_1^q\,dx < 0$ then $\varphi_1 \in L^-(\lambda) \cap B^-$, and hence $t(\varphi_1)\varphi_1 \in S^+(\lambda)$.

LEMMA 4.1. *Suppose that* $\int |x|^{-qb}Q(x)\varphi_1^q\,dx < 0$. *Then there exists* $\delta > 0$ *such that* $\overline{L^-}(\lambda) \cap \overline{B^+} = \emptyset$ *whenever* $\lambda_1(h) \leq \lambda < \lambda_1(h) + \delta$.

Proof. Arguing by contradiction we can find sequences $\lambda_m \to \lambda_1(h)^+$ and $\|u_m\|_E = 1$ such that

$$\int (|x|^{-pa}|\nabla u_m|^p - \lambda_m h(x)|x|^{-p(a+1)}|u_m|^p)\,dx \leq 0$$

and

$$\int |x|^{-qb}Q(x)|u_m|^q\,dx \geq 0.$$

As before, we may assume that $u_m \rightharpoonup u$ in E, $u_m \to u$ a.e., and

$$\int h(x)|x|^{-p(a+1)}|u_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|u|^p\,dx.$$

Now, if $u_m \not\rightarrow u$ in E we obtain

$$\int (|x|^{-pa}|\nabla u|^p - \lambda_1(h)h(x)|x|^{-p(a+1)}|u|^p)\,dx$$
$$< \lim_{m\to\infty} \int (|x|^{-pa}|\nabla u_m|^p - \lambda_m h(x)|x|^{-p(a+1)}|u_m|^p)\,dx \le 0,$$

which is impossible. Therefore, for some $k \in \mathbb{R}$, we have $u_m \to u = k\varphi_1$ in E. Since $\int |x|^{-qb}Q(x)|u|^q\,dx \ge 0$, we must have $k = 0$. Therefore, $u_m \to 0$ in E, which is again impossible. ∎

In the next proposition we present essential properties of the Nehari manifold under the assumption that $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$.

PROPOSITION 4.2. *Suppose that* $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$. *Then*

(i) $S^\circ(\lambda) = \{0\}$,
(ii) $0 \notin \overline{S^-(\lambda)}$ *and* $S^-(\lambda)$ *is closed,*
(iii) $S^-(\lambda) \cap \overline{S^+(\lambda)} = \emptyset$,
(iv) $S^+(\lambda)$ *is bounded.*

Proof. (i) If $u \in S^\circ(\lambda)\setminus\{0\}$ then $u/\|u\|_E \in L^\circ(\lambda)\cap B^\circ \subseteq \overline{L^-(\lambda)}\cap\overline{B^+} = \emptyset$, which gives a contradiction.

(ii) Arguing by contradiction, assume that there exists $\{u_m\} \subset S^-(\lambda)$ such that $u_m \to 0$ in E. Then

$$0 < \int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx = \int |x|^{-qb}Q(x)|u_m|^q\,dx \to 0.$$

Set $v_m = u_m/\|u_m\|_E$. We may assume that $v_m \rightharpoonup v$ in E, $v_m \to v$ a.e., and $\int h(x)|x|^{-p(a+1)}|v_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|v|^p\,dx$. We now observe that

$$0 < \int |x|^{-qb}Q(x)|v_m|^q\|u_m\|_E^{q-p}\,dx \le \|Q\|_{L^\infty}\int |x|^{-qb}|v_m|^q\|u_m\|_E^{q-p}\,dx$$
$$\le \|Q\|_{L^\infty}\hat{S}^{-q/p}\|u_m\|_E^{q-p} \to 0$$

and also that

$$0 < \int (|x|^{-pa}|\nabla v_m|^p - \lambda h(x)|x|^{-p(a+1)}|v_m|^p)\,dx$$
$$= \int |x|^{-qb}Q(x)|v_m|^q\|u_m\|_E^{q-p}\,dx \to 0.$$

This yields

$$\lim_{m\to\infty} \lambda\int h(x)|x|^{-p(a+1)}|v_m|^p\,dx = \lambda\int h(x)|x|^{-p(a+1)}|v|^p\,dx = 1,$$

so that $v \ne 0$. We also have

$$\int (|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\,dx$$
$$\le \lim_{m\to\infty} \int (|x|^{-pa}|\nabla v_m|^p - \lambda h(x)|x|^{-p(a+1)}|v_m|^p)\,dx = 0.$$

Thus $v/\|v\|_E \in \overline{L^-(\lambda)}$. Since $\int |x|^{-qb}Q(x)|v_m|^q\,dx \geq 0$, the concentration-compactness principle yields

$$0 \leq \int |x|^{-qb}Q(x)|v|^q\,dx + Q(0)\nu_0 + Q(\infty)\nu_\infty,$$

so that $v/\|v\|_E \in \overline{B^+}$. Therefore we have proved that $v/\|v\|_E \in \overline{L^-(\lambda)} \cap \overline{B^+}$, which is impossible. Hence $0 \notin \overline{S^-(\lambda)}$. Finally, since $\overline{S^-(\lambda)} \subseteq S^-(\lambda) \cup \{0\}$ and $0 \notin \overline{S^-(\lambda)}$, we conclude that $S^-(\lambda)$ is closed.

(iii) According to (i) and (ii) we have

$$\begin{aligned}\overline{S^-(\lambda)} \cap \overline{S^+(\lambda)} &\subseteq \overline{S^-(\lambda)} \cap (S^+(\lambda) \cup S^\circ(\lambda)) = S^-(\lambda) \cap (S^+(\lambda) \cup \{0\}) \\ &= (S^-(\lambda) \cap S^+(\lambda)) \cup (S^-(\lambda) \cap \{0\}) = \emptyset.\end{aligned}$$

(iv) If $S^+(\lambda)$ is unbounded we can find a sequence $\{u_m\} \subset S^+(\lambda)$ such that $\|u_m\|_E \to \infty$. We set $v_m = u_m/\|u_m\|_E$ and we may assume that $v_m \to v$ a.e. and $\int h(x)|x|^{-p(a+1)}|v_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|v|^p\,dx$. Now, since

$$\int (|x|^{-pa}|\nabla v_m|^p - \lambda h(x)|x|^{-p(a+1)}|v_m|^p)\,dx = \int |x|^{-qb}Q(x)|v_m|^q\|u_m\|^{q-p}\,dx$$

we deduce that

$$\lim_{m\to\infty} \int |x|^{-qb}Q(x)|v_m|^q\,dx = 0.$$

On the other hand, in view of the concentration-compactness principle, we obtain

$$0 = \lim_{m\to\infty} \int |x|^{-qb}Q(x)|v_m|^q\,dx = \int |x|^{-qb}Q(x)|v|^q\,dx + Q(0)\nu_0 + Q(\infty)\nu_\infty,$$

which yields $\int |x|^{-qb}Q(x)|v|^q\,dx \geq 0$, hence $v \in \overline{B^+}$. If $v_m \nrightarrow v$ in E, then

$$\begin{aligned}&\int (|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\,dx \\ &\qquad < \lim_{m\to\infty} \int (|x|^{-pa}|\nabla v_m|^p - \lambda h(x)|x|^{-p(a+1)}|v_m|^p)\,dx \leq 0.\end{aligned}$$

Therefore $v \in \overline{L^-(\lambda)} \cap \overline{B^+}$, which is impossible. Hence the case $v_m \to v$ in E prevails. Since $\|v\|_E = 1$ and $v \in \overline{L^-(\lambda)} \cap \overline{B^+}$, we again get a contradiction. The proof is complete. ∎

THEOREM 4.3. *Suppose that* $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$. *Then*

(i) *every minimizing sequence for* J_λ *in* $S^-(\lambda)$ *is bounded,*
(ii) $\inf_{u\in S^-(\lambda)} J_\lambda(u) > 0$,
(iii) *there exists* $u \in S^-(\lambda)$ *such that* $J_\lambda(u) = \inf_{v\in S^-(\lambda)} J_\lambda(v)$.

Proof. (i) Let $\{u_m\} \subset S^-(\lambda)$ be a minimizing sequence for J_λ. Suppose that $\{u_m\}$ is unbounded in E, say (without loss of generality) $\|u_m\| \to \infty$, and set $v_m = u_m/\|u_m\|_E$. We may assume that $v_m \rightharpoonup v$ in E and

$\int h(x)|x|^{-p(a+1)}|v_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|v|^p\,dx$. Since

$$\int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx \to \frac{pq}{q-p} \inf_{u\in S^-(\lambda)} J_\lambda(u),$$

we have

$$\int (|x|^{-pa}|\nabla v_m|^p - \lambda h(x)|x|^{-p(a+1)}|v_m|^p)\,dx = \int |x|^{-qb}Q(x)|v_m|^q \|u_m\|_E^{q-p}\,dx \to 0,$$

and this implies that $\lim_{m\to\infty}\int |x|^{-qb}Q(x)|v_m|^q\,dx = 0$. It then follows from the concentration-compactness principle that $\int |x|^{-qb}Q(x)|v|^q\,dx \geq 0$. If $v_m \not\to v$ in E, then

$$\int (|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\,dx < \lim_{m\to\infty}\int (|x|^{-pa}|\nabla v_m|^p - \lambda h(x)|x|^{-p(a+1)}|v_m|^p)\,dx = 0.$$

Hence, $v \in \overline{L^-(\lambda)} \cap \overline{B^+}$, which is impossible. Therefore, $v_m \to v$ in E. This yields again the impossibility $v \in \overline{L^-(\lambda)} \cap \overline{B^+}$.

(ii) We clearly have $\inf_{v\in S^-(\lambda)} J_\lambda(v) \geq 0$. We will now show, by contradiction, that $\inf_{v\in S^-(\lambda)} J_\lambda(v) > 0$. Indeed, suppose that a minimizing sequence $(u_m) \subset S^-(\lambda)$ for J_λ satisfies

$$\lim_{m\to\infty}\int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx = \lim_{m\to\infty}\int |x|^{-qb}Q(x)|u_m|^q\,dx = 0.$$

By (i), the sequence (u_m) is bounded in E. So we may assume that $u_m \rightharpoonup u$ in E, $u_m \to u$ a.e. and $\int h(x)|x|^{-p(a+1)}|u_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|u|^p\,dx$. By the concentration-compactness principle we have $\int |x|^{-qb}Q(x)|u|^q\,dx \geq 0$. If $u_m \not\to u$ in E, then

$$\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx < \lim_{m\to\infty}\int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx = 0.$$

Therefore $u \neq 0$ and $u/\|u\|_E \in \overline{L^-(\lambda)} \cap \overline{B^+}$, which is impossible.

(iii) Let (u_m) be a minimizing sequence for J_λ on $S^-(\lambda)$. By (i), the sequence (u_m) is bounded in E. We may assume that $u_m \rightharpoonup u$ in E, $u_m \to u$ a.e. and $\int h(x)|x|^{-p(a+1)}|u_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|u|^p\,dx$. Since

$$\left(\frac{1}{p} - \frac{1}{q}\right)\lim_{m\to\infty}\int |x|^{-qb}Q(x)|u_m|^q\,dx = \inf_{v\in S^-(\lambda)} J_\lambda(v) > 0,$$

the concentration-compactness principle implies $\int |x|^{-qb}Q(x)|u|^q\,dx > 0$. According to our assumption, we have $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$, hence $B^+ \subseteq L^+(\lambda)$, and consequently $\int(|x|^{-pa}|\nabla u|^p - \lambda|x|^{-p(a+1)}h(x)|u|^p)\,dx > 0$. Therefore $u/\|u\|_E \in L^+(\lambda) \cap B^+$, which yields $t(u)u \in S^-(\lambda)$ with

$$t(u) = \left(\frac{A_\lambda(u)}{B(u)}\right)^{1/(q-p)} = \left(\frac{\int(|x|^{-pa}|\nabla u|^p - \lambda|x|^{-p(a+1)}h(x)|u|^p)\,dx}{\int |x|^{-qb}Q(x)|u|^q\,dx}\right)^{1/(q-p)}.$$

If $u_m \nrightarrow u$ in E then, by the concentration-compactness principle,

$$\begin{aligned}&\int(|x|^{-pa}|\nabla u|^p - \lambda|x|^{-p(a+1)}h(x)|u|^p)\,dx\\ &\quad< \lim_{m\to\infty}\int(|x|^{-pa}|\nabla u_m|^p - \lambda|x|^{-p(a+1)}h(x)|u_m|^p)\,dx\\ &\quad= \lim_{m\to\infty}\int |x|^{-qb}Q(x)|u_m|^q\,dx = \int|x|^{-qb}Q(x)|u|^q\,dx + Q(0)\nu_0 + Q(\infty)\nu_\infty\\ &\quad\le \int |x|^{-qb}Q(x)|u|^q\,dx,\end{aligned}$$

and therefore $t(u) < 1$. We now observe that $t(u)u_m \nrightarrow t(u)u$ and the map $t \mapsto J_\lambda(tu_m)$ attains its maximum at $t = 1$, so that

$$J_\lambda(t(u)u) < \lim_{m\to\infty} J_\lambda(t(u)u_m) \le \lim_{m\to\infty} J_\lambda(u_m) = \inf_{v\in S^-} J_\lambda(v),$$

an impossibility. Thus $u_m \to u$ in E and u is a minimizer of J_λ on $S^-(\lambda)$. ∎

THEOREM 4.4. *If $L^-(\lambda) \neq \emptyset$ and $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$ then there exists $u \in S^+(\lambda)$ such that $J_\lambda(u) = \inf_{v\in S^+(\lambda)} J_\lambda(v)$.*

Proof. It follows from our assumptions that $L^-(\lambda) \cap B^- \neq \emptyset$. By Proposition 4.2(iv) there exists $M > 0$ such that $\|v\|_E \le M$ for every $v \in S^+(\lambda)$. Using this fact we obtain the following estimate from below for J_λ on $S^+(\lambda)$ (see (1.6) with $u = \phi = v$):

$$\begin{aligned}J_\lambda(v) &= \left(\frac{1}{p} - \frac{1}{q}\right)\int(|x|^{-pa}|\nabla v|^p - \lambda h(x)|x|^{-p(a+1)}|v|^p)\,dx\\ &\ge -\lambda\left(\frac{1}{p} - \frac{1}{q}\right)\int h(x)|x|^{-p(a+1)}|v|^p\,dx\\ &\ge -\lambda\left(\frac{1}{p} - \frac{1}{q}\right)\widehat{C}\|v\|_E^2 \ge -\lambda\left(\frac{1}{p} - \frac{1}{q}\right)\widehat{C}M^2.\end{aligned}$$

It is obvious that $B = \inf_{v\in S^+(\lambda)} J_\lambda(v) < 0$. Let $(u_m) \subset S^+(\lambda)$ be a minimizing sequence for J_λ. Then

$$J_\lambda(u_m) = \left(\frac{1}{p} - \frac{1}{q}\right) \int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx$$
$$= \left(\frac{1}{p} - \frac{1}{q}\right) \int |x|^{-qb} Q(x)|u_m|^q\,dx \to B < 0.$$

We can assume that $u_m \rightharpoonup u$ in E, $u_m \to u$ a.e. and $\int h(x)|x|^{-p(a+1)}|u_m|^p\,dx \to \int h(x)|x|^{-p(a+1)}|u|^p\,dx$. Since

$$\int |x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx$$
$$\leq \lim_{m\to\infty} \int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx < 0$$

and $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$, we see that $u/\|u\|_E \in L^-(\lambda) \cap B^-$ and $t(u)u \in S^+(\lambda)$ with

$$t(u) = \left(\frac{A_\lambda(u)}{B(u)}\right)^{1/(q-p)} = \left(\frac{\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx}{\int |x|^{-qb}Q(x)|u|^q\,dx}\right)^{1/(q-p)}.$$

We now claim that $u_m \to u$ in E. Otherwise, we obtain

$$\int (|x|^{-pa}|\nabla u|^p - \lambda h(x)|x|^{-p(a+1)}|u|^p)\,dx$$
$$< \lim_{m\to\infty} \int (|x|^{-pa}|\nabla u_m|^p - \lambda h(x)|x|^{-p(a+1)}|u_m|^p)\,dx$$
$$= \lim_{m\to\infty} \int |x|^{-qb}Q(x)|u_m|^q\,dx = \int |x|^{-qb}Q(x)|u|^q\,dx + Q(0)\nu_0 + Q(\infty)\nu_\infty$$
$$\leq \int |x|^{-qb}Q(x)|u|^q\,dx.$$

From this we derive that $t(u) > 1$. On the other hand, we have

$$J_\lambda(t(u)u) < J_\lambda(u) \leq \lim_{m\to\infty} J_\lambda(u_m) = B,$$

which is impossible. Thus, $u_m \to u$ in E and we conclude that u is a minimizer of J_λ on $S^+(\lambda)$. ■

Now, if $\int |x|^{-qb}Q(x)\varphi_1^q\,dx < 0$ then, by Lemma 4.1, there exists $\delta > 0$ such that $\overline{L^-(\lambda)} \cap \overline{B^+} = \emptyset$ for $\lambda_1(h) < \lambda < \lambda_1(h) + \delta$. By Theorems 4.3 and 4.4, J_λ has minimizers on $S^-(\lambda)$ and on $S^+(\lambda)$. These minimizers are clearly distinct and we have therefore proved the following:

THEOREM 4.5. *If* $\int |x|^{-qb}Q(x)\varphi_1^q\,dx < 0$ *then there exists* $\delta > 0$ *such that, for* $\lambda_1(h) < \lambda < \lambda_1(h) + \delta$*, problem* (2.1) *has two distinct positive solutions.*

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