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A NOTE ON THE SONG–ZHANG THEOREM FOR HAMILTONIAN GRAPHS

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Abstract. An independent set S of a graph G is said to be essential if S has a pair of vertices that are distance two apart in G. In 1994, Song and Zhang proved that if for each independent set S of cardinality $k+1$, one of the following condition holds:

(i) there exist $u \neq v \in S$ such that $d(u)+d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha(G)$;
(ii) for any distinct u and v in S, $|N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\}$,

then G is Hamiltonian. We prove that if for each essential independent set S of cardinality $k+1$, one of conditions (i) or (ii) holds, then G is Hamiltonian. A number of known results on Hamiltonian graphs are corollaries of this result.

1. Introduction. We consider only finite simple graphs in this paper; undefined notation and terminology can be found in [1]. In particular, we use $V(G)$, $E(G)$, $k(G)$, $\alpha(G)$ and $\delta(G)$ to denote the vertex set, edge set, connectivity, independence number and minimum degree of G, respectively. If G is a graph and $u, v \in V(G)$, then a path in G from u to v is called a (u,v)-*path* of G. If $v \in V(G)$ and H is a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v) = d_G(v)$ and $N(v) = N_G(v)$ when the graph in use is clear. If C and H are subgraphs of G, then $N_C(H) = \bigcup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of G induced by $V(G) - V(C)$. For vertices $u, v \in V(G)$, the distance between u and v, denoted $d(u,v)$, is the length of a shortest (u,v)-path in G, or ∞ if no such path exists.

Let $C_m = x_0x_1 \ldots x_{m-1}x_0$ denote a cycle of order m. Define $N^+_{C_m}(u) = \{x_{i+1} : x_i \in N_{C_m}(u)\}$, $N^-_{C_m}(u) = \{x_{i-1} : x_i \in N_{C_m}(u)\}$ and $N^{\pm}_{C_m}(u) = N^+_{C_m}(u) \cup N^-_{C_m}(u)$, where subscripts are taken modulo m. Let $S \subseteq V(G)$, and define $\Delta(S) = \max\{d(x) : x \in S\}$.

A subset $S \subseteq V(G)$ is said to be an *essential independent set* (EIS) if S is an independent set in G and there exist two distinct vertices $x, y \in S$ with $d(x,y) = 2$.

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Three classical results on Hamiltonian graphs are:

Theorem 1.1 (Dirac, [4]). *If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 1.2 (Ore, [11]). *If $d(u)+d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

Theorem 1.3 (Chvátal and Erdős, [3]). *If G is a graph with $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Theorem 1.2 was generalized by Fan [5] who showed that only pairs of vertices at distance 2 are essential. In 1996, Chen et al. [2] proved a Dirac-type result for essential independent sets with k vertices.

Theorem 1.4 (Chen et al., [2]). *Let G be a k-connected ($k \geq 2$) graph on $n \geq 3$ vertices. If $\max\{d(u) : u \in S\} \geq n/2$ for any essential independent set S with k vertices in G, then G is Hamiltonian.*

In 1997, Liu and Wei [10] considered essential independent sets with $k+1$ vertices in the following:

Theorem 1.5 (Liu and Wei, [10]). *Let G be a k-connected ($k \geq 2$) graph on $n \geq 3$ vertices. If $\max\{d(u) : u \in S\} \geq n/2$ for any essential independent set S with $k+1$ vertices in G, then G is Hamiltonian or is in one of three exceptional classes of graphs.*

In 2002, Hirohata [9] considered essential independent sets S with k vertices and showed that the length of a longest cycle depends on $\max\{d(u) : u \in S\}$. Recently, in [8] Theorem 1.5 as well as some other results were generalized.

Neighborhood unions have already been shown to be very useful in studying Hamiltonian graphs. The first use of this generalized degree condition was to provide another generalization of Dirac's theorem by Faudree et al. [7] in 1989.

Theorem 1.6 (Faudree et al. [7]). *If G is a 2-connected graph and if $|N(u) \cup N(v)| \geq (2n-1)/3$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

In 1991, Faudree et al. [6] considered the effect of $\delta(G)$.

Theorem 1.7 (Faudree et al., [6]). *If G is a 2-connected graph and if $|N(u) \cup N(v)| \geq n - \delta(G)$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

In 1994, Song and Zhang [12] considered independent sets with $k+1$ vertices and proved the following theorem.

Theorem 1.8 (Song and Zhang, [12]). *Let G be a k-connected graph ($k \geq 2$) with independence number α. If for each independent set S of cardinality $k+1$, one of the following conditions holds:*

(i) *there exist* $u \neq v \in S$ *such that* $d(u)+d(v) \geq n$ *or* $|N(u) \cap N(v)| \geq \alpha$;
(ii) *for any distinct* u *and* v *in* S, $|N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\}$,

then G *is Hamiltonian.*

The purpose of this paper is to unify and extend the theorems above through the use of essential independent sets by proving the following result.

Theorem 1.9. *Let* G *be a* k*-connected graph* $(k \geq 2)$ *with independence number* α. *If for each essential independent set* S *of cardinality* $k+1$, *one of the following conditions holds:*

(i) *there exist* $u \neq v \in S$ *such that* $d(u) + d(v) \geq n$ *or* $|N(u) \cap N(v)| \geq \alpha$;
(ii) *for any distinct* u *and* v *in* S, $|N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\}$,

then G *is Hamiltonian.*

Obviously, Theorem 1.9 generalizes Theorems 1.1, 1.2, 1.3, 1.6, 1.7 and 1.8. Next we present an example that shows that Theorem 1.9 is stronger than Theorem 1.8.

Let $k \geq 2$ and $n \geq (k+1)(k+3) + k + 2 + 1 = k^2 + 5k + 6$. Let $H = K_{n-(k+2)}$ and build a graph G as follows. Take H along with a disjoint set of vertices $S = \{x_1, \ldots, x_{k+2}\}$. Now join each $x_i \in S$, $1 \leq i \leq k+1$, to a distinct set of $k+3$ vertices of H. That is, make the neighborhoods of these vertices of S disjoint. Next join x_{k+2} to a set of k vertices of H in such a way that $N(x_{k+2}) \cap N(x_i) = \emptyset$ for $1 \leq i \leq k+1$.

Now the resulting graph G is clearly k-connected. Also $\alpha(G) = k+3$. If we consider the independent vertex set $S' = \{x_1, \ldots, x_{k+1}\}$ we see that $d(x_i) + d(x_j) = 2k + 6 < n$.

Also, for two vertices in S' we have $|N(x_i) \cap N(x_j)| = \emptyset$. Thus condition (i) of the Song–Zhang Theorem fails to hold. Further,

$$|N(x_i) \cup N(x_j)| = 2k + 6 < n - \max\{d(x) : x \in S\} = n - (k+3)$$

(using the bound on n). Thus, condition (ii) of the Song–Zhang Theorem also fails to hold. Hence, Theorem 1.8 cannot be applied to G.

However, the only essential independent sets of order $k+1$ contain a vertex y in H and k vertices from $S = \{x_1, \ldots, x_{k+2}\}$. For any such set, there exists some vertex x_i such that $d(y, x_i) = 2$ and

$$d(y) + d(x_i) = n - (k+2) - 1 + k + 3 = n.$$

Therefore, Theorem 1.9 does apply.

2. Proof of Theorem 1.9.

Before we begin the proof of Theorem 1.9, we need to establish a few basic facts. Within these facts we will also establish some useful inequalities.

For a cycle $C_m = x_0 x_1 \ldots x_{m-1} x_0$, we write $[x_i, x_j]$ to denote the subpath $x_i, x_{i+1}, \ldots, x_j$ of the cycle C_m, where subscripts are taken modulo m. For notational convenience, $[x_i, x_j]$ will denote the (x_i, x_j)-path of C_m as well as the vertex set of this path.

Claim. *Let G be a 2-connected non-Hamiltonian graph. Let $C_m = x_0x_1 \dots x_{m-1}x_0$ be a longest cycle of G, H a component of $G - C_m$, $x \in V(H)$. Suppose that $x_i \in N_{C_m}(x)$ and $x_j \in N_{C_m}(H)$ satisfy $\{x_{i+1}, x_{i+2}, \dots, x_{j-1}\} \cap N_{C_m}(H) = \emptyset$. Then Facts* (I)–(III) *and inequalities* (1)–(5) *below hold.*

Proof of Claim. Let P be a path in H with end-vertices adjacent to $x_i, x_j \in V(C_m)$, respectively.

Fact (I). *Suppose $x_h \in \{x_{i+1}, x_{i+2}, \dots, x_{j-2}\} - \{x_j, x_{j-1}\}$ is adjacent to x_{j+1}. Then x_{h+1} is adjacent to neither x_{i+1} nor x.*

First, since $\{x_{i+1}, x_{i+2}, \dots, x_{j-1}\} \cap N_{C_m}(H) = \emptyset$, it follows that x_{h+1} is not adjacent to x.

If x_{h+1} is adjacent to x_{i+1} we obtain the cycle

$$C^* = x_iPx_jx_{j-1} \dots x_{h+1}x_{i+1}x_{i+2} \dots x_hx_{j+1}x_{j+2} \dots x_i$$

which is longer than C_m, a contradiction.

Fact (II). *Suppose $x_h \in \{x_{j+1}, x_{j+2}, \dots, x_i\}$ is adjacent to x_{j+1}. Then x_{h-1} is adjacent to neither x_{i+1} nor x.*

Otherwise, if x_{h-1} is adjacent to x_{i+1}, then the cycle

$$C^* = x_iPx_jx_{j-1} \dots x_{i+1}x_{h-1}x_{h-2} \dots x_{j+1}x_hx_{h+1} \dots x_i$$

is longer than C_m, a contradiction. Also, suppose x_{h-1} is adjacent to x. Let P' be a path in H with end-vertices adjacent to x_{h-1}, x_j, respectively. Then

$$C^* = x_{h-1}P'x_jx_{j-1} \dots x_hx_{j+1}x_{j+2} \dots x_{h-1}$$

is a cycle longer than C_m, a contradiction.

Fact (III). *Suppose $y \in V(G - C_m)$ is adjacent to x_{j+1}. Then y is adjacent to neither x_{i+1} nor x.*

Clearly, y is not in H, so y is not adjacent to x. If y is adjacent to x_{i+1}, then the cycle

$$C^* = x_iPx_jx_{j-1} \dots x_{i+1}yx_{j+1}x_{j+2} \dots x_i$$

is longer than C_m, a contradiction. In Fact (I) above, we do not assume that the two vertices $\{x_j, x_{j-1}\}$ are adjacent to x_{j+1}. Hence, we have

$$\begin{aligned} (1) \qquad |N(x_{i+1}) \cup N(x)| &\le n - (d(x_{j+1}) - |\{x_{j-1}, x_j\}|) - |\{x_{i+1}, x\}| \\ &\le n - d(x_{j+1}). \end{aligned}$$

Clearly, all vertices in $N^+_{C_m}(H) \cup V(H)$ are nonadjacent to x_{i+1} and nonadjacent to x_{j+1}. Hence, we also have

$$(2) \qquad \begin{aligned} |N(x_{i+1}) \cup N(x_{j+1})| &\leq n - |N^+_{C_m}(H) \cup V(H)| \\ &\leq n - |N^+_{C_m}(x)| - |V(H)|. \end{aligned}$$

Moreover, if x_{j-1} is adjacent to x_{j+1}, then x_j is not adjacent to x_{i+1}. Combining this with the discussion in Facts (I), (II) and (III), there are at least $(d(x_{j+1}) - |\{x_j\}|) - |\{x_{i+1}\}| - |V(H)|$ vertices not adjacent to x_{i+1}. Hence,

$$d(x_{i+1}) \leq n - (d(x_{j+1}) - |\{x_j\}|) - |\{x_{i+1}\}| - |V(H)|,$$

which implies that

$$(3) \qquad d(x_{i+1}) + d(x_{j+1}) \leq n - |V(H)|.$$

Similarly, all vertices in $N^+_{C_m}(x_{j+1}) \cup N_{G-C_m}(x_{j+1}) \cup \{x\}$ are nonadjacent to x, thus we have

$$d(x) \leq n - |N^+_{C_m}(x_{j+1}) \cup N_{G-C_m}(x_{j+1}) \cup \{x\}|,$$

which implies

$$(4) \qquad d(x) + d(x_{j+1}) \leq n - 1.$$

Clearly, the common neighbors of x_{i+1} and x are all on C_m. Hence, we also have

$$(5) \qquad |N(x_{i+1}) \cap N(x)| \leq \alpha - 1.$$

Proof of Theorem 1.9. Assume that G is not Hamiltonian. Let $C_m = x_0x_1 \ldots x_{m-1}x_0$ be a longest cycle of G, and H a component of $G - C_m$. Since G is k-connected, we have $|N_{C_m}(H)| \geq k$. Let P be a path in H whose end-vertices x^*, y^* are adjacent to x_i and x_j on C_m respectively. Let $x \in V(H)$ and $x_i \in N_{C_m}(x)$. Let S^* denote k vertices of $N^+_{C_m}(H)$ containing x_{i+1}, and let $S = S^* \cup \{x\}$. Clearly, S is an EIS. Now, G satisfies conditions (i) or (ii) of the Theorem.

Suppose (i) holds, that is, there exist $u, v \in S$ with $u \neq v$ such that $d(u) + d(v) \geq n$ or $|N(u) \cap N(v)| \geq \alpha$.

Since S is an EIS, we have $\alpha(G) \geq k+1$. Then, by inequalities (3) and (4), $d(u) + d(v) \geq n$ is impossible. Together with (i), this implies $|N(u) \cap N(v)| \geq \alpha$. By (5), if $|N(u) \cap N(v)| \geq \alpha(G)$, then $u, v \in N^+_{C_m}(H)$. Without loss of generality, assume $\{u, v\} = \{x_{i+1}, x_{j+1}\}$.

Since C_m is a longest cycle of G, the vertices of $N(x_{i+1}) \cap N(x_{j+1})$ are not in $G - C_m$, for otherwise a cycle longer than C_m is easily found. Thus, $N(x_{i+1}) \cap N(x_{j+1}) \subseteq V(C_m)$, which implies $|N(x_{i+1}) \cap N(x_{j+1})| = |N_{C_m}(x_{i+1}) \cap N_{C_m}(x_{j+1})| = |N^-_{C_m}(x_{i+1}) \cap N^-_{C_m}(x_{j+1})|$. Since C_m is a longest cycle of G, $N^-_{C_m}(x_{i+1}) \cap N^-_{C_m}(x_{j+1})$ is an independent set (or again, using P, a longer cycle is easily found). Let w be a vertex of H. Then $\{w\} \cup (N^-_{C_m}(x_{i+1}) \cap N^-_{C_m}(x_{j+1}))$ is also an independent set. By condition (i) of the Theorem, $|N^-_{C_m}(x_{i+1}) \cap N^-_{C_m}(x_{j+1})| \geq \alpha$. This implies that $|\{w\} \cup (N^-_{C_m}(x_{i+1}) \cap N^-_{C_m}(x_{j+1}))| \geq \alpha + 1$, contradicting the fact that the independence number of G is α.

Now suppose (ii) holds, that is, for any distinct $u, v \in S$, $|N(u) \cup N(v)| \geq n - \max\{d(x) : x \in S\} = n - \Delta(S)$.

If $\Delta(S) = d(x)$, then by inequality (2), we have $|N(x_{i+1}) \cup N(x_{j+1})| \leq n - d(x) - 1$, while condition (ii) says that $|N(u) \cup N(v)| \geq n - \Delta(S)$, a contradiction.

We now consider the following two cases.

Case 1: $k \geq 3$ and $|N_{C_m}(H)| \geq k$. In this case, let $x_{i1}, \dots, x_{ik} \in N_{C_m}(H)$ be such that there are no neighbors of H in the intervals $[x_{it+1}, \dots, x_{i(t+1)-1}]$ for $t = 1, \dots, k-1$. Let $z \in V(H)$ be adjacent to some vertex of $\{x_{i1}, x_{i2}, \dots, x_{ik}\}$ and let

$$S = \{z, x_{i1+1}, \dots, x_{ik+1}\}.$$

Clearly, S is an EIS. Without loss of generality, $\Delta(S) = d(x_{ik+1})$.

If $x_{ik-1}x_{ik+1} \notin E(G)$, then by (1), there exist

$$(d(x_{ik+1}) - |\{x_{ik}\}|) + |\{x_{i(k-1)+1}, z\}|$$

vertices that are nonadjacent to $x_{i(k-1)+1}$ and nonadjacent to z, hence we have

$$\begin{aligned}|N(x_{i(k-1)+1}) \cup N(z)| &\leq n - (d(x_{ik+1}) - |\{x_{ik}\}|) - |\{x_{i(k-1)+1}, z\}| \\ &\leq n - d(x_{ik+1}) - 1,\end{aligned}$$

a contradiction.

Suppose $x_{ik-1}x_{ik+1} \in E(G)$. Without loss of generality, x_{ik+t} is not adjacent to x_{ik-1}, and all of $\{x_{ik+1}, x_{ik+2}, \dots, x_{ik+t-2}, x_{ik+t-1}\}$ are adjacent to x_{ik-1} (clearly, x_{ik+t} must exist in the set $\{x_{ik+1}, x_{ik+2}, \dots, x_{i(k+1)-1}\}$, since $x_{i(k+1)-1}$ is not adjacent to x_{ik-1}). Then, without loss of generality, $x \in V(H)$ is adjacent to some vertex of $\{x_{i1}, x_{i2}, \dots, x_{i(k-1)}\}$. Let $S^* = \{x, x_{i1+1}, x_{i2+1}, \dots, x_{i(k-1)+1}, x_{ik+t}\}$. Clearly, S^* is an EIS (for otherwise a longer cycle clearly exists, a contradiction). For the EIS S^* we will prove that conditions (i) and (ii) of the Theorem fail to hold.

First, when $w, y \in \{x, x_{i1+1}, x_{i2+1}, \dots, x_{i(k-1)+1}, x_{ik+t}\} \setminus \{x_{ik+t}\}$, we can easily check that $d(w) + d(y) \leq n - 1$ and $|N(w) \cap N(y)| \leq \alpha - 1$.

Next, suppose $w = x_{ik+t}$ and $y = x$. Clearly we also have $d(w) + d(y) \leq n - 1$ and $|N(w) \cap N(y)| \leq \alpha - 1$.

Now suppose $w = x_{ik+t}$ and $y \in \{x_{i1+1}, x_{i2+1}, \dots, x_{i(k-1)+1}\}$.

Since $x_{ik-1}x_{ik+1} \in E(G)$ and each vertex of $\{x_{ik+1}, x_{ik+2}, \dots, x_{ik+t-1}\}$ is adjacent to x_{ik-1}, it follows that each vertex of $\{x, x_{i1+1}, x_{i2+1}, \dots, x_{ik+t}\} \setminus \{x_{ik+t}\}$ is nonadjacent to any vertex of $\{x_{ik}, x_{ik+1}, \dots, x_{ik+t-2}, x_{ik+t-1}\}$ (otherwise, we easily get a longer cycle). Then clearly, for any $x_{i(k-r)+1}$ $(1 \leq r \leq k-1)$,

(F1) if $x_h \in \{x_{i(k-r)+1}, x_{i(k-r)+2}, \dots, x_{ik-1}\}$ is adjacent to $x_{i(k-r)+1}$, then x_{h-1} is not adjacent to x_{ik+t}

(otherwise, the cycle

$$x_{ik}x_{ik+1}x_{ik+2}\dots x_{ik+t-1}x_{ik-1}x_{ik-2}\dots x_h x_{i(k-r)+1}x_{i(k-r)+2} \\ \dots x_{h-1}x_{ik+t}x_{ik+t+1}\dots x_{i(k-r)}Px_{ik}$$

is longer than C_m, a contradiction). Similarly,

(F2) If $x_h \in \{x_{ik+t+1}, x_{ik+t+2}, \dots, x_{i(k-r)}\} \setminus \{x_{i(k-r)}\}$ is adjacent to $x_{i(k-r)+1}$, then x_{h+1} is not adjacent to x_{ik+t}.

If there exist p vertices of $C_m \setminus \{x_{i(k-i)}\}$ adjacent to $x_{i(k-r)+1}$ or x_{ik+t}, then there must also exist p vertices of $C_m \setminus \{x_{i(k-r)}\}$ not adjacent to x_{ik+t} or $x_{i(k-r)+1}$. Moreover, no vertex of H is adjacent to both $x_{i(k-r)+1}$ and x_{ik+t}, and every vertex of $G - C_m - H$ is adjacent to at most one of $\{x_{i(k-r)+1}, x_{ik+t}\}$, and $x_{i(k-r)+1}$ and x_{ik+t} are not adjacent to both $x_{i(k-r)+1}$ and x_{ik+t}. Hence, we have $d(x_{i(k-r)+1}) + d(x_{ik+t}) \leq n-1$.

It follows that $|N^-(x_{i(k-r)+1}) \cap N^-(x_{ik+t})| \leq \alpha - 1$ (otherwise, by a proof similar to case (i), we must get a longer cycle). Thus, $|N(x_{i(k-r)+1}) \cap N(x_{ik+t})| \leq \alpha(G) - 1$.

Therefore, when $w = x_{ik+t}$, $y \in \{x_{i1+1}, x_{i2+1}, \dots, x_{i(k-1)+1}\}$, we also have $d(w) + d(y) \leq n-1$ and $|N(w) \cap N(y)| \leq \alpha(G) - 1$.

Now, we consider condition (ii) of the Theorem.

Suppose $d(x_{ik+t}) \leq \max\{d(x_{ih+1}) : h = 1, \dots, k-1\}$. Without loss of generality, assume $\Delta(S^*) = d(x_{ih+1})$, where $h \in \{1, \dots, k-1\}$. Clearly x_{ih+1} is not adjacent to x_{ik+2} (otherwise, the cycle $C^* = x_{ih}Px_{ik}x_{ik+1}x_{ik-1}x_{ik-2}$ $\dots x_{ih+1}x_{ik+2}x_{ik+3}\dots x_{ih}$ is longer than C_m). Further, $x_{i(h-1)+1}$ and x are not both adjacent to x_{ik+1} (otherwise, we must get a longer cycle). Thus, by (1), we have

$$\begin{aligned} |N(x_{i(h-1)+1}) \cup N(x)| &\leq n - (d(x_{ih+1}) - |\{x_{ih-1}, x_{ih}\}|) \\ &\quad - |\{x_{i(h-1)+1}, x\}| - |\{x_{ik+1}\}| \\ &\leq n - d(x_{ih+1}) - 1, \end{aligned}$$

a contradiction.

Suppose $d(x_{ik+t}) > \max\{d(x_{ih+1}) : h = 1, \dots, k-1\}$.

In this case, clearly, none of $\{x_{i(k-1)+1}, x_{i(k-1)+2}, \dots, x_{ik-1}\}$ is adjacent to x. By the choice of x_{ik+t}, none of $\{x_{ik+1}, x_{ik+2}, \dots, x_{ik+t}\}$ is adjacent to $x_{i(k-1)+1}$ and none to x (otherwise, we obtain a cycle longer than C_m).

Since C_m is a longest cycle of G, we have:

- If $x_{i(k-1)+r} \in \{x_{i(k-1)+2}, x_{i(k-1)+3}, \dots, x_{ik-2}\}$ is adjacent to x_{ik+t}, then $x_{i(k-1)+r+1}$ is adjacent to neither $x_{i(k-1)+1}$ nor x.
- $x_{ik+r} \in \{x_{ik+t}, x_{ik+t+1}, \dots, x_{i(k-1)}\}$ is adjacent to x_{ik+t}, then x_{ik+r-1} is adjacent to neither $x_{i(k-1)+1}$ nor x.
- $x_{ik+r} \in \{x_{ik}, x_{ik+1}, \dots, x_{ik+t-1}\} \setminus \{x_{ik}\}$ is adjacent to x_{ik+t}, then x_{ik+r} is adjacent to neither $x_{i(k-1)+1}$ nor x.

Similar to the discussion of inequality (1), we have

$$\begin{aligned} |N(x_{i(k-1)+1}) \cup N(x)| &\leq n - (d(x_{ik+t}) - |\{x_{ik}\}|) - |\{x_{i(k-1)+1}, x\}| \\ &\leq n - d(x_{ik+t}) - 1, \end{aligned}$$

a contradiction.

Case 2: $|N_{C_m}(H)| = |\{x_i, x_j\}| = 2$. In this case, without loss of generality, assume $d(x_{i+1}) \leq d(x_{j+1})$.

Claim (a). *Let x, y be two vertices of H which are adjacent to x_i, x_j, respectively. If $d(x_{i+1}, x_{j+1}) = 2$, then there is a Hamilton path in the subgraph H with end-vertices x, y.*

Proof of Claim (a). Let P' be a longest path of H with end-vertices x, y. If P' is not a Hamilton path of H, let w be a vertex of $H - P'$ which is adjacent to some vertex of P'. Clearly, $\{x_{i+1}, x_{j+1}, w\}$ is an EIS. Further, we know that condition (i) of the Theorem does not hold. Thus, (ii) must hold. Then we can check that w must be adjacent to every vertex of $H - \{w\}$, for otherwise, by (1), we again reach a contradiction. Thus, we get a path in H longer than P' with end-vertices x, y, a contradiction.

Claim (b). *If $u \in V(H)$ is adjacent to x_i, then u must be adjacent to x_j.*

Proof of Claim (b). If u is not adjacent to x_j, then, by a proof similar to that of (1), $|N(x_{i+1}) \cup N(x)| \leq n - (d(x_{j+1}) - |\{x_j\}| - |\{x_{i+1}, x\}| \leq n - d(x_{j+1}) - 1$, a contradiction.

Subcase 2.1: $|V(H)| \geq 2$. Let $\{x_i, x_j\} = N_{C_m}(H)$, and let $x, y \in V(H)$ be adjacent to x_i, x_j, respectively. Moreover, let $|V(H)| = h$.

Subcase 2.1.1: $d(x_{i+1}, x_{j+1}) \geq 3$.

Subcase 2.1.1.1: $d(x) \geq \max\{d(x_{i+1}), d(x_{j+1})\}$ or $d(y) \geq \max\{d(x_{i+1}), d(x_{j+1})\}$. Without loss of generality, assume $d(x) \geq \max\{d(x_{i+1}), d(x_{j+1})\}$.

Clearly $\{x, x_{i+1}, x_{j+1}\}$ is an EIS. Further, we know that condition (i) of the Theorem does not hold. Thus, (ii) must hold. But we can check that

$$|N(x_{i+1}) \cup N(x_{j+1})| \leq n - |N(x)| \leq n - \max\{d(x) : x \in S\} - 1,$$

contrary to (ii).

SUBCASE 2.1.1.2: Subcase 2.1.1.1 fails to hold. Without loss of generality, $d(x_{j+1}) = \max\{d(x_{i+1}), d(x_{j+1}), d(x), d(y)\}$. Since $d(x_{i+1}, x_{j+1}) \geq 3$, let $x_r \in \{x_{j+1}, x_{j+2}, \ldots, x_i\}$ be adjacent to x_{j+1} with r as large as possible. Then x_r is not adjacent to x_{i+1}. Let $x_h \in \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}$ be adjacent to x_{j+1} with h as small as possible. Then x_h is not adjacent to x_{i+1}. Hence, one can check that

$$\begin{aligned}|N(x_{i+1}) \cup N(x)| &\leq n - (d(x_{j+1}) - |\{x_j, x_{j-1}\}|) - |\{x_{i+1}, x\}| - |\{x_k, x_h\}| \\ &\leq n - d(x_{j+1}) - 2,\end{aligned}$$

a contradiction.

SUBCASE 2.1.2: $d(x_{i+1}, x_{j+1}) = 2$. By Claim (a), H has a Hamilton path with end-vertices x, y. Suppose that

($*$) $x_f \in \{x_{j+1}, x_{j+2}, \ldots, x_i\}$ is adjacent to x_{i+1} and $x_{f+r} \in \{x_{j+1}, x_{j+2}, \ldots, x_i\}$ is adjacent to x_{j+1} (where $r \geq 1$ and x_{f+1} is not adjacent to x_{i+1}).

Then none of $\{x_{f+1}, x_{f+2}, \ldots, x_{f+h}\}$ is adjacent to x_{j+1} (otherwise, together with Claim (a) that H has a Hamilton path with end-vertices x, y, we get a cycle longer than C_m). Hence, we have

$$\begin{aligned}|N(x_{i+1}) \cup N(x)| &\leq n - (d(x_{j+1}) - |\{x_{j-1}, x_j\}|) - |\{x_{i+1}, x\}| \\ &\quad - (|\{x_{f+1}, x_{f+2}, \ldots, x_{f+h}\}| - 1) \\ &\leq n - d(x_{j+1}) - 1,\end{aligned}$$

a contradiction.

Similarly, if $x_f \in \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}$ is adjacent to x_{j+1}, and x_{f+r} is adjacent to x_{i+1}, we also get a contradiction. Now suppose that ($*$) fails to hold. Namely, when $x_f \in \{x_{j+1}, x_{j+2}, \ldots, x_i\}$ is adjacent to x_{j+1}, then none of $\{x_{j+1}, x_{j+2}, \ldots, x_{f-1}\}$ is adjacent to x_{i+1}. If $x_f \in \{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\}$ is adjacent to x_{j+1}, then none of $\{x_{f+1}, x_{f+2}, \ldots, x_j\}$ is adjacent to x_{i+1}. In this case, under the conditions of the Theorem, if $x_f \in \{x_{j+1}, x_{j+2}, \ldots, x_i\}$ is adjacent to x_{j+1}, and none of $\{x_{f+1}, x_{f+2}, \ldots, x_i\}$ is adjacent to x_{j+1}, then all of $\{x_{j+1}, x_{j+2}, \ldots, x_f\}$ are adjacent to x_{j+1}, and every vertex of $\{x_f, x_{f+1}, \ldots, x_i\}$ is adjacent to x_{i+1}. Similarly, when $x_t \in \{x_{i+1}, x_{i+2}, \ldots, x_j\}$ is adjacent to x_{i+1}, and none of $\{x_{t+1}, x_{t+2}, \ldots, x_j\}$ is adjacent to x_{i+1}, then every vertex of $\{x_{i+1}, x_{i+2}, \ldots, x_t\}$ is adjacent to x_{i+1} (otherwise we obtain the contradiction that $|N(x_{i+1}) \cup N(x)| \leq n - d(j+1) - 1$). Clearly x_{f-1} is not adjacent to any of $\{x_f, x_{f+1}, \ldots, x_t\} - \{x_f, x_t\}$

and x_{f-1} is not adjacent to x_j (otherwise, we again obtain a longer cycle). Thus, if $d(x_{i+1}) \le d(x_{f-1})$, we have

$$\begin{aligned}|N(x_{i+1}) \cup N(x)| &\le n - (d(x_{f-1}) - |\{x_f, x_t\}|) - |\{x_{i+1}, x, x_j\}| \\ &\le n - d(x_{f-1}) - 1,\end{aligned}$$

a contradiction. If $d(x_{i+1}) > d(x_{f-1})$, we have

$$|N(x_{f-1}) \cup N(x)| \le n - (d(x_{i+1}) - |\{x_f, x_t\}|) - |\{x_{i-1}, x, x_j\}| \le n - d(x_{i+1}) - 1,$$

a contradiction.

SUBCASE 2.2: $|V(H)| = 1$. Let $V(H) = \{x\}$ and $|N_{C_m}(x)| = 2 = |\{x_1, x_f\}|$. In this case, we have $C_m = C_{n-1}$, since otherwise, as we are not in Subcase 2.1, any component H' of $G - C_m - H$ has $|V(H')| = 1$. Without loss of generality, let $V(H') = \{y\}$, so $|N_{C_m}(y)| = 2$. This implies that $|N(x) \cup N(y)| \le 4$. Since C_m is a longest cycle, y is not adjacent to at least one of $\{x_2, x_{f+1}\}$. Without loss of generality, y is not adjacent to x_2. Then $d(x_2) \le n - |\{x, y, x_2, x_{f+1}\}| = n - 4$. Clearly, $S = \{x, y, x_2\}$ is an EIS. By condition (ii) of the Theorem, for any distinct u and v in S, $|N(u) \cup N(v)| \ge n - \Delta(S)$. Together with $d(x_2) \le n - 4$, we see that $|N(x) \cup N(y)| = 4$ implies $m \ge 4$ and $d(x_2) = n - 4$. Since C_m is a longest cycle we can easily check that $m \ge 6$ and $n \ge 8$. By inequality (3), we have $d(x_{f+1}) \le n - |V(H)| - d(x_2) = 3$. If y is not adjacent to x_{f+1}, then $S = \{x, y, x_{f+1}\}$ is an EIS. Then condition (ii) of the Theorem implies that $|N(x) \cup N(y)| \ge n\Delta(S)$ fails, a contradiction. Suppose y is adjacent to x_{f+1}. Let $N(y) = \{x_i, x_j\}$, say $i < j$. Since C_m is a longest cycle of G, if $x_{h+1} = x_i$, then x_{j-1} is not adjacent to x_2. Thus we have $d(x_2) < n - 4$, which contradicts the above result that $d(x_2) = n - 4$. If $x_{f+1} = x_j$ then since C_m is a longest cycle of G; it follows that x_{i+1} is not adjacent to x_2 and we have $d(x_2) < n - 4$, again contradicting $d(x_2) = n - 4$. Therefore, $C_m = C_{n-1}$.

Now, without loss of generality, assume $d(x_2) \le d(x_{f+1})$. Let $S = \{x, x_2, x_{f+1}\}$.

CLAIM (I). *The vertex x_2 is not adjacent to x_f.*

For otherwise, we have $|N(x_2) \cup N(x)| = d(x_2)$. By condition (ii) of the Theorem, that implies $|N(x_2) \cup N(x)| \ge n - \Delta(S) = n - d(x_{f+1})$, and we have $d(x_2) \ge n - d(x_{f+1})$. This contradicts inequality (3).

CLAIM (II). *The vertex x_{f-1} is adjacent to x_{f+1}.*

For otherwise, by inequality (3) and Claim (I), we have

$$d(x_2) \le n - (d(x_{f+1}) - |\{x_f\}|) - |\{x_2\}| - |\{x_f\}| - |V(H)| \le n - d(x_{f+1}) - 2.$$

But by condition (ii) of the Theorem and Claim (I), we have

$$d(x_2) = |N(x_2) \cup N(x)| - 1 \geq n - \Delta(S) - 1 = n - d(x_{f+1}) - 1,$$

a contradiction.

CLAIM (III). *The vertex x_2 is adjacent to x_{n-1}.*

For otherwise, if x_2 is not adjacent to x_{n-1}, when $d(x_2) = d(x_{f+1})$, we can apply Claim (II) to deduce that x_2 is adjacent to x_{n-1}, a contradiction. Suppose $d(x_2) < d(x_{f+1})$. If $d(x_{n-1}) \geq d(x_{f-1})$, Claim (II) implies that x_2 is adjacent to x_{n-1}, again a contradiction. If $d(x_{n-1}) < d(x_{f-1})$, together with inequality (3) we have $\max\{d(x_2), d(x_{n-1})\} < (n-1)/2$. Let $S = \{x, x_2, x_{n-1}\}$. Clearly, this contradicts condition (ii) of the Theorem.

CLAIM (IV). *Let $x_t \in \{x_{f+1}, x_{f+2}, \ldots, x_{n-1}\}$ be adjacent to x_2 and suppose none of $\{x_{f+1}, x_{f+2}, \ldots, x_{t-2}, x_{t-1}\}$ is adjacent to x_2. Let $x_t \in \{x_2, x_3, \ldots, x_{f-1}\}$ be adjacent to x_{f+1} and none of $\{x_2, x_3, \ldots, x_k - 1\}$ be adjacent to x_{f+1}. Then $d(x_{t-1}) + d(x_{k-1}) \leq n-3$.*

In this case, x_t is adjacent to x_{f+1} (otherwise, by inequality (1), we have

$$\begin{aligned} |N(x_2) \cup N(x)| &\leq n - (d(x_{f+1}) - |\{x_{f-1}, x_f\}|) - |\{x_2, x\}| - |\{x_{t-1}\}| \\ &= n - d(x_{f+1}) - 1, \end{aligned}$$

contradicting condition (ii) of the Theorem).

Since C_{n-1} is a longest cycle of G, when $x_r \in \{x_2, x_3, \ldots, x_{f-1}\}$ is adjacent to x_2, then x_{r-1} is not adjacent to x_{t-1}. If $x_r \in \{x_t, x_{t+1}, \ldots, x_{n-1}, x_1\} - \{x_{n-1}, x_1\}$ is adjacent to x_2, then x_{r+1} is not adjacent to x_{t-1}. Clearly, x_2 is adjacent to neither x_f nor x_{f+1}, and x_{t-1} is adjacent to neither x_{f-1} nor x_f. Hence, we have

$$d(x_{t-1}) \leq n - (d(x_2) - |\{x_{n-1}, x_1\}|) - |\{x_{f-1}, x_f, x_{t-1}, x\}| = n - d(x_2) - 2.$$

Similarly, $d(x_{k-1}) \leq n - d(x_{f+1}) - 2$. This implies

$$d(x_{t-1}) + d(x_{k-1}) \leq [n - d(x_2) - 2] + [n - d(x_{f+1}) - 2]. \tag{6}$$

Without loss of generality, assume $d(x_2) \geq d(x_{f+1})$. By condition (ii) of the Theorem, we have $d(x_2) + 1 = |N(x_2) \cup N(x)| \geq n - d(x_{f+1})$, which implies that $d(x_2) + d(x_{f+1}) \geq n - 1$. By inequality (3), we have $d(x_2) + d(x_{f+1}) \leq n - 1$. This implies $d(x_2) + d(x_{f+1}) = n - 1$. Together with (6), we have

$$d(x_{t-1}) + d(x_{k-1}) \leq [n - d(x_2) - 2] + [n - d(x_{f+1}) - 2] = n - 3. \tag{7}$$

In what follows we will show that $d(x_{k-1}) + d(x_{t-1}) \geq n-2$, which contradicts the above inequality. First we must establish the following claims.

CLAIM (A). *If $x_{k-1}x_f \in E(G)$, then $d(x_{k-1}) \geq d(x_{f-1})$. If $x_{k-1}x_{f-1} \notin E(G)$, then $d(x_{k-1}) \geq d(x_{f+1}) - 1$.*

Clearly, $\{x, x_{n-1}, x_{k-1}\}$ is an EIS. If the Claim fails to hold, then by condition (ii) of the Theorem, we have

$$d(x_{n-1}) + 1 = |N(x_{n-1}) \cup N(x)| \geq n - \Delta\{x, x_{n-1}, x_{k-1}\}. \tag{8}$$

If $d(x_{k-1}) \geq d(x_{n-1})$, then inequality (8) becomes

$$d(x_{n-1}) + 1 = |N(x_{n-1}) \cup N(x)| \geq n - \Delta\{x, x_{n-1}, x_{k-1}\} = n - d(x_{k-1}).$$

Since Claim (A) fails to hold,

$$n - d(x_{f-1}) > n - d(x_{f-1}).$$

Thus, $d(x_{n-1}) + 1 > n - d(x_{h-1})$, which implies that $d(x_{n-1}) + d(x_{h-1}) > n - 1$, which contradicts inequality (3).

If $d(x_{k-1}) < d(x_{n-1})$, then combined with the above hypothesis that $d(x_{n-1}) \leq d(x_{f-1})$, and that Claim (A) fails, we find that when $x_{k-1}x_f \in E(G)$,

$$d(x_{f-1}) + 1 > d(x_{k-1}) + 1 \geq |N(x_{k-1}) \cup N(x)|. \tag{9}$$

When $x_{k-1}x_f \notin E(G)$, we get

$$d(x_{f-1}) + 1 > d(x_{k-1}) + 2 \geq |N(x_{k-1}) \cup N(x)|. \tag{10}$$

Now $|N(x_{k-1}) \cup N(x)| \geq n - \Delta\{x, x_{n-1}, x_{k-1}\} = n - d(x_{n-1})$, which implies that $d(x_{n-1}) + d(x_{f-1}) > n - 1$. However, this contradicts (3). Thus, Claim (A) is proved.

Claim (B). *If $x_{t-1}x_n \in E(G)$, then $d(x_{t-1}) \geq d(x_{n-1})$. If $x_{t-1}x_n \notin E(G)$, then $d(x_{t-1}) \geq d(x_{n-1}) - 1$.*

The proof of Claim (B) is similar to that of (A) and is omitted.

Thus, when x_{k-1} is adjacent to x_f or x_{t-1} is adjacent to x_n, we have

$$d(x_{k-1}) + d(x_{t-1}) \geq d(x_{n-1}) + d(x_{f-1}) - 1 = n - 2.$$

This contradicts (7).

When x_{k-1} is not adjacent to x_f and x_{t-1} is not adjacent to x_n, we have

$$d(x_{k-1}) + d(x_{t-1}) \geq d(x_{n-1}) + d(x_{f-1}) - 2 = n - 3.$$

Together with inequality (7), we have $d(x_{k-1}) + d(x_{t-1}) = n - 3$. Then clearly $x_{k-1}x_{t-1} \notin E(G)$ and $x_{k-1}x_1 \notin E(G)$ and $x_{k-1}x_f \notin E(G)$. Since $x_{k-1}x_f \in E(G)$ and $x_{t-1}x_n \notin E(G)$, none of the vertices of $\{x_{k-1}, x_{t-1}, x_1, x_f\}$ is adjacent to both x_{k-1} and x_{t-1}. Together with the fact that $d(x_{k-1}) + d(x_{t-1}) = n - 3$, we see that x_{k-1} and x_{t-1} must have at least one common neighbor. Thus, $\{x, x_{t-1}, x_{k-1}\}$ is an EIS. Without loss of generality, $d(x_{t-1}) \geq d(x_{k-1})$.

By condition (ii) of the Theorem, we have

$$d(x_{k-1}) + 2 = |N(x_{k-1}) \cup N(x)| \geq n - \Delta\{x, x_{t-1}, x_{k-1}\} = n - d(x_{t-1}),$$

which implies $d(x_{k-1}) + d(x_{t-1}) \geq n - 2$, a contradiction to inequality (7), completing the proof of the Theorem.

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