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ON MULTILINEAR GENERALIZATIONS OF THE CONCEPT OF NUCLEAR OPERATORS

BY

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Abstract. This paper introduces the class of Cohen p-nuclear m-linear operators between Banach spaces. A characterization in terms of Pietsch's domination theorem is proved. The interpretation in terms of factorization gives a factorization theorem similar to Kwapień's factorization theorem for dominated linear operators. Connections with the theory of absolutely summing m-linear operators are established. As a consequence of our results, we show that every Cohen p-nuclear $(1 < p \leq \infty)$ m-linear mapping on arbitrary Banach spaces is weakly compact.

1. Introduction and notation. The success of the theory of absolutely summing linear operators has motivated the investigation of new classes of multilinear mappings and polynomials between Banach spaces. The first possible directions of a multilinear theory of absolutely summing multilinear mappings were outlined by several authors (we mention, for example, [1, 2, 13–16, 19, 22, 23, 25, 26]).

The aim of this paper is to introduce and study a new class of multilinear operators, the Cohen p-nuclear multilinear operators. The space $\mathcal{N}_p^m$ of Cohen p-nuclear multilinear operators defined on Banach spaces is a Banach space and this kind of m-linear operators satisfy a natural analog of the Pietsch domination theorem. The original motivation for our research is to give a multilinear version of Kwapień's factorization theorem: $\mathcal{N}_p^m = \mathcal{D}_p^m \circ (\Pi_p, \ldots, \Pi_p)$ where Π_p is the Banach space of all p-summing linear operators and $\mathcal{D}_p^m$ the Banach space of all Cohen strongly p-summing multilinear operators. We also show that every Cohen p-nuclear $(1 < p \leq \infty)$ m-linear mapping on arbitrary Banach spaces is weakly compact.

This paper is organized as follows. In Section 1, we give some basic definitions and properties. In Section 2, we introduce a multilinear version of Cohen p-nuclear operators for which the resulting vector space is a Banach space. We prove a natural analog of the Pietsch domination theorem for such operators similar to the linear case. In Section 3, we characterize

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the class of Cohen p-nuclear m-linear operators as products of absolutely p-summing and Cohen strongly p-summing m-linear operators, generalizing a linear result of Kwapień. Finally, in Section 4, we obtain certain connections between the classes investigated in this paper (for other recent papers comparing different classes of multilinear mappings related to summability, we refer to [4–7, 20, 21]) and apply our results to prove that every Cohen p-nuclear ($1 < p \leq \infty$) m-linear mapping on arbitrary Banach spaces is weakly compact.

Now, we fix the notation used in this paper. Let $m \in \mathbb{N}$ and $X_1, \ldots, X_m$, Y be Banach spaces over $\mathbb{K}$ (real or complex scalar field). We denote by $\mathcal{L}(X_1, \ldots, X_m; Y)$ the Banach space of all continuous m-linear mappings from $X_1 \times \cdots \times X_m$ to Y, under the norm $\|T\| = \sup_{x_k \in B_{X_k}} \|T(x_1, \ldots, x_m)\|$, where B_{X_k} denotes the closed unit ball of X_k. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, \ldots, X_m)$. In the case $X_1 = \cdots = X_n = X$, we simply write $\mathcal{L}(^m X; Y)$.

Let now X be a Banach space and $1 \leq p < \infty$. We denote by $l_p^n(X)$ the space of all sequences $(x_i)_{1 \leq i \leq n}$ in X with the norm

$$\|(x_i)_{1 \leq i \leq n}\|_{l_p^n(X)} = \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p},$$

and by $(l_p^n)^\omega(X)$ the space of all sequences $(x_i)_{1 \leq i \leq n}$ in X with the norm

$$\|(x_i)_{1 \leq i \leq n}\|_{(l_p^n)^\omega}(X) = \sup_{\|\xi\|_{X^*}=1} \Big(\sum_{i=1}^{n} |\langle x_i, \xi \rangle|^p \Big)^{1/p},$$

where X^* denotes the topological dual of X.

Let $l_p(X)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\|(x_i)_{i=1}^\infty\|_{l_p(X)} = \Big(\sum_{i=1}^{\infty} \|x_i\|^p \Big)^{1/p}.$$

We denote by $l_p^\omega(X)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\|(x_i)_{i=1}^\infty\|_{l_p^\omega(X)} = \sup_{\|\xi\|_{X^*}=1} \|(\xi(x_i))_{i=1}^\infty\|_{l_p(X)}.$$

If $p = \infty$ we consider bounded sequences and in $l_\infty(X)$ we use the sup norm.

We know (see [12]) that $l_p(X) = l_p^\omega(X)$ for some $1 \leq p < \infty$ if, and only if, $\dim(X)$ is finite. If $p = \infty$, we have $l_\infty(X) = l_\infty^\omega(X)$. If K is a Hausdorff compact topological space, $C(K)$ denotes the Banach space, under the supremum norm, of all continuous functions on K. We denote by $\mathcal{L}_{\mathrm{f}}(X_1, \ldots, X_m; Y)$ the space of all m-linear mappings *of finite type*, which is generated by the mappings of the special form

$$T_{y\otimes_{j=1}^m x_j^*} = x_1^* \otimes \cdots \otimes x_m^* \otimes y : (x^1, \ldots, x^m) \mapsto x_1^*(x^1) \ldots x_m^*(x^m) y$$

for some non-zero $x_j^* \in X_j^*$ $(1 \leq j \leq m)$ and $y \in Y$. In [16], the *adjoint* of an m-linear operator is defined as follows: $T^* : Y^* \to \mathcal{L}(X_1, \ldots, X_m)$, $y^* \mapsto T^*(y^*) : X_1 \times \cdots \times X_m \to \mathbb{K}$, with $T^*(y^*)(x^1, \ldots, x^m) = y^*(T(x^1, \ldots, x^m))$. Recall that a *p-summing linear operator* $u : X \to Y$ (notation: $u \in \Pi_p(X;Y)$) between Banach spaces transforms p-weakly summing sequences into p-strongly summing sequences, i.e.

$$\|(u(x_i))_{1\leq i\leq n}\|_{l_p^n(X)} \leq C \|(x_n)_{1\leq i\leq n}\|_{(l_p^n)^\omega(X)}.$$

The infimum of the C defines a norm π_p on $\Pi_p(X;Y)$ (see [24, 12]).

• Following [23], an *ideal of multilinear mappings* (or *multi-ideal*) is a subclass $\mathcal{M}$ of all continuous multilinear mappings between Banach spaces such that for all $m \in \mathbb{N}$ and Banach spaces $X_1, \ldots, X_m$ and Y, the components $\mathcal{M}(X_1, \ldots, X_m; Y) := \mathcal{L}(X_1, \ldots, X_m; Y) \cap \mathcal{M}$ satisfy:

(i) $\mathcal{M}(X_1, \ldots, X_m; Y)$ is a linear subspace of $\mathcal{L}(X_1, \ldots, X_m; Y)$ which contains the m-linear mappings of finite type.
(ii) The ideal property: If $T \in \mathcal{M}(G_1, \ldots, G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \ldots, m$ and $v \in \mathcal{L}(F, Y)$, then $v \circ T \circ (u_1, \ldots, u_m)$ is in $\mathcal{M}(X_1, \ldots, X_m; Y)$.

If $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ satisfies

(i′) $(\mathcal{M}(X_1, \ldots, X_m; Y), \|\cdot\|_{\mathcal{M}})$ is a normed (Banach) space for all Banach spaces $X_1, \ldots, X_m$ and Y and all m,
(ii″) $\|T^m : \mathbb{K}^m \to \mathbb{K} : T^m(x^1, \ldots, x^m) = x^1 \ldots x^m\|_{\mathcal{M}} = 1$ for all m,
(iii‴) if $T \in \mathcal{M}(G_1, \ldots, G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, \ldots, m$ and $v \in \mathcal{L}(F, Y)$, then $\|v \circ T \circ (u_1, \ldots, u_m)\|_{\mathcal{M}} \leq \|v\| \, \|T\|_{\mathcal{M}} \|u_1\| \ldots \|u_m\|$,

then $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is called a *normed* (*Banach*) *multi-ideal*.

We begin by presenting different classes of ideals of multilinear mappings related to the concept of absolutely summing operator:

• Let $m \in \mathbb{N}$. An m-linear operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is *Cohen strongly p-summing* $(1 < p \leq \infty)$ if there exists a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j$ $(1 \leq j \leq m)$ and any $y_1^*, \ldots, y_n^* \in Y^*$, we have

$$\|(\langle T(x_i^1, \ldots, x_i^m), y_i^* \rangle)_{1\leq i\leq n}\|_{l_1^n} \leq C \Big(\sum_{i=1}^n \prod_{j=1}^m \|x_i^j\|_{X_j}^p \Big)^{1/p} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}.$$

Again the class of all Cohen strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into Y, denoted by $\mathcal{D}_p^m(X_1, \ldots, X_m; Y)$, is a Banach space with the norm $d_p^m(T)$ which is the smallest constant C as above. For $p = 1$, we have $\mathcal{D}_1^m(X_1, \ldots, X_m, Y) = \mathcal{L}(X_1, \ldots, X_m; Y)$.

It is well known (see [1, Theorem 2.4]) that T is Cohen strongly p-summing ($1 < p \leq \infty$) if, and only if, there exists a constant $C > 0$ and a Radon probability measure μ on $B_{Y^{**}}$ such that for all $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$ and $y^* \in Y^*$, we have

$$|\langle T(x^1, \ldots, x^m), y^* \rangle| \leq C \prod_{j=1}^{m} \|x^j\| \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu \Big)^{1/p^*}. \tag{1}$$

• A multilinear operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is *r-dominated* ($1 \leq r < \infty$) if there exists a constant $C > 0$ and Borel probabilities μ_j on $B_{X_j^*}$ ($1 \leq j \leq m$) such that

$$\|T(x^1, \ldots, x^m)\| \leq C \prod_{j=1}^{m} \Big(\int_{B_{X_j^*}} |x^j(x^*)|^r d\mu_j(x^*) \Big)^{1/r} \tag{2}$$

for every $x^j \in X_j$. Moreover, in this case we define

$$\delta_r(T) = \inf\{C > 0 : C \text{ satisfies } (2)\}.$$

Consequently, r_1-dominated implies r_2-dominated for $r_1 \leq r_2$. We denote by $\mathcal{L}_{\mathrm{d}}^r(X_1, \ldots, X_m; Y)$ the vector space of all r-dominated m-linear operators T from $X_1 \times \cdots \times X_m$ into Y, which is a quasi-Banach space with the quasi-norm $\delta_r(T)$. If $r > m$, then $\delta_r(T)$ is a norm.

• We say that an m-linear operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is *absolutely p-summing* ($1 \leq p < \infty$) if there is a constant $C > 0$ such that for any $n \in \mathbb{N}$ and $(x_i^j)_{1 \leq i \leq n} \subset X_j$ ($1 \leq j \leq m$), we have

$$\Big(\sum_{i=1}^{n} \|T(x_i^1, \ldots, x_i^m)\|^p \Big)^{1/p} \leq C \prod_{j=1}^{m} \|(x_i^j)_{1 \leq i \leq n}\|_{(l_p^n)^\omega}(X_j). \tag{3}$$

The space of all absolutely p-summing m-linear mappings from $X_1 \times \cdots \times X_m$ into Y will be denoted by $\mathcal{L}_{\mathrm{as},p}(X_1, \ldots, X_m; Y)$, and the infimum of the C for which (3) always holds defines a norm $\|T\|_{\mathrm{as},p}$ on $\mathcal{L}_{\mathrm{as},p}(X_1, \ldots, X_m; Y)$.

• Let $1 \leq p \leq \infty$. Then $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is *strongly p-summing* if there exists a constant $C > 0$ such that for every $x_1^j, \ldots, x_n^j \in X_j$ ($1 \leq j \leq m$) we have

$$\Big(\sum_{i=1}^{n} \|T(x_i^1, \ldots, x_i^m)\|^p \Big)^{1/p} \leq C \sup_{\phi \in B_{\mathcal{L}(X_1, \ldots, X_m)}} \Big(\sum_{i=1}^{n} |\phi(x_i^1, \ldots, x_i^m)|^p \Big)^{1/p}. \tag{4}$$

Again the class of all strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into Y, denoted by $\mathcal{L}_{\mathrm{sas},p}(X_1, \ldots, X_m; Y)$, is a Banach space with the norm $\|T\|_{\mathrm{sas},p}$ which is the smallest constant C such that (4) holds.

• An m-linear operator $T : X_1 \times \cdots \times X_m \to Y$ is *fully* (or *multiple*) *p-summing* if there is a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j$ $(1 \le j \le m)$ we have

$$(5) \qquad \Big(\sum_{i_1,\ldots,i_m=1}^{\infty} \|T(x_{i_1}^1, \ldots, x_{i_m}^m)\|^p \Big)^{1/p} \le C \prod_{j=1}^{m} \|(x_i^j)_{i=1}^{\infty}\|_{l_p^w(X_j)}.$$

We denote the vector space of all such mappings by $\mathcal{L}_{\mathrm{fas},p}(X_1, \ldots, X_m; Y)$, and the smallest C satisfying (5) by $\|T\|_{\mathrm{fas},p}$. This defines a norm on $\mathcal{L}_{\mathrm{fas},p}(X_1, \ldots, X_m; Y)$.

• We say that $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is *p-semi-integral* (notation: $T \in \mathcal{L}_{\mathrm{si},p}(X_1, \ldots, X_m; Y)$) if there exists a constant $C \ge 0$ and a regular probability measure μ on the Borel σ-algebra of $B_{X_1^*} \times \cdots \times B_{X_m^*}$ endowed with the product of the weak star topologies $\sigma(X_j^*, X_j)$, $1 \le j \le m$, such that

$$\|T(x^1, \ldots, x^m)\| \le C \Big(\int_{B_{X_1^*} \times \cdots \times B_{X_m^*}} |\varphi_1(x^1) \ldots \varphi_m(x^m)|^p d\mu(\varphi_1, \ldots, \varphi_m) \Big)^{1/p}$$

for every $x^j \in X_j$ and $j = 1, \ldots, m$. The infimum of the C defines a norm $\|\cdot\|_{\mathrm{si},p}$ on the space of p-semi-integral mappings.

It is well known [7, Theorem 1] that $T \in \mathcal{L}_{\mathrm{si},p}(X_1, \ldots, X_m; Y)$ if and only if there exists $C \ge 0$ such that

$$(6) \qquad \Big(\sum_{i=1}^{n} \|T(x_i^1, \ldots, x_i^m)\|^p \Big)^{1/p} \le C \sup_{\phi^j \in B_{X_j^*}, j=1,\ldots,m} \Big(\sum_{i=1}^{n} |\phi^1(x_i^1) \ldots \phi^m(x_i^m)|^p \Big)^{1/p}.$$

Since Pietsch's paper [23], several generalizations of absolutely summing operators to the multilinear setting have been investigated. The ideal of Cohen strongly p-summing multilinear operators was introduced by of Achour–Mezrag [1]. Dominated mappings were first explored by Geiss [14], Schneider [26] and Matos [15]. The ideal of strongly p-summing multilinear operators was introduced by Dimant [13]. The ideal of multiple summing, also called fully summing, multilinear mappings was first vaguely sketched by Ramanujan and Schock [25], and introduced independently by Matos [16] and Pérez-García and Villanueva [22], and exhaustively explored in recent years (we mention, for example, [17, 20]). The semi-integral mappings were introduced by Alencar–Matos [2].

2. Cohen p-nuclear mappings. We will extend to multilinear operators the class of p-nuclear operators introduced by Cohen [10] and generalized to Cohen (p,q)-nuclear operators by Apiola [3]. We prove directly the principal result of this section, which is the domination theorem.

For the convenience of the reader we start by recalling the linear case. A linear operator T between Banach spaces X, Y is *Cohen p-nuclear* (for $1 < p < \infty$) if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1^*, \ldots, y_n^* \in Y^*$ we have

$$\Big|\sum_{i=1}^{n} \langle T(x_i), y_i^*\rangle\Big| \le C \sup_{x^* \in B_{X^*}} \|(x^*(x_i))\|_{l_p^n} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}.$$

The smallest constant C, denoted by $n_p(T)$, such that the above inequality holds, is called the *Cohen p-nuclear norm* on the space $\mathcal{N}_p(X,Y)$ of all Cohen p-nuclear operators from X into Y, which is a Banach space. For $p = 1$ and $p = \infty$ we have $\mathcal{N}_1(X,Y) = \Pi_1(X,Y)$ and $\mathcal{N}_\infty(X,Y) = \mathcal{D}_\infty(X,Y)$ (for $1 < p \le \infty$, $\mathcal{D}_p(X,Y)$ is the Banach space of all strongly p-summing linear operators, see [10]).

We now give our definition.

Definition 2.1. An m-linear operator $T : X_1 \times \cdots \times X_m \to Y$ is *Cohen p-nuclear* $(1 < p < \infty)$ if there is a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j$ $(1 \le j \le m)$ and any $y_1^*, \ldots, y_n^* \in Y^*$, we have

$$\begin{aligned}(7)\qquad &\Big|\sum_{i=1}^{n} \langle T(x_i^1, \ldots, x_i^m), y_i^*\rangle\Big| \\ &\qquad \le C\Big(\sup_{\substack{x^{j*} \in B_{X_j^*} \\ 1 \le j \le m}} \sum_{i=1}^{n} \prod_{j=1}^{m} |\langle x_i^j, x^{j*}\rangle|^p\Big)^{1/p} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}.\end{aligned}$$

Again the class of all Cohen p-nuclear m-linear operators from $X_1 \times \cdots \times X_m$ into Y, which is denoted by $\mathcal{N}_p^m(X_1, \ldots, X_m; Y)$, is a Banach space with the norm $n_p^m(T)$, which is the smallest constant C such that (7) holds.

For $p = \infty$, (7) becomes

$$\Big|\sum_{i=1}^{n} \langle T(x_i^1, \ldots, x_i^m), y_i^*\rangle\Big| \le C\Big(\sup_{1 \le i \le n} \prod_{j=1}^{m} \|x_i^j\|_{X_j}\Big) \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_1^n}.$$

It is clear that every $T \in \mathcal{N}_p^m(X_1, \ldots, X_m; Y)$ is continuous and $\|T\| \le n_p^m(T)$.

Proposition 2.2.

(a) $\mathcal{N}_1^m(X_1, \ldots, X_m; Y) = \mathcal{L}_{\mathrm{si},1}(X_1, \ldots, X_m; Y)$.

(b) $\mathcal{N}_\infty^m(X_1, \ldots, X_m; Y) = \mathcal{D}_\infty^m(X_1, \ldots, X_m; Y)$.

Proof. (a) Let $T \in \mathcal{N}_1^m(X_1,\dots,X_m;Y)$. Then

$$\Big|\sum_{i=1}^n \langle T(x_i^1,\dots,x_i^m),y_i^*\rangle\Big| \le n_1^m(T)\Big(\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|\Big)\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_\infty^n}$$
$$\le n_1^m(T)\Big(\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|\Big)\sup_i\|y_i^*\|.$$

On the other hand, we have

$$\sum_{i=1}^n\|T(x_i^1,\dots,x_i^m)\| = \sup\Big\{\Big|\sum_{i=1}^n\langle T(x_i^1,\dots,x_i^m),y_i^*\rangle\Big| : \sup_i\|y_i^*\|\le 1\Big\}.$$

This implies

$$\sum_{i=1}^n\|T(x_i^1,\dots,x_i^m)\| \le n_1^m(T)\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|.$$

Thus by (6), T is 1-semi-integral and $\|T\|_{\mathrm{si},1}\le n_1^m(T)$.

Conversely, let T be a 1-semi-integral m-linear operator. We have

$$\Big|\sum_{i=1}^n\langle T(x_i^1,\dots,x_i^m),y_i^*\rangle\Big| \le \sup_i\|y_i^*\|\sum_{i=1}^n\|T(x_i^1,\dots,x_i^m)\|$$
$$\le \|T\|_{\mathrm{si},1}\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|\sup_i\|y_i^*\|$$
$$\le \|T\|_{\mathrm{si},1}\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_\infty^n}.$$

Thus T is a Cohen 1-nuclear m-linear operator and $n_1^m(T)\le\|T\|_{\mathrm{si},1}$.

(b) is obvious. ∎

EXAMPLE 2.3. Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K and let $1\le p<\infty$. Each $g\in L_p(\mu)$ defines an m-linear multiplication operator $T_g\in\mathcal{L}({}^mC(K);L_1(\mu))$ with $T_g(f^1,\dots,f^m)=g\cdot f^1\cdot\dots\cdot f^m$. This map is Cohen p-nuclear and $n_p^m(T_g)=\|g\|_{L_p(\mu)}$.

The next proposition asserts that $(\mathcal{N}_p^m(X_1,\dots,X_m;Y),n_p^m(T))$ is a normed (Banach) multi-ideal. We omit the proof.

PROPOSITION 2.4.

(i) *Every m-linear mapping of finite type is Cohen p-nuclear, that is,* $\mathcal{L}_{\mathrm{f}}(X_1,\dots,X_m;Y)\subset\mathcal{N}_p^m(X_1,\dots,X_m;Y)$.

(ii) (Ideal property) *If* $T\in\mathcal{N}_p^m(X_1,\dots,X_m;Y)$, $u_j\in\mathcal{L}(E_j,X_j)$, $j=1,\dots,m$, *and* $w\in\mathcal{L}(Y,Z)$, *then* $w\circ T\circ(u_1,\dots,u_m)$ *is Cohen p-nuclear and*

$$n_p^m(w\circ T\circ(u_1,\dots,u_m))\le\|w\|n_p^m(T)\prod_{j=1}^m\|u_j\|.$$

(iii) $n_p^m(T^m:\mathbb{K}^m\to\mathbb{K}:T^m(x^1,\dots,x^m)=x^1\dots x^m)=1$ *for all* m.

This class satisfies a Pietsch domination theorem which is the principal result of this section. For the proof we will use Ky Fan's lemma (see [12, p. 190]).

KY FAN'S LEMMA. *Let E be a Hausdorff topological vector space, and let $\mathcal{C}$ be a compact convex subset of E. Let M be a set of functions on $\mathcal{C}$ with values in $(-\infty,\infty]$ having the following properties:*

(a) *each $f\in M$ is convex and lower semicontinuous;*

(b) *if $g\in\operatorname{conv}(M)$, then there is an $f\in M$ with $g(x)\le f(x)$ for every $x\in\mathcal{C}$;*

(c) *there is an $r\in\mathbb{R}$ such that each $f\in M$ has a value not greater than r.*

Then there is an $x_0\in\mathcal{C}$ such that $f(x_0)\le r$ for all $f\in M$.

THEOREM 2.5. *For $T\in\mathcal{L}(X_1,\dots,X_m;Y)$ and $1<p<\infty$, the following conditions are equivalent:*

(i) *The operator T is Cohen p-nuclear.*

(ii) *No matter how we choose finitely many vectors $x_1^j,\dots,x_n^j$ in X_j $(1\le j\le m)$ and $y_1^*,\dots,y_n^*$ in Y^*, we have*

$$\sum_{i=1}^n|\langle T(x_i^1,\dots,x_i^m),y_i^*\rangle|\le n_p^m(T)\Big(\sup_{\substack{x^{j*}\in B_{X_j^*}\\1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|^p\Big)^{1/p}\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}.$$

(iii) *There exist Radon probability measures $\mu_j\in C(B_{X_j^*})^*$ $(1\le j\le m)$ and $\lambda\in C(B_{Y^{**}})^*$ such that for all $(x^1,\dots,x^m)\in X_1\times\dots\times X_m$ and $y^*\in Y^*$,*

(8) $$|\langle T(x^1,\dots,x^m),y^*\rangle| \le C\prod_{j=1}^m \|x_j\|_{L_p(B_{X_j^*},\mu_j)}\|y^*\|_{L_{p^*}(B_{Y^{**}},\lambda)}.$$

Proof. The implication (i)⇒(ii) is trivial. The proof is omitted.

The main point of the proof, the implication (ii)⇒(iii), follows the ideas of [14] and [1]. We consider the sets $P(B_{X_j^*})$ $(1\le j\le m)$ and $P(B_{Y^{**}})$ of probability measures in $C(B_{X_j^*})^*$ and $C(B_{Y^{**}})^*$, respectively. These are convex sets which are compact when we endow $C(B_{X_j^*})^*$ and $C(B_{Y^{**}})^*$ with their weak* topologies. We are going to apply Ky Fan's lemma with $E=C(B_{X_1^*})^*\times\cdots\times C(B_{X_m^*})^*\times C(B_{Y^{**}})^*$ and $\mathcal{C}=P(B_{X_1^*})\times\cdots\times P(B_{X_m^*})\times P(B_{Y^{**}})$.

Consider the set M of all functions $f:\mathcal{C}\to\mathbb{R}$ for which there exist $x_1^j,\dots,x_n^j\in X_j$ $(j=1,\dots,m)$ and $y_1^*,\dots,y_n^*\in Y^*$ such that

$$\begin{aligned}&f(\mu_1,\dots,\mu_m,\lambda)\\&\quad:=\sum_{i=1}^n|\langle T(x_i^1,\dots,x_i^m),y_i^*\rangle|-\frac{C}{p}\sum_{i=1}^n\prod_{j=1}^m\int_{B_{X_j^*}}|\langle x_i^j,x^{j*}\rangle|^p\,d\mu_j(x^{*j})\\&\qquad-\frac{C}{p^*}\sum_{i=1}^n\int_{B_{Y^{**}}}|\langle y_i^*,y^{**}\rangle|^{p^*}\,d\lambda(y^{**})\end{aligned}$$

for all $(\mu_1,\dots,\mu_m,\lambda)\in\mathcal{C}$. It is clear that all such f are continuous and affine and that the set M is a convex cone and consequently conditions (a) and (b) of Ky Fan's lemma are satisfied.

For condition (c), since $B_{X_j^*}$ and $B_{Y^{**}}$ are weak* compact and norming, there exist for $f\in M$ elements $x_0^{*j}\in B_{X_j^*}$ and $y_0\in B_{Y^{**}}$ such that

$$\sup_{\substack{x^{j*}\in B_{X_j^*}\\1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|^p=\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x_0^{j*}\rangle|^p$$

and

$$\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}^{p^*}=\sum_{i=1}^n|\langle y_i^*,y_0\rangle|^{p^*}.$$

Using the elementary identity

(9) $$\alpha\beta=\inf_{\epsilon>0}\left\{\frac1p\left(\frac{\alpha}{\epsilon}\right)^p+\frac{1}{p^*}(\epsilon\beta)^{p^*}\right\},\quad\forall\alpha,\beta\in\mathbb{R}_+^*,$$

we find by taking

$$\alpha=\Big(\sup_{\substack{x^{j*}\in B_{X_j^*}\\1\le j\le m}}\sum_{i=1}^n\prod_{j=1}^m|\langle x_i^j,x^{j*}\rangle|^p\Big)^{1/p},\qquad\beta=\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}$$

and $\epsilon = 1$ that

$$\begin{aligned} & f(\delta_{x_0^{*1}}, \ldots, \delta_{x_0^{*m}}, \delta_{y_0}) \\ &= \sum_{i=1}^{n} |\langle T(x_i^1, \ldots, x_i^m), y_i^* \rangle| - \frac{C}{p}\Big(\sup_{\substack{x^{j*} \in B_{X_j^*} \\ \leq j \leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} |\langle x_i^j, x^{j*} \rangle|^p \Big) \\ &\quad - \frac{C}{p^*} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}^{p^*} \\ &\leq \sum_{i=1}^{n} |\langle T(x_i^1, \ldots, x_i^m), y_i^* \rangle| \\ &\quad - C\Big(\sup_{\substack{x^{j*} \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} |\langle x_i^j, x^{j*} \rangle|^p \Big)^{1/p} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}, \end{aligned}$$

where δ_x is the Dirac measure at x. The last quantity is less than or equal to zero (by hypothesis (ii)) and hence condition (c) is satisfied with $r = 0$. By Ky Fan's lemma, there is $(\mu_1, \ldots, \mu_m, \lambda) \in \mathcal{C}$ with $f(\mu_1, \ldots, \mu_m, \lambda) \leq 0$ for all $f \in M$. Then, if f is generated by the single elements $(x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$ and $y^* \in Y^*$,

$$\begin{aligned} & |\langle T(x^1, \ldots, x^m), y^* \rangle| \\ &\leq \frac{C}{p} \prod_{j=1}^{m} \int_{B_{X_j^*}} |\langle x_i^j, x^{*j} \rangle|^p d\mu_j(x^{*j}) + \frac{C}{p^*} \int_{B_{Y^{**}}} |\langle y_i^*, y^{**} \rangle|^{p^*} d\lambda(y^{**}). \end{aligned}$$

Fix $\epsilon > 0$. Replacing x^j by $\epsilon^{-1/m} x^j$, y^* by ϵy^* and taking the infimum over all $\epsilon > 0$ (using the elementary identity (9)), we find

$$\begin{aligned} & |\langle T(x^1, \ldots, x^m), y^* \rangle| \\ &\leq C\Big[\frac{1}{p} \Big(\Big(\prod_{j=1}^{m} \int_{B_{X_j^*}} |\langle x_i^j, x^{*j} \rangle|^p \, d\mu_j(x^{*j}) \Big)^{1/p} / \epsilon \Big)^p \\ &\quad + \frac{1}{p^*} \Big(\epsilon \Big(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} \, d\lambda(y^{**}) \Big)^{1/p^*} \Big)^{p^*} \Big] \\ &\leq C \prod_{j=1}^{m} \Big(\int_{B_{X_j^*}} |\langle x^j, x^{*j} \rangle|^p \, d\mu_j(x^{*j}) \Big)^{1/p} \Big(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} \, d\lambda(y^{**}) \Big)^{1/p^*}. \end{aligned}$$

Now we prove that (iii) implies (i). Let $(x_i^1, \ldots, x_i^m) \in X_1 \times \cdots \times X_m$ and $y_i^* \in Y^*$. By (8), we have

$$|\langle T(x_i^1,\ldots,x_i^m),y_i^*\rangle| \le C\prod_{j=1}^m \|x_i^j\|_{L_p(B_{X_j^*},\mu_j)}\|y_i^*\|_{L_{p^*}(B_{Y^{**}},\lambda)}$$

for all $1\le i\le n$, and so

$$\Big|\sum_{i=1}^n\langle T(x_i^1,\ldots,x_i^m),y_i^*\rangle\Big| \le \sum_{i=1}^n|\langle T(x_i^1,\ldots,x_i^m),y_i^*\rangle|$$
$$\le C\sum_{i=1}^n\Big(\prod_{j=1}^m\|x_i^j\|_{L_p(B_{X_j^*},\mu_j)}\|y_i^*\|_{L_{p^*}(B_{Y^{**}},\lambda)}\Big).$$

We use Hölder's inequality to obtain

$$\Big|\sum_{i=1}^n\langle T(x_i^1,\ldots,x_i^m),y_i^*\rangle\Big|$$
$$\le C\Big(\sum_{i=1}^n\prod_{j=1}^m\|x_i^j\|^p_{L_p(B_{X_j^*},\mu_j)}\Big)^{1/p}\Big(\sum_{i=1}^n\|y_i^*\|_{L_{p^*}(B_{Y^{**}},\lambda)}{}^{p^*}\Big)^{1/p^*}$$
$$=C\Big(\sum_{i=1}^n\int\limits_{B_{X_1^*}\times\cdots\times B_{X_m^*}}|x^{1*}(x_i^1)\ldots x^{m*}(x_i^m)|^p d(\mu_1\otimes\cdots\otimes\mu_m)(x^{1*},\ldots,x^{m*})\Big)^{1/p}$$
$$\cdot\Big(\sum_{i=1}^n\int\limits_{B_{Y^{**}}}|y_i^*(y^{**})|^{p^*}d\lambda(y^{**})\Big)^{1/p^*}$$
$$\le C\Big(\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\le j\le m}}\sum_{i=1}^n|x^{1*}(x_i^1)\ldots x^{m*}(x_i^m)|^p\Big)^{1/p}\sup_{y\in B_Y}\Big(\sum_{i=1}^n|y_i^*(y)|^{p^*}\Big)^{1/p^*}.$$

Therefore T is Cohen p-nuclear and $n_p^m(T)\le C$, as we wanted to prove. ∎

3. Kwapień's factorization theorem. Comparing condition (iii) of Theorem 2.5 with condition (b) of [11, Corollary 19.2], it is legitimate to say that Cohen p-nuclear multilinear operators are a generalization of $(p;p^*)$-dominated linear operators. Therefore the following theorem can be regarded as a multilinear version of Kwapień's factorization theorem.

THEOREM 3.1 (Kwapień's Factorization Theorem). *Let $1<p<\infty$. Then $T\in\mathcal{L}(X_1,\ldots,X_m;Y)$ is Cohen p-nuclear if and only if there exist Banach spaces $G_1,\ldots,G_m$, absolutely p-summing linear operators $u_j\in\mathcal{L}(X_j,G_j)$ and a Cohen strongly p-summing m-linear mapping $S\in\mathcal{L}(G_1,\ldots,G_m;Y)$ such that $T=S(u_1,\ldots,u_m)$ Moreover,*

$$n_p^m(T)=\sup\Big\{d_p^m(S)\prod_{j=1}^m\pi_p(u_j):T=S\circ(u_1,\ldots,u_m)\Big\}$$

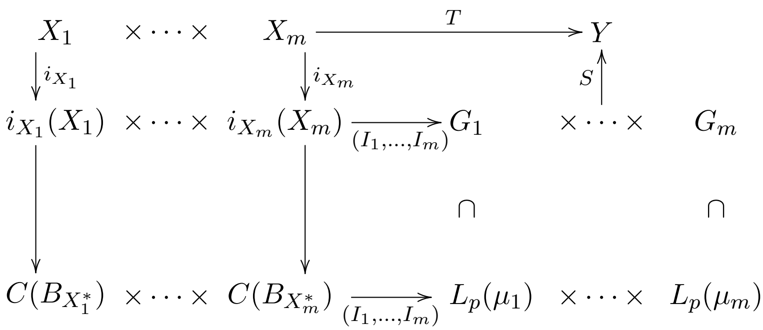
(i.e. $\mathcal{N}_p^m=\mathcal{D}_p^m\circ(\Pi_p,\ldots,\Pi_p)$ isometrically).

Proof. The "if" part follows from a straightforward combination of Theorem 2.4 with Pietsch's domination theorem for absolutely p-summing linear operators.

To prove the "only if" part, take $T \in \mathcal{N}_p^m(X_1, \dots, X_m; Y)$. Then, by (8), there exist Radon probability measures $\mu_j \in C(B_{X_j^*})^*$ $(1 \le j \le m)$ and $\lambda \in C(B_{Y^{**}})^*$ such that for all $(x^1, \dots, x^m) \in X_1 \times \cdots \times X_m$ and $y^* \in Y^*$,

$$|\langle T(x^1, \dots, x^m), y^* \rangle| \le C \prod_{j=1}^{m} \|x_j\|_{L_p(B_{X_j^*}, \mu_j)} \|y^*\|_{L_{p^*}(B_{Y^{**}}, \lambda)}.$$

Let $(x^1, \dots, x^m) \in X_1 \times \cdots \times X_m$. Define $u_j^0(x^j) := \langle \cdot, x^j \rangle \in C(B_{X_j^*})$ and consider the diagram



where $I_j : C(B_{X_j^*}) \to L_p(\mu_j)$ is the canonical injection, $i_{X_j} : X_j \to C(K_j)$ is the natural isometric injection and G_j is the closure of the space $I_j \circ u_j^0(X_j)$, $u_j(x^j) := I_j(u_j^0(x^j))$. Since $\pi_p(I_j) = 1$ and $\|u_j^0\| = 1$, it follows that $\pi_p(u_j) \le 1$.

The operator S is defined on $u_1(X_1) \times \cdots \times u_m(X_m)$, $u_j(X_j) = I_j(u_j^0(x^j))$ $(1 \le j \le m)$, by

$$S(u_1(x^1), \dots, u_m(x^m)) := T(x^1, \dots, x^m),$$

and this definition makes sense because

$$|\langle S(u_1(x^1), \dots, u_m(x^m)), y^* \rangle| \le n_p^m(T) \prod_{j=1}^{m} \|u_j(x^j)\|_{G_j} \Big(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} d\lambda(y^{**}) \Big)^{1/p^*}.$$

It follows that S is continuous on $u_1(X_1) \times \cdots \times u_m(X_m)$ and has a unique extension to $\overline{u_1(X_1)} \times \cdots \times \overline{u_m(X_m)} = G_1 \times \cdots \times G_m$; moreover, the inequality implies that

$$\|S^*(y^*)\| = \sup\{|\langle S^*(y^*), (u_1(x^1), \ldots, u_m(x^m))\rangle| : \|u_j(x^j)\| \leq 1\}$$
$$\leq n_p^m(T)\Big(\int_{B_{Y^{**}}} |\langle y^*, y^{**}\rangle|^{p^*} d\lambda(y^{**})\Big)^{1/p^*},$$

which means that S^* is absolutely p^*-summing. From [18, Theorem 2.7], S is a Cohen strongly p-summing m-linear operator and $d_p^m(S) = \pi_{p^*}(S^*) \leq n_p^m(T)$. This ends the proof. ∎

EXAMPLE 3.2. The operator $T : l_1 \times l_1 \to l_1$ given by $T((x_k^1)_k, (x_k^2)_k) = (x_k^1 x_k^2)_k$ is 1-nuclear.

Proof. Let

$$S : l_2 \times l_2 \to l_1, \quad ((x_k^1)_k, (x_k^2)_k) \mapsto (x_k^1 x_k^2)_k,$$

for all $(x_k^1)_k, (x_k^2)_k \in l_2$. Then, for all $n \in \mathbb{N}$ and all $(x_{1,k}^j)_k, \ldots, (x_{n,k}^j)_k \in l_2$ $(1 \leq j \leq 2)$, $y_1^*, \ldots, y_n^* \in l_\infty$, we have

$$\begin{aligned}
&\sum_{i=1}^n |\langle S((x_{i,k}^1)_k, (x_{i,k}^2)_k), y_i^*\rangle| \\
&\qquad \leq \sum_{i=1}^n \|S((x_{i,k}^1)_k, (x_{i,k}^2)_k)\| \, \|y_i^*\| \\
&\qquad \leq \sum_{i=1}^n \Big(\sum_{k=1}^\infty |x_{i,k}^1|^2\Big)^{1/2} \Big(\sum_{k=1}^\infty |x_{i,k}^2|^2\Big)^{1/2} \sup_i \sup_{y \in B_{l_1}} |y_i^*(y)| \\
&\qquad \leq \sum_{i=1}^n \prod_{j=1}^2 \|(x_{i,k}^j)_k\|_{l_2} \sup_{y \in B_{l_1}} \sup_i |y_i^*(y)| \\
&\qquad \leq \sum_{i=1}^n \prod_{j=1}^2 \|(x_{i,k}^j)_k\|_{l_2} \sup_{y \in B_{l_1}} \|y_i^*(y)\|_{l_\infty}.
\end{aligned}$$

Thus S is Cohen strongly 1-summing. On the other hand, the canonical operator $I_j : l_1 \to l_2$ $(1 \leq j \leq 2)$ is 1-summing. We conclude by Theorem 3.1 that $T = S(I_1, I_2) : l_1 \times l_1 \to l_1$ is Cohen 1-nuclear. ∎

4. Relations between different classes of summability. In this section we will obtain certain inclusions between different classes investigated in this paper and establish the position of Cohen p-nuclear mappings with respect to other concepts. As a consequence of our results, we show that every Cohen p-nuclear $(1 < p \leq \infty)$ m-linear mapping on arbitrary Banach spaces is weakly compact.

We also need the definition of integral multilinear operators.

Definition 4.1. [8] We say that $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is *integral* (notation: $T \in \mathcal{I}(X_1, \ldots, X_m; Y)$) if there exists a constant $C \geq 0$ such that for every $m \in \mathbb{N}$, and all families $(x_i^j)_{1 \leq i \leq n} \subset X_j$ and $(y_i^*)_{1 \leq i \leq n} \subset Y^*$, we have

$$\Big| \sum_{i=1}^{n} \langle T(x_i^1, \ldots, x_i^m), y_i^* \rangle \Big| \leq C \sup_{\substack{x^{j*} \in B_{X_j^*} \\ 1 \leq j \leq m}} \Big\| \sum_{i=1}^{n} x^{1*}(x_i^1) \ldots x^{m*}(x_i^m) y_i^* \Big\|_{Y^*}.$$

The infimum of the C defines a norm $\| \cdot \|_I$ on the space of integral mappings. In the case $Y = \mathbb{K}$, this definition was given in [23] (see also [14]).

In [27], the author introduces the ideal of integral multilinear mappings as those satisfying a certain integral condition. Cilia and Gutiérrez [9] prove that the various definitions of integral multilinear mappings are equivalent.

Theorem 4.2 ([7, Theorem 3]).

(i) $\mathcal{L}_{\mathrm{d}}^{p}(X_1, \ldots, X_m; Y) \subset \mathcal{L}_{\mathrm{si},p}(X_1, \ldots, X_m; Y)$.
(ii) $\mathcal{L}_{\mathrm{si},p}(X_1, \ldots, X_m; Y) \subset \mathcal{L}_{\mathrm{fas},p}(X_1, \ldots, X_m; Y) \subset \mathcal{L}_{\mathrm{as},p}(X_1, \ldots, X_m; Y)$.
(iii) $\mathcal{L}_{\mathrm{si},p}(X_1, \ldots, X_m; Y) \subset \mathcal{L}_{\mathrm{sas},p}(X_1, \ldots, X_m; Y)$.

The following theorem yields inclusions of the class of Cohen p-nuclear mappings in other classes of multilinear mappings investigated in this paper.

Theorem 4.3. *Let $1 < p < \infty$, let $X_1, \ldots, X_m, Y$ be Banach spaces and let $T : X_1 \times \cdots \times X_m \to Y$ be an m-linear operator.*

(i) *If $T \in \mathcal{N}_p^m(X_1, \ldots, X_m; Y)$ then $T \in \mathcal{D}_p^m(X_1, \ldots, X_m; Y)$ and $d_p^m(T) \leq n_p^m(T)$.*
(ii) *If $T \in \mathcal{N}_p^m(X_1, \ldots, X_m; Y)$ then $T \in \mathcal{L}_{\mathrm{d}}^{r}(X_1, \ldots, X_m; Y)$ for all $r \geq p$ and $\delta_r(T) \leq n_p^m(T)$.*
(iii) *If $1 < p \leq 2$ and $T \in \mathcal{N}_p^m(X_1, \ldots, X_m; Y)$, then T is absolutely q-summing for all q.*

Proof. (i) If T is Cohen p-nuclear, then

$$\begin{aligned} &|\langle T(x^1, \ldots, x^m), y^* \rangle| \\ &\quad \leq n_p^m(T) \prod_{j=1}^{m} \Big(\int_{B_{X_j^*}} |\langle x^j, \xi^j \rangle|^p d\mu_j(\xi^j) \Big)^{1/p} \Big(\int_{B_{Y^{**}}} |\langle y^{**}, y^* \rangle|^{p^*} d\lambda(y^{**}) \Big)^{1/p^*} \\ &\quad \leq n_p^m(T) \prod_{j=1}^{m} \big(\sup_{\xi^j \in B_{X_j^*}} |\langle x^j, \xi^j \rangle| \big) \Big(\int_{B_{Y^{**}}} |\langle y^{**}, y^* \rangle|^{p^*} d\mu(y^{**}) \Big)^{1/p^*} \\ &\quad \leq n_p^m(T) \prod_{j=1}^{m} \|x^j\|_{X_j} \Big(\int_{B_{Y^{**}}} |\langle y^{**}, y^* \rangle|^{p^*} d\mu(y^{**}) \Big)^{1/p^*}. \end{aligned}$$

Thus, by (1), $T \in \mathcal{D}_p^m(X_1, \ldots, X_m; Y)$ and $d_p^m(T) \leq n_p^m(T)$.

(ii) If T is Cohen p-nuclear, then

$$\begin{aligned}
&\|T(x^1,\dots,x^m)\| \\
&\qquad = \sup_{y^*\in B_{Y^*}} |\langle T(x^1,\dots,x^m),y^*\rangle| \\
&\qquad \le n_p^m(T)\prod_{j=1}^m \|x^j\|_{L_p(B_{X_j^*},\mu_j)} \sup_{y^*\in B_{Y^*}}\Big(\int_{B_{Y^{**}}} |\langle y^{**},y^*\rangle|^{p^*}\,d\lambda(y^{**})\Big)^{1/p^*} \\
&\qquad \le n_p^m(T)\prod_{j=1}^m \|x^j\|_{L_p(B_{X_j^*},\mu_j)} \sup_{y^*\in B_{Y^*}} \|y^*\| \\
&\qquad \le n_p^m(T)\prod_{j=1}^m \|x^j\|_{L_r(B_{X_j^*},\mu_j)}.
\end{aligned}$$

Thus, by (2), T is r-dominated and $\delta_r(T)\le n_p^m(T)$.

(iii) By (ii), T is 2-dominated ($r=2$); hence Proposition 4.3 in [6] shows that T is absolutely q-summing for all q. ■

THEOREM 4.4. *Every integral m-linear operator is Cohen p-nuclear.*

Proof. Let $T\in\mathcal{L}(X_1,\dots,X_m;Y)$. If T is integral, we can use Hölder's inequality to write

$$\begin{aligned}
&\Big|\sum_{i=1}^n \langle T(x_i^1,\dots,x_i^m),y_i^*\rangle\Big| \\
&\quad \le \|T\|_I \sup_{\substack{x_j^*\in B_{X_j^*}\\ 1\le j\le m}} \Big(\sup_{y\in B_Y}\Big|\sum_{i=1}^n x^{1*}(x_i^1)\dots x^{m*}(x_i^m)y_i^*(y)\Big|\Big) \\
&\quad \le \|T\|_I \sup_{\substack{x_j^*\in B_{X_j^*}\\ 1\le j\le m}} \Big(\sum_{i=1}^n |x^{1*}(x_i^1)\dots x^{m*}(x_i^m)|^p\Big)^{1/p} \sup_{y\in B_Y}\Big(\sum_{i=1}^n |y_i^*(y)|^{p^*}\Big)^{1/p^*}.
\end{aligned}$$

Thus T is Cohen p-nuclear. ■

COROLLARY 4.5.

(i) $\mathcal{I}(X_1,\dots,X_m;Y)\subset\mathcal{N}_p^m(X_1,\dots,X_m;Y)\subset\mathcal{L}_{\mathrm{si},p}(X_1,\dots,X_m;Y)\subset\mathcal{L}_{\mathrm{sas},p}(X_1,\dots,X_m;Y)$.

(ii) $\mathcal{N}_p^m(X_1,\dots,X_m;Y)\subset\mathcal{L}_{\mathrm{as},p}(X_1,\dots,X_m;Y)$.

(iii) $\mathcal{I}(X_1,\dots,X_m;Y)$ *and* $\mathcal{N}_p^m(X_1,\dots,X_m;Y)\subset\mathcal{L}_{\mathrm{fas},p}(X_1,\dots,X_m;Y)$.

As a consequence of Proposition 2.2(a) and [7, Remarks 1, 2 and Theorem 4] we have

Remark 4.6.

1) The inclusion $\mathcal{N}_1^m(X_1,\ldots,X_m;Y)\subset\mathcal{L}_{\text{sas},1}(X_1,\ldots,X_m;Y)$ is sometimes strict.
2) The inclusion $\mathcal{N}_1^m(X_1,\ldots,X_m;Y)\subset\mathcal{L}_{\text{fas},1}(X_1,\ldots,X_m;Y)$ is sometimes strict.
3) If X_j $(1\le j\le m)$ has cotype 2, then we have $\mathcal{N}_1^m(X_1,\ldots,X_m;Y)=\mathcal{L}_{\text{d}}^1(X_1,\ldots,X_m;Y)$ for every Y.
4) The inclusion $\mathcal{N}_1^m(X_1,\ldots,X_m;Y)\subset\mathcal{L}_{\text{as},1}(X_1,\ldots,X_m;Y)$ is sometimes strict.

A multilinear mapping T between Banach spaces is *weakly compact* if T can be written as $T=u\circ R$ where R is a multilinear mapping and u is a weakly compact linear operator. By $\mathcal{L}_{\text{w}}(X_1,\ldots,X_m;Y)$ we denote the closed subspace of $\mathcal{L}(X_1,\ldots,X_m;Y)$ formed by the weakly compact mappings.

Let $X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m$ denote the completed projective tensor product of $X_1,\ldots,X_m$. Recall that every m-linear operator $T\in\mathcal{L}(X_1,\ldots,X_m;Y)$ has an associated linear operator $T_L\in\mathcal{L}(X_1\widehat{\otimes}_\pi\ldots\widehat{\otimes}_\pi X_m;Y)$. For $1<p\le\infty$, it is not difficult to prove that T is Cohen strongly p-summing if and only if T_L is strongly p-summing.

Since strongly p-summing linear operators are weakly compact [10, Corollary 2.2.5(i)], we conclude by Proposition 3.2(a) in [5] that every $T\in\mathcal{D}_p^m(X_1,\ldots,X_m;Y)$ is weakly compact.

Our main result of this section is the following corollary, which is a straightforward consequence of Theorem 4.3(i) and [5, Proposition 5.7 and Remark 5.9].

Corollary 4.7.

1) *Every Cohen p-nuclear $(1<p\le\infty)$ m-linear mapping on arbitrary Banach spaces is weakly compact.*
2) *Let $K_1,\ldots,K_m$ be compact Hausdorff spaces. For m-linear mappings from $C(K_1)\times\cdots\times C(K_m)$ to an arbitrary Banach space, we have*

$$\mathcal{N}_1^m\subseteq\Pi\circ\mathcal{L}\subseteq\mathcal{L}_{\text{sas},p}\cap\mathcal{L}_{\text{w}}.$$

3) *The statement of* 1) *is not true for $p=1$.*

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