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GENERALIZED CALDERÓN CONDITIONS AND REGULAR ORBIT SPACES

BY

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Abstract. The construction of generalized continuous wavelet transforms on locally compact abelian groups A from quasi-regular representations of a semidirect product group $G = A \rtimes H$ acting on $\mathrm{L}^2(A)$ requires the existence of a square-integrable function whose Plancherel transform satisfies a Calderón-type resolution of the identity. The question then arises under what conditions such square-integrable functions exist.

The existing literature on this subject leaves a gap between sufficient and necessary criteria. In this paper, we give a characterization in terms of the natural action of the dilation group H on the character group of A. We first prove that a Calderón-type resolution of the identity gives rise to a decomposition of Plancherel measure of A into measures on the dual orbits, and then show that the latter property is equivalent to regularity conditions on the orbit space of the dual action.

Thus we obtain, for the first time, sharp necessary and sufficient criteria for the existence of a wavelet inversion formula associated to a quasi-regular representation. As a byproduct and special case of our results we deduce that discrete series subrepresentations of the quasi-regular representation correspond precisely to dual orbits with positive Plancherel measure and associated compact stabilizers. Only sufficiency of the conditions was previously known.

1. Introduction.

The continuous wavelet transform of $f \in \mathrm{L}^2(\mathbb{R})$ is obtained by picking a suitable $\psi \in \mathrm{L}^2(\mathbb{R})$ and letting

$$V_\psi f(b,a) = \int_{\mathbb{R}} f(t)|a|^{-1/2}\overline{\psi\left(\frac{t-b}{a}\right)}\,dt \quad \text{for } b \in \mathbb{R},\ a \in \mathbb{R}\setminus\{0\}.$$

Among the many useful aspects of wavelets, probably the most fundamental one is wavelet inversion, usually formulated as

$$f(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} V_\psi f(b,a)|a|^{-1/2}\psi\left(\frac{t-b}{a}\right) db\,\frac{da}{|a|^2},$$

to be read in the weak sense (rather than pointwise). This remarkable identity holds precisely if ψ is chosen to be an *admissible vector*, satisfying the

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Calderón condition

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi = 1. \tag{1}$$

The generalization of this construction, in particular to higher-dimensional euclidean space, has been studied early on (see e.g. [18, 3]). In the euclidean setting, the role of the dilations $a \neq 0$ is assumed by the elements of a matrix group H, and various sources have studied which properties of H ensure the existence of an inversion formula (see e.g. [2, 8, 11, 16]). A further extension, replacing $\mathbb{R}^d$ by a general locally compact group A and H by a group of topological automorphisms, was considered in [4].

The wavelet inversion formula is closely related to a suitable generalization of the Calderón condition. As will be seen in the next section, this condition is quite easy to write down. However, it is not at all trivial to decide whether there actually exist L^2-functions satisfying it. Sufficient conditions for dilation groups acting on $\mathbb{R}^d$ were derived in [11, 16], along with some necessary conditions. However, a complete characterization of these groups in terms of necessary and sufficient conditions has been missing. The chief contribution of this paper is to provide such a characterization in terms of the natural action on the dual group.

The paper is structured as follows: Section 2 contains a more detailed exposition of the group-theoretic construction of continuous wavelet transforms from the action of an automorphism group on a locally compact abelian group. We investigate wavelet inversion formulae valid for elements from a proper closed invariant subspace. For this purpose, we introduce the dual action of the dilation group, and formulate the Calderón condition for admissible vectors. A useful auxiliary notion for the discussion of admissible vectors is “weak admissiblity”. We formulate a full characterization of dilation groups admitting weakly admissible vectors (Theorem 6), which is the central result of this paper. The following two sections are devoted to a proof of this theorem. As it turns out, the core result is of a predominantly measure-theoretic nature, and our treatment highlights these aspects. The main result of these sections is Theorem 12. In the final section we resume the discussion of admissible vectors. Theorems 6 and 19 provide a complete characterization of invariant subspaces allowing a wavelet inversion formula. We also comment on irreducible subspaces with a wavelet inversion formula, which necessarily correspond to orbits of the dual action with positive measure and compact fixed groups (Corollary 21).

2. Wavelet transforms from semidirect products. Let us briefly sketch the group-theoretic framework for the construction of continuous wavelet transforms on locally compact abelian groups. The case where the underlying group is $\mathbb{R}^n$ has been studied e.g. in [2, 11, 16], the generalization to arbitrary LCA groups was considered in [4].

Let A denote a second countable locally compact abelian group (with group structure written additively), and let H be a group of topological automorphisms of A, endowed with a second countable locally compact group topology making the natural action of H on A continuous. The semidirect product group $G = A \rtimes H$ consists of elements $(a, h) \in A \times H$, with the group law $(a, h) \cdot (b, g) = (a + h(b), hg)$. When endowed with the product topology, G is a second countable locally compact group as well.

For any locally compact group S, integration against (left) Haar measure is denoted as $\int_S g(s)\, ds$. Haar measure of a Borel set $B \subset S$ is denoted by $|B| = \int_S \mathbf{1}_B(s)\, ds$. Here, as below, we use the notation $\mathbf{1}_B$ for the indicator function of B.

The action of H on A induces a continuous homomorphism $\delta : H \to \mathbb{R}^+$ by $\delta(h) = |h(B)|/|B|$, where $B \subset A$ is any Borel set of positive measure. The left Haar integral on G is given by

$$\int\limits_G f(x, h)\, d(x, h) = \int\limits_H \int\limits_A f(x, h)\, dx\, \frac{dh}{\delta(h)},$$

and the modular function of G is $\Delta_G(a, h) = \Delta_H(h)/\delta(h)$.

The group G has a natural unitary representation acting on $\mathrm{L}^2(A)$ via

$$\pi(a, h)f(t) = \delta(h)^{-1/2} f(h^{-1}(t - a)) \quad (t \in A).$$

Given a function $g \in \mathrm{L}^2(A)$, the associated wavelet transform is an operator V_ψ mapping $f \in \mathrm{L}^2(A)$ to its coefficient function $V_\psi f$, defined on G as

$$V_\psi f(a, h) = \langle f, \pi(a, h)\psi \rangle.$$

Definition 1. Let $\mathcal{H} \subset \mathrm{L}^2(A)$ be a closed π-invariant subspace. Then $g \in \mathcal{H}$ is called *weakly admissible* (*for* $\mathcal{H}$) if $V_\psi : \mathcal{H} \to \mathrm{L}^2(G)$ is a (well-defined) bounded injective map. It is called *admissible* (*for* $\mathcal{H}$) if $V_\psi : \mathcal{H} \to \mathrm{L}^2(G)$ is an isometric embedding.

Remark 2. Admissibility is equivalent to a weak-sense inversion formula: ψ is admissible for $\mathcal{H}$ iff for all $f \in \mathcal{H}$,

$$f = \int\limits_H \int\limits_A V_\psi f(a, h) \pi(a, h)\psi\, da\, \frac{dh}{\delta(h)},$$

holds in the weak sense (see e.g. [10, Section 2.2]).

Remark 3. Note that V_ψ intertwines the action of π with left translation. In particular, g is weakly admissible iff V_ψ is a bounded injective intertwining operator between the restriction of π to $\mathcal{H}$ and the left regular representation acting on $\mathrm{L}^2(G)$. In fact, the existence of weakly admissible vectors is equivalent to unitary containment in the regular representation [10, 2.21].

We denote by $\widehat{A}$ the dual group of A. It is a second countable locally compact abelian group as well. (For this and the following facts concerning locally compact abelian groups, see [6].) The Fourier transform of $f \in \mathrm{L}^1(A)$ is defined as

$$\widehat{f}(\xi) = \int_A f(x)\overline{\xi(x)}\,dx.$$

We normalize Haar measure on $\widehat{A}$ so that for all $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, $\|f\|_2 = \|\widehat{f}\|_2$. The Plancherel theorem implies that the Fourier transform extends to a unitary operator $\mathrm{L}^2(A) \to \mathrm{L}^2(\widehat{A})$. For this reason, Haar measure on $\widehat{A}$ is also called *Plancherel measure* of A.

The action of H on A gives rise to the dual action on $\widehat{A}$, which is a right action defined by $(\xi.h)(x) = \xi(h(x))$. The behaviour of Haar measure on $\widehat{A}$ is similar to that of Haar measure on A, i.e., $|B.h| = \delta(h)|B|$ for all $B \subset \widehat{A}$ Borel.

For the study of (weakly) admissible vectors for invariant subspaces, the dual action is an indispensable tool. To begin with, invariant subspaces are in one-to-one correspondence to H-invariant Borel subsets of $\widehat{A}$, by the following result.

LEMMA 4. *Let $X \subset \widehat{A}$ be an H-invariant Borel subset. Let*

$$\mathcal{H}_X = \{f \in \mathrm{L}^2(A) : \widehat{f} \cdot \mathbf{1}_X = \widehat{f}\}.$$

Then $\mathcal{H}_X \subset \mathrm{L}^2(A)$ is a π-invariant closed subspace. We write π_X for the restriction of π to $\mathcal{H}_X$.

Conversely, if $\mathcal{H} \subset \mathrm{L}^2(A)$ is a π-invariant closed subspace, then $\mathcal{H} = \mathcal{H}_X$ for a suitably chosen H-invariant Borel set X.

Proof. First note that if $\mathcal{H}$ is invariant under shifts, i.e. all operators of the type $\pi(a, e_H)$, then necessarily $\mathcal{H} = \mathcal{H}_X$ for some Borel set X. This follows from the characterization of the commuting algebra by the Fourier transform, e.g. in [6, 4.44]. If, in addition, $\mathcal{H}$ is also invariant under $\pi(0, h)$ for all h, it necessarily follows that, possibly after removing a set of measure zero, X is in addition H-invariant. The proof given in [8] for this fact in the case $A = \mathbb{R}^n$ carries over verbatim. ∎

We next turn to the derivation of admissibility criteria. Direct calculation employing the Plancherel theorem for A allows us to derive the crucial equality

$$\|V_\psi f\|_2^2 = \int_{\widehat{A}} |\widehat{f}(\xi)|^2 \int_H |\widehat{\psi}(\xi.h)|^2\,dh\,d\xi.$$

See e.g. [11, 16] for the proof in the case $A = \mathbb{R}^d$, which immediately carries over to the general setting. From this, one easily derives the following criteria for strong and weak admissibility, generalizing the Calderón condition for wavelets over the reals:

LEMMA 5. *Let $\mathcal{H} \subset \mathrm{L}^2(A)$ be closed and π-invariant. Hence $\mathcal{H} = \mathcal{H}_X$ for a suitable H-invariant Borel set $X \subset \widehat{A}$. Then $\psi \in \mathcal{H}$ is weakly admissible iff the function*

$$\xi \mapsto \int_H |\widehat{\psi}(\xi.h)|^2 \, dh \tag{2}$$

is a.e. bounded and nonzero on X. Moreover, ψ is admissible iff this function equals 1 *a.e.*

Furthermore, it is easily verified that for

$$\int_H |\widehat{\psi}(\xi.h)|^2 \, dh$$

to be finite, the stabilizer of ξ, defined by $H_\xi = \{h \in H : \xi.h = \xi\}$, must be compact (see Lemma 11 below). Hence Lemma 5 implies that almost all stabilizers must be compact for H to be weakly admissible. However, it has been noticed early on that this necessary condition is not sufficient: The relevant counterexample is provided by letting $A = \mathbb{R}^2$ and $H = \mathrm{SL}(2, \mathbb{Z})$. It turns out that almost all stabilizers are trivial, but H is not weakly admissible (see [8] for a related example).

Additional sufficient criteria were provided in [10, 11, 16] for the case of $A = \mathbb{R}^n$ and a matrix group H, but the results in those papers do not yield a full characterization. The authors of [16] studied the condition that for almost every ξ there exists $\epsilon > 0$ such that the ϵ-stabilizer $H_{\epsilon,\xi} = \{h \in H : |\xi - \xi.h| < \epsilon\}$ is a compact subset of H. Here $|\cdot|$ denotes the Euclidean distance. It is shown in [16] that this condition ensures weak admissibility. Necessity of this condition was conjectured, but not shown in [16]. By contrast, [10, 11] studied regularity conditions on the orbit spaces, somewhat similar to the properties that will be considered in the next section. However, no necessary condition was derived.

The following theorem is the chief result of this paper. It characterizes the groups H allowing a weakly admissible vector in terms of regularity properties of the orbit space.

THEOREM 6. *Let $\mathcal{H} = \mathcal{H}_X$ for $X \subset \widehat{A}$ Borel and H-invariant. Then $\mathcal{H}$ has a weakly admissible vector iff there exists a conull H-invariant Borel subset $B \subset X$ such that:*

(i) *For all $\xi \in B$, the stabilizer H_ξ is compact.*

(ii) *There exists a Borel set $C \subset B$ such that for all $\xi \in B$, the set $C \cap \xi.H$ is a singleton.*

This result is a direct consequence of the purely measure-theoretic Theorem 12 below. We have chosen to remove (almost) all references to wavelets and harmonic analysis from the following two sections, because we believe that the central problem is measure-theoretic in nature, and of a certain independent interest.

3. Measure-theoretic setup and main result. Let us begin by fixing terminology. A useful survey of the relevant definitions and results concerning Borel spaces can be found in [1].

A *Borel space* is a set X endowed with a σ-algebra on X. The elements of the σ-algebra are called *Borel sets*. A measure defined on the σ-algebra is called a *Borel measure*. A map between Borel spaces is called *Borel* if the preimage of Borel sets are Borel again. A *Borel isomorphism* is a bijection between Borel spaces that is Borel in both directions. In the following, the Borel structures of locally compact groups and metric spaces are understood to be generated by the respective topologies. A Borel space is called *standard* if it is Borel isomorphic to a Borel subset of a separable complete metric space. We note that second countable locally compact groups are completely metrizable, and therefore standard. This applies to H, but also to A and $\widehat{A}$. Also, Borel subsets of standard spaces are clearly standard. Throughout the next two sections, X denotes a standard Borel measure space, on which a locally compact second countable group H acts jointly measurably from the right.

We assume that a fixed σ-finite Borel measure λ on X is given, which is quasi-invariant under H. This means that for all $h \in H$, the measure $\lambda_h : A \mapsto \lambda(A.h)$ is equivalent to λ. By the Radon–Nikodym theorem, this assumption implies the existence of a function $\rho : X \times H \to \mathbb{R}$ such that

$$\frac{d\lambda_h}{d\lambda}(\xi) = e^{\rho(\xi,h)}$$

holds for all $(\xi, h) \in X \times H$. This ρ is called the *cocycle* of the measure; it can be assumed measurable on $X \times H$, and such that it satisfies the following cocycle conditions, for all $g, h \in H$ and $\xi \in X$:

$$\rho(\xi, gh) = \rho(\xi.g, h) + \rho(\xi, g), \tag{3}$$

$$\rho(\xi, h) = 0 \quad \text{if } \xi.h = \xi; \tag{4}$$

see e.g. [13] or [20, Appendix B]. We note that the definition of the cocycle entails the following two formulae for integration:

$$\lambda(B.h) = \int_B e^{\rho(\xi,h)}\, d\lambda(\xi), \tag{5}$$

$$\int_X f(\xi.h^{-1})\, d\lambda(\xi) = \int_X f(\xi) e^{\rho(\xi,h)}\, d\lambda(\xi), \tag{6}$$

where the second equation holds for all positive Borel functions f, in the extended sense that one side is infinite iff the other is.

We denote by X/H the space of all H-orbits in X. Let $q : X \to X/H$ denote the quotient map. X/H is endowed with the quotient Borel structure: a subset $B \subset X/H$ is declared Borel if $q^{-1}(B) = \bigcup\{W : W \in B\} \subset X$ is Borel.

Definition 7. The action of H on X is called *weakly admissible* if there exists a Borel function $\varphi : X \to \mathbb{R}^+$ satisfying

$$0 < \int_H \varphi(\xi.h)dh < \infty \qquad \text{for } \lambda\text{-a.e. } \xi \in X. \tag{7}$$

The definition is a clear analogy to the Calderón condition. If $X \subset \widehat{A}$ is an invariant Borel subset, we will see shortly that the existence of weakly admissible vectors for a representation is equivalent to weak admissibility of the dual action. The following lemma spells out the technical details.

Lemma 8. *If the action of H is weakly admissible, there exists a function φ such that*

$$\varphi \geq 0, \qquad 0 < \int_H \varphi(\xi.h)dh \leq 1 \quad (\lambda\text{-}a.e.), \qquad \varphi \in \mathrm{L}^1(X, \lambda). \tag{8}$$

Proof. Write X as a disjoint union of sets of finite measure, $X = \bigcup_{n\in\mathbb{N}} X_n$. Assume that φ_1 satisfies (7), and define

$$\varphi_2(\xi) = \frac{\min(1, \varphi_1(\xi))}{2^n(1 + \lambda(X_n))}, \qquad \xi \in X_n.$$

Then φ_2 is integrable, and also satisfies (7). The same is then true for

$$\varphi(\xi) = \frac{\varphi_2(\xi)}{1 + \int_H \varphi_2(\xi.h)\, dh},$$

which is the desired function. ∎

Corollary 9. *Let $\mathcal{H} = \mathcal{H}_X \subset \mathrm{L}^2(A)$ for a suitable H-invariant $X \subset \widehat{A}$. There exists a weakly admissible vector for $\mathcal{H}$ iff the dual action of H on $\widehat{X}$ is weakly admissible.*

Proof. The "only if" part is clear. For the other direction, we let $\widehat{\psi}(\xi) = \varphi(\xi)^{1/2}$, where φ satisfies (8). ∎

As already indicated in the title, the structure of the orbit space X/H is of central interest. Such spaces can be quite pathological. By contrast, the situation for each individual orbit is quite simple, as the following lemma shows.

LEMMA 10. *For all $\xi \in X$, the orbit $\xi.H \subset X$ is Borel. Furthermore, the stabilizer $H_\xi = \{h \in H : \xi.h = \xi\}$ is a closed subgroup of H, and the quotient map $H \ni h \mapsto \xi.h$ induces a Borel isomorphism $H_\xi \setminus H \to \xi.H$.*

Proof. See [1, Chapter I, Proposition 3.7]. ∎

Now the first necessary condition for weakly admissible actions is easily proved.

LEMMA 11. *The set*

$$X_c := \{\xi \in X : H_\xi \text{ is compact}\} \subset X$$

is Borel and H-invariant. If H is weakly admissible, then $X_c \subset X$ is conull.

Proof. The stabilizer map $x \mapsto H_\xi$ is Borel, if one endows the set of closed subgroups with the Fell topology [1, Chapter II, Proposition 2.3]. Moreover, the set of compact subgroups is Borel (see [10, Proposition 5.5]), hence X_c is Borel. H-invariance is immediate from the observation that all stabilizers associated to a given orbit are conjugate.

If φ is a positive Borel function on X and $\xi \in X$ is such that

$$0 < \int_H \varphi(\xi.h)\, dh < \infty$$

then the fact that the function $h \mapsto \varphi(\xi.h)$ is integrable (with nonzero integral) and left invariant under the closed subgroup H_ξ at the same time forces H_ξ to be compact: for instance, pick $\epsilon > 0$ such that $|\{h : \varphi(\xi.h) > \epsilon\}| > 0$. This set has finite Haar measure, and is left H_ξ-invariant, thus [8, Lemma 11] yields compactness of H_ξ.

Thus, if the action of H is weakly admissible, then $|X \setminus X_c| = 0$. ∎

We will characterize weak admissibility in terms of measure-theoretic properties of X/H, which are closely related to standardness. A Borel space is called *countably generated* if the σ-algebra is generated by a countable subset. It is called *separated* if single points are Borel. A Borel space is called *countably separated* if there is a sequence of Borel sets separating the points. All these properties are inherited by products and Borel subspaces. A Borel space is called *analytic* if it is (Borel-isomorphic to) the Borel image of a standard space in a countably generated space.

We say that X/H *admits a λ-transversal* if there exists an H-invariant λ-conull Borel set $Y \subset X$ and a Borel set $C \subset Y$ meeting each orbit in Y in precisely one point.

A *pseudo-image* of λ is a measure $\overline{\lambda}$ on X/H obtained as image measure of an equivalent finite measure under the quotient map q; clearly, all pseudoimages are equivalent. We call $\overline{\lambda}$ *standard* if there exists $Y \subset X$ Borel, H-invariant, conull, such that Y/H is standard.

Finally, we need the notion of a *measure decomposition*: A *measurable family of measures* is a family $(\beta_{\mathcal{O}})_{\mathcal{O}\subset X}$ indexed by the orbits in X, such that for all Borel sets $B \subset X$, the map $\mathcal{O} \mapsto \beta_{\mathcal{O}}(B)$ is Borel on X/H.

A *measure decomposition of* λ consists of a pair $(\overline{\lambda}, (\beta_{\mathcal{O}})_{\mathcal{O}\subset X})$, where $\overline{\lambda}$ is a pseudo-image of λ on X/H, or a σ-finite measure equivalent to such a pseudo-image, and a measurable family $(\beta_{\mathcal{O}})_{\mathcal{O}\subset X}$ such that for all $B \subset X$ Borel,

$$\lambda(B) = \int\limits_{X/H} \beta_{\mathcal{O}}(B)\, d\overline{\lambda}(\mathcal{O}).$$

Note that this entails, for all positive Borel functions f on X, that

$$\int\limits_X f(\xi)\, d\lambda(\xi) = \int\limits_{X/H} \int\limits_{\mathcal{O}} f(\xi)\, d\beta_{\mathcal{O}}(\xi)\, d\overline{\lambda}(\mathcal{O}).$$

We say that λ *decomposes over the orbits* if there exists a measure decomposition with the additional requirement that, for $\overline{\lambda}$-almost every $\mathcal{O} \in X/H$, the measure $\beta_{\mathcal{O}}$ is supported in $\mathcal{O}$, meaning $\beta_{\mathcal{O}}(X \setminus \mathcal{O}) = 0$.

THEOREM 12. *Let X be a standard Borel space, and H a second countable group acting measurably on X. Assume that λ is a quasi-invariant σ-finite measure on X. Consider the following statements:*

(a) *The action of H is weakly admissible.*
(b) *λ decomposes over the orbits.*
(c) *$\overline{\lambda}$ is standard.*
(d) *X/H admits a λ-transversal.*

Then (a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d), *and $\lambda(X \setminus X_c) = 0$.*

Conversely, if $\lambda(X \setminus X_c) = 0$, then (d)$\Rightarrow$(a).

The equivalence of (b) through (d) is possibly folklore, although we have not been able to locate a handy reference for the "almost everywhere" version that we consider here. Also, the proof of (b)$\Rightarrow$(c) turned out to be rather more technical than initially expected. We include detailed arguments for the sake of reference.

Note that Corollary 9 and Theorem 12, applied to the dual action of H on the invariant set X, indeed imply Theorem 6.

4. Proof of Theorem 12. If the action of H is weakly admissible, then $\lambda(X \setminus X_c) = 0$ by Lemma 11. Without loss of generality, we will therefore assume in the following that $X = X_c$.

4.1. Proof of (a)⇒(b)**.** Assume that the action of H is weakly admissible. The proof strategy will be to define a measure μ that decomposes over the orbits, and to show that μ is σ-finite and equivalent to λ.

LEMMA 13. *Let $\overline{\lambda}$ be a pseudo-image of λ on X/H. Let $\varphi : X \to \mathbb{R}_0^+$ be a Borel function satisfying* (8)*. For Borel sets $U \subset X/H$ let*

$$\overline{\lambda_\varphi}(U) = \int\limits_{q^{-1}(U)} \varphi(\xi)\, d\lambda(\xi).$$

Then $\overline{\lambda_\varphi}$ is a finite measure on X that is equivalent to $\overline{\lambda}$, satisfying for all Borel functions $g : X/H \to \mathbb{R}_0^+$ the equality

$$\int\limits_{X/H} g(\mathcal{O})\, d\overline{\lambda}_\varphi = \int\limits_X g(\xi.H)\varphi(\xi)\, d\lambda(\xi). \tag{9}$$

Proof. We need to show for an H-invariant Borel set $V \subset X$ that

$$\lambda(V) = 0 \iff \int\limits_V \varphi(\xi)\, d\lambda(\xi) = 0.$$

The direction "⇒" is clear; after all, the right-hand side is an integral over a λ-null set.

For the other direction, we employ the quasi-invariance of λ and invariance of V to note that for all $h \in H$,

$$\int\limits_V \varphi(\xi.h)\, d\lambda(\xi) = 0.$$

Integrating over H and applying Tonelli's theorem, we obtain

$$\int\limits_V \int\limits_H \varphi(\xi.h)\, dh\, d\lambda(\xi) = 0.$$

By assumption, the inner integral vanishes λ-almost nowhere, hence $\lambda(V) = 0$ follows.

By definition, equation (9) holds for indicator functions, and extends to nonnegative Borel maps by standard arguments. ∎

LEMMA 14.

(a) *Let $\mathcal{O} = \xi.H$, and assume that H_ξ is compact. Then*

$$\mu_{\mathcal{O}}(B) = |\{h \in H : \xi.h \in B\}|$$

defines a σ-finite measure supported on $\mathcal{O}$. Furthermore, $\mu_{\mathcal{O}}$ is independent of the choice of $\xi \in \mathcal{O}$.

(b) *Let φ be a Borel function satisfying* (8)*. For Borel sets $B \subset X$, define*

$$\mu(B) = \int\limits_{X/H} \mu_{\mathcal{O}}(B)\, d\overline{\lambda}_\varphi(\mathcal{O}) = \int\limits_X \varphi(\xi) \int\limits_H \mathbf{1}_B(\xi.h)\, dh\, d\xi. \tag{10}$$

Then μ is a well-defined Borel measure.

Proof. Since $\mathcal{O}$ is Borel, $\mu_{\mathcal{O}}$ is a well-defined Borel measure. Furthermore, since H_ξ is compact, $\mu_{\mathcal{O}}$ is finite on sets of the form $\xi.C$, with C compact. In particular, since H is σ-compact, $\mu_{\mathcal{O}}$ is σ-finite. The measure $\mu_{\mathcal{O}}$ is independent of the choice of ξ, since the action is on the right, and Haar-measure on H is left invariant. The well-definedness of μ follows from Fubini's theorem and the measurability of $(h,\xi) \mapsto \mathbf{1}_A(\xi.h)$. The second equation of (10) is obtained directly from (9). ∎

The following result will allow us to establish equivalence of μ and λ.

LEMMA 15. *Let φ be a positive Borel function satisfying* (8), *and let*

$$\Phi(\xi) = \int_H \varphi(\xi.h)\,dh.$$

Then, for all Borel functions $f : X \to \mathbb{R}_0^+$,

$$\int_X f(\xi)\,d\lambda(\xi) = \int_X \frac{\varphi(\xi)}{\Phi(\xi)} \int_H f(\xi.h)e^{\rho(\xi,h)}\Delta_H(h)^{-1}\,dh\,d\lambda(\xi). \tag{11}$$

Proof. The proof is a straightforward computation, using Tonelli's theorem:

$$\begin{aligned}
\int_X f(\xi)\,d\lambda(\xi) &= \int_X \frac{f(\xi)}{\Phi(\xi)} \int_H \varphi(\xi.h)\,dh\,d\lambda(\xi) = \int_H\int_X \frac{f(\xi)}{\Phi(\xi)}\varphi(\xi.h)\,d\lambda(\xi)\,dh \\
&= \int_H\int_X \frac{f(\xi)}{\Phi(\xi)}\varphi(\xi.h)\,d\lambda(\xi)\,dh = \int_H\int_X \frac{f(\xi.h^{-1})}{\Phi(\xi.h^{-1})}\varphi(\xi)e^{\rho(\xi,h^{-1})}\,d\lambda(\xi)\,dh \\
&= \int_X \frac{\varphi(\xi)}{\Phi(\xi)} \int_H f(\xi.h^{-1})e^{\rho(\xi,h^{-1})}\,dh\,d\lambda(\xi) \\
&= \int_X \frac{\varphi(\xi)}{\Phi(\xi)} \int_H f(\xi.h)e^{\rho(\xi,h)}\Delta_H(h)^{-1}\,dh,
\end{aligned}$$

where the penultimate equality used H-invariance of Φ. ∎

The next lemma establishes σ-finiteness of μ:

LEMMA 16. *Let H be weakly admissible.*

(a) *There exists $\varphi : X \to \mathbb{R}_0^+$ satisfying* (8), *and an associated H-invariant conull Borel subset Ω_φ, such that, for all $\xi \in \Omega_\varphi$, the map $H \ni h \mapsto \varphi(\xi.h)$ is continuous.*

(b) *Let φ satisfy* (2). *For $k \in \mathbb{N}$, define*

$$A_k = \{\xi \in X : \varphi(\xi) > 1/k\}.$$

Then, for all $k \in \mathbb{N}$ and $g \in H$,

$$\mu(A_k.g) = \Delta_H(g)\mu(A_k) \le \Delta_H(g)k\overline{\lambda_\varphi}(X/H) < \infty. \tag{12}$$

(c) *With φ, Ω_φ as in part* (a)*, and A_k as in part* (b)*: If $(h_n)_{n\in\mathbb{N}} \subset H$ is dense, then $\Omega_\varphi \subset \bigcup_{n,k\in\mathbb{N}} A_k.h_n$. In particular, μ is σ-finite.*

Proof. For the proof of (a) pick φ_0 satisfying (8). Pick a continuous, compactly supported $\nu : H \to \mathbb{R}_0^+$ satisfying

$$\int_H \nu(g)\Delta_H(g)\,dg = 1.$$

Letting

$$\Omega_\varphi = \left\{\xi \in X : \int_H \varphi_0(\xi.h)\,dh < \infty\right\}$$

defines an H-invariant conull Borel subset. For $\xi \in \Omega_\varphi$, we define

$$\varphi(\xi) = \int_H \varphi_0(\xi.g)\nu(g)\,dg,$$

and obtain

$$\varphi(\xi.h) = \int_H \varphi_0(\xi.hg)\nu(g)dg.$$

The assumption $\xi \in \Omega_\varphi$ amounts to saying that the map $g \mapsto \varphi_0(\xi.g)$ is in $\mathrm{L}^1(H)$. Now strong continuity of the left action of H on $\mathrm{L}^1(H)$ and boundedness of ν imply that $h \mapsto \varphi(\xi.h)$ is continuous.

Integrability of φ is a straightforward consequence of $\varphi_0 \in \mathrm{L}^1(X,\lambda)$, $\nu \in \mathrm{L}^1(H)$ and Fubini's theorem. Finally,

$$\begin{aligned}\int_H \varphi(\xi.h)dh &= \int_H\int_H \varphi_0(\xi.hg)\nu(g)\,dg\,dh = \int_H \nu(g)\int_H \varphi_0(\xi.hg)\,dh\,dg \\ &= \int_H \nu(g)\Delta_H(g)\int_H \varphi_0(\xi.h)\,dh\,dg = \int_H \varphi_0(\xi.h)\,dh,\end{aligned}$$

where the last equation was due to our choice of ν. Hence (8) for φ_0 implies the same for φ, and (a) is shown.

The first equality of (12) follows from

$$\int_H \mathbf{1}_{A_k.g}(\xi.h)\,dh = \int_H \mathbf{1}_{A_k}(\xi.hg^{-1})\,dh = \Delta_H(g)\int_H \mathbf{1}_{A_k}(\xi.h)\,dh\ ,$$

and integration over X/H. For the inequality, observe that by definition of A_k, we have $\mathbf{1}_{A_k}(\xi) < k\varphi(\xi)$, and thus, by our choice of φ,

$$\mu_{\xi.H}(A_k) = \int_H \mathbf{1}_{A_k}(\xi.h)\,dh \le k\int_H \varphi(\xi.h)\,dh \le k.$$

But then

$$\mu(A_k) = \int_{X/H} \mu_{\mathcal{O}}(A_k)\,d\overline{\lambda}_\varphi(\mathcal{O}) \le k\overline{\lambda_\varphi}(X/H).$$

For part (c) let $\xi \in \Omega_\varphi$, hence $0 < \int_H \varphi(\xi.h)\, d\lambda(\xi) \le 1$. Hence the integrand cannot be identically zero, and there exists $k \in \mathbb{N}$ such that

$$B = \{g \in H : \varphi(\xi.g^{-1}) > 1/k\}$$

is nonempty. By the choice of φ, the set B is open, hence there exists $n \in \mathbb{N}$ such that $h_n \in B$, implying $\varphi(\xi.h_n^{-1}) > 1/k$. But this means $\xi \in A_k.h_n$, as desired. ■

Now the implication (a)⇒(b) is easily proved. We pick φ according to Lemma 16(a), and consider the measure μ defined in Lemma 14(b), using $\overline{\lambda} = \overline{\lambda}_\varphi$. Then μ is equivalent to λ: On the one hand, Lemma 15 provides, for an arbitrary Borel set $A \subset X$,

$$\lambda(A) = \int\limits_X \frac{\varphi(\xi)}{\Phi(\xi)} \int\limits_H \mathbf{1}_A(\xi.h) e^{\rho(\xi,h)} \Delta_H(h)^{-1}\, dh\, d\lambda(\xi), \tag{13}$$

whereas, by Lemma 13,

$$\mu(A) = \int\limits_X \varphi(\xi) \int\limits_H \mathbf{1}_A(\xi.h)\, dh\, d\lambda(\xi). \tag{14}$$

Hence, by (13), $\lambda(A) = 0$ iff $\varphi(\xi) \int_H \mathbf{1}_A(\xi.h) e^{\rho(\xi,h)} \Delta_H(h)^{-1}\, dh = 0$ for λ-a.e. ξ. Both Δ_H and the exponential function are strictly positive, hence this is the case precisely when $\varphi(\xi) \int_H \mathbf{1}_A(\xi.h)\, dh = 0$ for λ-a.e. ξ. But by (14), the latter condition is equivalent to $\mu(A) = 0$. Hence λ and μ are equivalent.

Recall that by definition, $d\mu(\xi) = d\mu_{\mathcal{O}}(\xi) d\overline{\lambda}(\mathcal{O})$. By Lemma 16(c), μ is σ-finite. Hence the Radon–Nikodym theorem applies, and yields

$$d\lambda(\xi) = \frac{d\lambda}{d\mu}(\xi) d\mu(\xi) = \frac{d\lambda}{d\mu}(\xi) d\mu_{\mathcal{O}}(\xi) d\overline{\lambda}(\mathcal{O}),$$

which shows that letting

$$d\beta_{\mathcal{O}}(\xi) = \frac{d\lambda}{d\mu}(\xi) d\mu_{\mathcal{O}}(\xi)$$

yields the desired measure decomposition.

4.2. Proof of (b)⇒(c). For this step, we first replace λ by an equivalent probability measure α. Then α decomposes over the orbits as well, by the same argument as in the proof of (a)⇒(b). In the decomposition of α, almost every $\beta_{\mathcal{O}}$ is finite, and can thus be normalized to be a probability measure. Then the measure on the quotient space effecting the decomposition of α into the normalized measures turns out to be a probability measure as well.

In short, λ can be assumed to be a probability measure, together with all measures involved in the decomposition. Furthermore, we may assume that $\overline{\lambda}$ is the image measure of λ under q. The following argument relates the decomposition to the ergodic decomposition constructed in [13], and then uses properties of the latter. For this purpose, let ρ denote the cocycle of λ. Let $M_\rho(X)$ denote the set of Borel probability measures on X with cocycle ρ.

We endow $M_\rho(X)$ with the coarsest σ-algebra such that, for all Borel sets $B \subset X$, the mapping $M_\rho(X) \ni \nu \mapsto \nu(B)$ is Borel. Let $\mathcal{S}$ denote the σ-algebra of X, and let $\mathcal{S}^H$ be the subalgebra of H-invariant Borel sets. Clearly, $\mathcal{S}^H$ is a subalgebra of $\mathcal{S}$. The conditional expectation of f with respect to $\nu \in M_\rho(X)$ is a Borel function

$$E_\nu(f \mid \mathcal{S}^H) : X \to \mathbb{R}_0^+$$

which is H-invariant and satisfies

$$\int_B f(\xi)\, d\nu(\xi) = \int_B E_\nu(f \mid \mathcal{S}^H)(\xi)\, d\nu(\xi)$$

for all H-invariant Borel sets B. The conditional expectation always exists and is ν-a.e. unique [19, 5.1.15].

By [13, Theorem 5.2], there exists an H-invariant map $p : X \to M_\rho(X)$, $\xi \mapsto p_\xi$, such that p_ξ is ergodic, and in addition, for every $\nu \in M_\rho(X)$ and every positive Borel function f on X,

$$E_\nu(f \mid \mathcal{S}^H)(\xi) = \int_X f(\omega)\, dp_\xi(\omega) \tag{15}$$

holds for ν-almost all $\xi \in X$.

Lemma 17. *Assume that $(\overline{\lambda}, (\beta_{\mathcal{O}})_{\mathcal{O}\subset X})$ is a decomposition of λ into probability measures over the orbits. Let ρ be the cocycle of λ, and let $p : X \to M_\rho(X)$ denote the ergodic decomposition associated to ρ. There exists a conull, H-invariant Borel set $Y \subset X$ such that $p_\xi = \beta_{\xi.H}$ for all $\xi \in Y$.*

Proof. We first prove that for almost all orbits $\mathcal{O}$, $\beta_{\mathcal{O}} \in M_\rho(X)$. For Borel subsets $B \subset X$ and H-invariant $C \subset X$,

$$\begin{aligned}\int_C \beta_{\mathcal{O}}(B.h)\, d\overline{\lambda}(\mathcal{O}) = \lambda(B.h \cap C) &= \int_C \mathbf{1}_B(\xi) e^{\rho(\xi,h)}\, d\lambda(\xi)\\ &= \int_C \int_B e^{\rho(\xi,h)}\, d\beta_{\mathcal{O}}(\xi)\, d\overline{\lambda}(\mathcal{O}),\end{aligned}$$

and thus, for all $h \in H$,

$$\beta_{\mathcal{O}}(B.h) = \int_B e^{\rho(\xi,h)}\, d\beta_{\mathcal{O}}(\xi) \tag{16}$$

is valid for a $\overline{\lambda}$-conull set of orbits $\mathcal{O}$ that may still depend on $h \in H$ and B.

By Fubini's theorem, for each $B \in \mathcal{S}$ there exists $Y(B) \subset X$ Borel, H-invariant and conull such that (16) holds for all orbits $\mathcal{O} \subset Y(B)$ and all $h \in T(\mathcal{O}, B)$, with $T(\mathcal{O}, B) \subset H$ Borel, conull. Next pick a generating sequence $(B_k)_{k \in \mathbb{N}}$ of $\mathcal{S}$, and define

$$Y = \bigcap_{k \in \mathbb{N}} Y(B_k), \qquad \forall \mathcal{O} \subset Y : T(\mathcal{O}) = \bigcap_{k \in \mathbb{N}} T(\mathcal{O}, B_k).$$

Then (16) holds for all $B \in \mathcal{S}$, $\mathcal{O} \subset Y$ and $h \in T(\mathcal{O})$, since both sides of (16) define a Borel measure, hence coincide on a σ-algebra.

Now fix $\mathcal{O} \subset Y$, and define

$$H(\mathcal{O}) = \left\{ h \in H : \forall B \in \mathcal{S} : \beta_{\mathcal{O}}(B.h) = \int_B e^{\rho(\xi,h)}\, d\beta_{\mathcal{O}}(\xi) \right\}.$$

We claim that $H(\mathcal{O})$ is a subgroup of H: Assume that $h \in H(\mathcal{O})$. Then (16) extends to positive Borel functions f, yielding

$$\int_X f(\xi.h^{-1})\, d\beta_{\mathcal{O}}(\xi) = \int_X f(\xi) e^{\rho(\xi,h)}\, d\beta_{\mathcal{O}}(\xi). \tag{17}$$

Furthermore, the cocycle properties (3) and (4) entail that $\rho(\xi, h^{-1}) = -\rho(\xi.h^{-1}, h)$. Using this, we can compute

$$\begin{aligned}
\int_B e^{\rho(\xi,h^{-1})}\, d\beta_{\mathcal{O}}(\xi) &= \int_B e^{-\rho(\xi.h^{-1},h)}\, d\beta_{\mathcal{O}}(\xi) = \int_X \mathbf{1}_B(\xi) e^{\rho(\xi.h^{-1},h)}\, d\beta_{\mathcal{O}}(\xi) \\
&\overset{(17)}{=} \int_X \mathbf{1}_B(\xi.h) e^{-\rho(\xi,h)} e^{\rho(\xi,h)}\, d\beta_{\mathcal{O}}(\xi) \\
&= \int_{B.h^{-1}} d\beta_{\mathcal{O}}(\xi) = \beta_{\mathcal{O}}(B.h^{-1}),
\end{aligned}$$

which proves $h^{-1} \in H(\mathcal{O})$.

Next let $g, h \in H(\mathcal{O})$. Then, since $g \in \mathcal{O}$,

$$\begin{aligned}
\beta_{\mathcal{O}}(B.hg) &= \int_{B.h} e^{\rho(\xi,g)}\, d\beta_{\mathcal{O}}(\xi) = \int_X \mathbf{1}_B(\xi.h^{-1}) e^{\rho(\xi,g)}\, d\beta_{\mathcal{O}}(\xi) \\
&\overset{(17)}{=} \int_X \mathbf{1}_B(\xi) e^{\rho(\xi,h)} e^{\rho(\xi.h,g)}\, d\beta_{\mathcal{O}}(\xi) \overset{(3)}{=} \int_X \mathbf{1}_B(\xi) e^{\rho(\xi,hg)}\, d\beta_{\mathcal{O}}(\xi),
\end{aligned}$$

and therefore $hg \in H(\mathcal{O})$.

Hence $H(\mathcal{O}) \subset H$ is a subgroup, with $H(\mathcal{O}) \supset T(\mathcal{O})$. In particular, $H(\mathcal{O}) \supset T(\mathcal{O})T(\mathcal{O})^{-1}$, and since $T(\mathcal{O})$ has positive Haar measure, $H(\mathcal{O})$ contains a nonempty open subset [5, Proposition III.12.3]. Hence $H(\mathcal{O})$ is an open subgroup, and therefore closed. On the other hand, $T(\mathcal{O})$ is conull and thus dense in H, whence finally $H = H(\mathcal{O})$. But this shows $\beta_{\mathcal{O}} \in M_\rho(X)$ for all $\mathcal{O} \subset Y$.

Then, since $\beta_{\mathcal{O}}$ is supported in $\mathcal{O}$, it follows for every nonnegative Borel function f and $\xi \in \mathcal{O}$ that

$$\int_X f(\omega)\, d\beta_{\mathcal{O}}(\omega) = E_{\beta_{\mathcal{O}}}(f \,|\, \mathcal{S}^H)(\xi) = \int_X f(\omega)\, dp_\xi(\omega),$$

where the second equation is due to (15). But this means that $\beta_{\mathcal{O}} = p_\xi$. ∎

Hence, after passing to a suitable conull H-invariant subset, we may assume that $\beta_{\xi.H} = p_\xi$ holds for all $\xi \in X$. In particular, we may assume in the following that p separates the orbits in X.

Denote by $\mathcal{T}$ the coarsest σ-algebra on X making p a Borel map. Since the $\beta_{\mathcal{O}}$ are a measurable family, $p : X \to M_\rho(X)$ is clearly Borel, thus $\mathcal{T} \subset \mathcal{S}$. On the other hand, by [13, Theorem 5.2], $\mathcal{T}$ is countably generated.

Since p is H-invariant, the elements of $\mathcal{T}$ are H-invariant as well. Hence $q : X \to X/H$ induces an isomorphism of σ-algebras between $\mathcal{T}$ and its image $\overline{\mathcal{T}} = \{q(A) : A \in \mathcal{T}\}$. In particular, the latter is countably generated as well, and it is contained in the quotient σ-algebra on X/H. Furthermore, it is clearly separated, since p separates the orbits. But then the quotient σ-algebra, being finer than $\overline{\mathcal{T}}$, is countably separated. Hence, by [1, Chapter I, Proposition 2.9], it follows that X/H is an analytic Borel space. But then there exists a conull Borel subset $A \subset X/H$ which is standard (see [1, remarks following Chapter I, 2.13]). This shows (c).

4.3. Proof of (c)⇒(d)⇒(b)**.** For (c)⇒(d) we may assume, after passing to a suitable conull subset, that X/H is standard. Then [1, Chapter I, Proposition 2.15], yields a $\overline{\lambda}$-conull Borel set $V \subset X/H$ and a Borel cross-section $\sigma : V \to q^{-1}(V)$. Then σ is injective, and V is standard, as a Borel subset of X/H. But then $\sigma(V)$ is Borel, by [1, Chapter I, Proposition 2.5], and it meets every orbit contained in V in precisely one point.

Finally, (d)⇒(b) follows by [17, Lemma 11.1].

4.4. Proof of (d)⇒(a)**.** Now assume (d), and that all stabilizers are compact. Let $Y \subset X$ be H-invariant and conull, and let $C \subset Y$ be a Borel transversal for the orbits in Y. Let $K \subset H$ denote a compact neighbourhood of the identity, and $V = C.K = \{\xi.h : \xi \in C, h \in K\}$. Then V is an analytic subset of X, as the Borel image of the standard set $C \times K$ in the countably generated space Y. Since analytic sets are universally measurable (cf. [1, p. 11]), V is λ-measurable. Hence there exist sets $U \subset V \subset W$ with U, W Borel and $\lambda(W \setminus U) = 0$.

We intend to use $\varphi = \mathbf{1}_W$ to show weak admissibility. This amounts to showing, for almost all $\xi \in X$, that

$$0 < \mu_{\mathcal{O}}(W) = \mu_H(\{h : \xi.h \in W\}) < \infty, \tag{18}$$

for $\mathcal{O} \ni \xi$. In order to do this, we first consider $\mathbf{1}_V$. Note that for every $\xi \in X$,

$$\{h \in H : \xi.h \in V\} = H_\xi K$$

is compact. Since the canonical map $H_\xi \setminus H \to \mathcal{O}$ is a Borel isomorphism, it follows that $V \cap \xi.H$ is in fact a Borel set. In addition, since $H_\xi K$ is a compact neighborhood of the identity element,

$$0 < \mu_{\mathcal{O}}(V \cap \xi.H) < \infty. \tag{19}$$

In order to deduce (18) from this, we use (d)⇒(b) and decompose λ into a family $(\beta_{\mathcal{O}})_{\mathcal{O}\subset X}$ of measures supported on the orbits. Then almost every $\beta_{\mathcal{O}}$ is equivalent to a finite quasi-invariant measure $\tilde{\beta}_{\mathcal{O}}$. With respect to the topology induced by the canonical bijection $H_\xi \setminus H \to \mathcal{O}$, the finite measure $\tilde{\beta}_{\mathcal{O}}$ becomes regular [7, Theorem 7.8]. On the other hand, $\mu_{\mathcal{O}}$ is also a regular quasi-invariant measure, hence $\mu_{\mathcal{O}}$ is equivalent to $\tilde{\beta}_{\mathcal{O}}$ by [5, III.14.9], and thus finally to $\beta_{\mathcal{O}}$.

Now $\lambda(W \setminus U) = 0$ entails $\beta_{\mathcal{O}}(W \setminus U) = 0$ for almost all orbits $\mathcal{O}$. Since $\beta_{\mathcal{O}}$ is equivalent to $\mu_{\mathcal{O}}$, it follows for these orbits that

$$\mu_{\mathcal{O}}((W \cap \xi.H) \setminus (V \cap \xi.H)) \le \mu_{\mathcal{O}}((W \cap \xi.H) \setminus (U \cap \xi.H)) = 0,$$

with $\xi \in \mathcal{O}$. Thus $\mu_{\mathcal{O}}(W) = \mu_{\mathcal{O}}(W \cap \xi.H) = \mu_{\mathcal{O}}(V \cap \xi.H)$, and thus (19) implies (18).

5. Admissible vectors versus weakly admissible vectors. We return to the discussion of Section 2, dealing with the quasi-regular representation of $G = A \rtimes H$ on $\mathrm{L}^2(A)$. We consider suitable Borel-measurable H-invariant $X \subset \widehat{A}$ and the associated invariant subspace $\mathcal{H} = \mathcal{H}_X$. We assume the existence of a weakly admissible vector in $\mathcal{H}$, and want to clarify which additional criteria must be met to ensure the existence of an admissible vector. For explicit reference to the results of the previous two sections, let λ denote Haar measure on $\widehat{A}$.

The main tool will be the decomposition of Haar measure on X. The discussion in this section closely follows [10, Section 5.2], but we have chosen to spell out most details for two reasons: first, we start from somewhat more general assumptions, and secondly, the arguments in [10, Section 5.2] are partly flawed. This applies in particular to [10, Lemma 5.9], which is an analog of Lemma 18 below. Thus the following serves both as an erratum to some of the results in [10] and a generalization thereof.

LEMMA 18. *Assume that $\mathcal{H}$ has a weakly admissible vector.*

(a) *Fix any pseudo-image $\overline{\lambda}$ of Plancherel measure on X. There exists an essentially unique family of measures $(\beta_{\mathcal{O}})_{\mathcal{O}\subset X}$ such that $d\xi = d\beta_{\mathcal{O}}(\xi)d\overline{\lambda}(\mathcal{O})$.*
(b) *For every orbit $\mathcal{O} \subset X$ let $\mu_{\mathcal{O}}$ be as in Lemma 14. There exists an essentially unique Borel function $\kappa : X \to \mathbb{R}_0^+$ such that, for $\overline{\lambda}$-almost*

all orbits,

$$\frac{d\beta_{\mathcal{O}}}{d\mu_{\mathcal{O}}}(\xi) = \kappa(\xi).$$

(c) *The function κ can be chosen in such a way that for all $h \in H$ and all ξ in a fixed H-invariant conull set,*

$$\kappa(\xi.h) = \kappa(\xi) \Delta_G(0,h)^{-1}.$$

In particular, κ is H-invariant iff G is unimodular. In this case, λ has a decomposition $(\overline{\lambda}, (\mu_{\mathcal{O}})_{\mathcal{O} \subset X})$, where $\overline{\lambda}$ is a suitable σ-finite measure.

Proof. Part (a) is Theorem 12 (a)⇒(b). For part (b) let μ be as defined in Lemma 14. Then μ and λ are equivalent σ-finite measures, as was shown in the proof of 12(a)⇒(b), and we find that

$$\kappa(\xi) = \frac{d\lambda}{d\mu}(\xi) = \frac{d\beta_{\mathcal{O}}}{d\mu_{\mathcal{O}}}(\xi)$$

is the desired global Radon–Nikodym derivative. Thus (b) follows. For part (c), we let $\mu_h(B) = \mu(B.h)$, and $\lambda_h(B) = \lambda(B.h)$. Then

$$\frac{d\mu_h}{d\mu}(\xi) = \Delta_H(h), \qquad \frac{d\lambda_h}{d\lambda}(\xi) = \delta(h).$$

For any nonnegative Borel map f on X, the definition of μ_h entails

$$\int_X f(\xi)\, d\mu_h(\xi) = \int_X f(\xi.h^{-1})\, d\mu(\xi).$$

It follows for $h \in H$ and arbitrary Borel sets $B \subset X$ that

$$\begin{aligned}\int_B \frac{d\lambda}{d\mu}(\xi.h)\, d\mu_h(\xi) &= \int_X \mathbf{1}_B(\xi)\, \frac{d\lambda}{d\mu}(\xi.h)\, d\mu_h(\xi) = \int_X \mathbf{1}_B(\xi.h^{-1}) \frac{d\lambda}{d\mu}(\xi)\, d\mu(\xi)\\ &= \int_{B.h} \frac{d\lambda}{d\mu}(\xi)\, d\mu(\xi) = \lambda_h(B) = \int_B \frac{d\lambda_h}{d\mu_h}(\xi)\, d\mu_h(\xi),\end{aligned}$$

and thus

$$\frac{d\lambda_h}{d\mu_h}(\xi) = \frac{d\lambda}{d\mu}(\xi.h) \qquad (\lambda\text{-a.e.})$$

Hence, for a.e. $\xi \in X$, the chain rule for Radon–Nikodym derivatives yields

$$\kappa(\xi.h) = \frac{d\lambda}{d\mu}(\xi.h) = \frac{d\lambda_h}{d\mu_h}(\xi) = \frac{d\lambda_h}{d\lambda}(\xi)\, \frac{d\lambda}{d\mu}(\xi)\, \frac{d\mu}{d\mu_h}(\xi) = \kappa(\xi)\, \frac{\delta(h)}{\Delta_H(h)},$$

which is the desired equality, except that the conull subset of X on which it holds may still depend on h. However, by [20, B.5], one finds a conull invariant Borel subset of X on which the relation holds everywhere, independent of h.

If κ is constant on the orbits, it defines a Borel mapping $\overline{\kappa}$ on X/H. Replacing each $\beta_{\mathcal{O}}$ by $\mu_{\mathcal{O}}$, we can make up for it by taking $\overline{\kappa}(\mathcal{O})d\overline{\lambda}(\mathcal{O})$ as the new measure on the orbit space. The result is a σ-finite measure $\overline{\kappa}d\overline{\lambda}$. ∎

The next result clarifies the role of the specific choice of $\overline{\lambda}$.

THEOREM 19. *Let $\mathcal{H} = \mathcal{H}_X \subset \mathrm{L}^2(A)$ be closed and π-invariant. There exists an admissible vector for $\mathcal{H}$ iff there exists a weakly admissible vector, and in addition,*

(i) *G is nonunimodular, or*
(ii) *G is unimodular, and with $\overline{\lambda}$ chosen according to Lemma* 18(c),

$$\overline{\lambda}(X/H) < \infty.$$

Proof. First assume that G is unimodular, and that ψ is an admissible vector. Then the Plancherel theorem and the measure decomposition over the orbits, with $\overline{\lambda}$ as in Lemma 18(c) and $\beta_{\mathcal{O}} = \mu_{\mathcal{O}}$, allow us to compute

$$\begin{aligned}\|\psi\|_2^2 &= \int\limits_X |\widehat{\psi}(\xi)|^2\, d\lambda(\xi) = \int\limits_{X/H}\int\limits_H |\widehat{\psi}(\xi)|^2\, d\mu_{\mathcal{O}}(\xi)\, d\overline{\lambda}(\mathcal{O}) \\ &= \int\limits_{X/H}\int\limits_H |\widehat{\psi}(\xi.h)|^2\, dh\, d\overline{\lambda}(\mathcal{O}) = \int\limits_{X/H} 1\, d\overline{\lambda}(\mathcal{O}) = \overline{\lambda}(X/H).\end{aligned}$$

Here the penultimate equality is due to admissibility of ψ. In particular, $\overline{\lambda}(X/H) < \infty$.

For the converse, assume that ψ_0 is a weakly admissible vector. Define

$$\Phi(\xi) = \Big(\int\limits_H |\widehat{\psi_0}(\xi.h)|^2\, dh\Big)^{1/2}.$$

By assumption, $0 < \Phi(\xi) < 1$ a.e. Let $\varphi(\xi) = \widehat{\psi_0}(\xi)/\Phi(\xi)$. It follows that

$$\int\limits_H |\varphi(\xi.h)|^2\, d\xi = 1 \qquad (\lambda\text{-a.e.}).$$

If G is unimodular, the measure decomposition allows us to compute

$$\|\varphi\|_2^2 = \int\limits_{X/H}\int\limits_H |\varphi(\xi.h)|^2\, d\xi\, d\overline{\lambda}(\mathcal{O}) = \overline{\lambda}(X/H).$$

Thus, if $\overline{\lambda}(X/H) < \infty$, the inverse Plancherel transform of φ is admissible for $\mathcal{H}_X$.

Finally, assume that G is nonunimodular. Then Δ_G is nontrivial on H, and there exists $h_0 \in H$ such that $\Delta_G(h_0) < 1/2$. Since $\overline{\lambda}$ is σ-finite, we can write X as a disjoint union $X = \bigcup_{n\in\mathbb{N}} V_n$, with $V_n \subset X$ Borel, H-invariant and with $\overline{\lambda}(V_n/H) < \infty$. Since $\widehat{\psi} \in \mathrm{L}^2(\widehat{A})$,

$$\Psi : \xi \mapsto \Big(\int\limits_{\xi.H} |\widehat{\psi}(\xi)|^2\, d\beta_{\xi.H}(\xi)\Big)^{1/2}$$

is finite a.e., and we may in addition assume that Ψ is bounded on each V_n; in particular, the functions $(\mathbf{1}_{V_n} \cdot \Psi)_{n\in\mathbb{N}}$ are square-integrable.

Now pick a sequence $(k_n)_{n\in\mathbb{N}}$ of integers satisfying

$$2^{-k_n}\|\mathbf{1}_{V_n} \cdot \Psi\|_2^2 < 2^{-n},$$

and let

$$\nu(\xi) = \sum_{n\in\mathbb{N}} \Delta_H(h_0)^{k_n}\varphi(\xi.h_0^{k_n}).$$

On the one hand,

$$\begin{aligned}
\int_X |\nu(\xi)|^2\,d\lambda(\xi) &= \int_{X/H}\int_{\mathcal{O}} |\nu(\xi)|^2\,d\beta_{\mathcal{O}}(\xi)\,d\overline{\lambda}(\mathcal{O}) \\
&= \sum_{n\in\mathbb{N}}\int_{V_n}\int_{\mathcal{O}} \Delta_H(h_0)^{k_n}|\varphi(\xi.h_0^{k_n})|^2\,d\beta_{\mathcal{O}}(\xi)\,d\overline{\lambda}(\mathcal{O}) \\
&= \sum_{n\in\mathbb{N}}\int_{V_n}\int_{\mathcal{O}} \Delta_H(h_0)^{k_n}\delta(h_0)^{-k_n}|\varphi(\xi)|^2\,d\beta_{\mathcal{O}}(\xi)\,d\overline{\lambda}(\mathcal{O}) \\
&= \sum_{n\in\mathbb{N}} \Delta_G(h_0)^{k_n}\int_{V_n}\int_{\mathcal{O}} |\varphi(\xi)|^2\,d\beta_{\mathcal{O}}(\xi)\,d\overline{\lambda}(\mathcal{O}) \\
&= \sum_{n\in\mathbb{N}} \Delta_G(h_0)^{k_n}\|\mathbf{1}_{V_n} \cdot \Psi\|_2^2 \le \sum_{n\in\mathbb{N}} 2^{-k_n}\|\mathbf{1}_{V_n} \cdot \Psi\|_2^2 < \infty,
\end{aligned}$$

by choice of the k_n. Hence ν is square-integrable. Moreover, the Calderón condition is also easily verified: For $x \in V_n$,

$$\int_H |\nu(\xi.h)|^2\,dh = \int_H |\varphi(\xi.hh_0^{k_n})|^2\Delta_H(h_0)^{k_n}\,dh = \int_H |\varphi(\xi.h)|^2\,dh = 1$$

by construction of φ. Thus the inverse Plancherel transform of ν is the desired admissible vector. ∎

Remark 20. For unimodular semidirect products, we do not have a clean-cut and complete characterization of the groups having an admissible vector for all of $\mathrm{L}^2(A)$. A straightforward adaptation of the proof for [10, Proposition 5.14] allows us to describe a rather general setting in which $\mathrm{L}^2(A)$ does not have an admissible vector:

Suppose that $G = A \rtimes H$ is unimodular, and has a weakly admissible vector. Let r be a topological automorphism of A. We assume that r has the following properties:

(i) r normalizes H.

(ii) For any (hence all) $B \subset H$ and $C \subset A$ of positive finite Haar measure,

$$\frac{|rBr^{-1}|}{|B|} \neq \frac{|r(C)|}{|C|}.$$

Then $\overline{\lambda}(X/H) = \infty$. In particular, there exists no admissible vector for $L^2(A)$.

This result applies in particular to $A = \mathbb{R}^d$: Choose $r = s \cdot \mathrm{Id}_{\mathbb{R}^d}$ with $s \neq 1$. Then r commutes with all elements of the matrix group H. In particular, conjugation with r leaves Haar measure on H invariant, whereas $|r(C)|/|C| = s^d$. Thus (ii) is ensured, which proves that there exist no admissible vectors in this case.

The final result concerns irreducible representations. Recall that irreducible representations with admissible vectors are called *discrete series representations*. Most early sources generalizing wavelets to higher dimension, e.g. [14, 18, 2, 8], restricted their attention to the discrete series case. The implication (b)⇒(a) of the following result has been proved for $A = \mathbb{R}^n$ in [8]. However, the converse was previously known only for special cases in the setting $A = \mathbb{R}^n$ and $H \subset \mathrm{GL}(n, \mathbb{R})$: For H discrete, it boils down to stating that no discrete series representation of that type exists, which was observed in [8, Remark 12]. For G unimodular, the converse was proved in [9, Proposition 2.7.1].

COROLLARY 21. *Let $\mathcal{H}_X \subset L^2(A)$ be a nontrivial closed π-invariant subspace. The following are equivalent:*

(a) *The restriction of π to $\mathcal{H}_X$ is a discrete series representation.*
(b) *There exists an orbit $\mathcal{O} \subset X$ such that $|X \setminus \mathcal{O}| = 0$, with associated compact stabilizers.*

Proof. For (b)⇒(a), the arguments given in [8] immediately carry over; see also [4].

Conversely, assume that π restricted to $\mathcal{H}_X$ is in the discrete series. If $X = W \cup V$ with disjoint, H-invariant Borel sets U, W of positive measure, then $\mathcal{H}_X = \mathcal{H}_W \oplus \mathcal{H}_V$ contradicts irreducibility. Thus the action of H on X is ergodic with respect to Haar measure. Since $\overline{\lambda}$ is standard on X/H, it follows by [1, Chapter I, Proposition 3.9] that there exists a conull orbit. The associated stabilizers must be compact by Theorem 6. ∎

REMARK 22. The measure decompositions discussed in this paper are closely related to direct integral theory. In order to see this connection, first note that the quasi-regular representation π is type I: Its commuting algebra is contained in the commuting algebra of the regular representation of A on $L^2(A)$; A being abelian, the latter algebra is commutative. Hence π is multiplicity-free, in particular type I. It therefore has a unique direct integral decomposition into irreducibles, which is closely related to the ergodic decomposition of λ.

For the sake of simplicity, let us assume that there exists a weakly admissible vector, so that the ergodic decomposition is in fact a decomposition over the orbits. Then the measure decomposition $d\lambda(\xi) = d\beta_{\mathcal{O}}(\xi) d\overline{\lambda}(\mathcal{O})$ gives rise to a direct integral decomposition

$$\mathrm{L}^2(\widehat{A}) \simeq \int_{X/H}^{\oplus} \mathrm{L}^2(\mathcal{O}, d\beta_{\mathcal{O}})\, d\overline{\lambda}(A).$$

It can be shown that this decomposition also applies to the representation, yielding

$$\pi \simeq \int_{X/H}^{\oplus} \mathrm{Ind}_{A \rtimes H_\xi}^{G}(\xi \times \mathbf{1})\, d\overline{\lambda}(\mathcal{O}),$$

where $\mathbf{1}$ denotes the trivial representation of H_ξ. By Mackey's theory, the induced representations are irreducible (and pairwise inequivalent), thus we have decomposed π into irreducibles.

But the orbit space $\widehat{A}/H$ also occurs in the direct integral decomposition of the regular representation of G. In fact, the existence of a weakly admissible vector for $\mathrm{L}^2(A)$ implies that the regular representation of G is type I: By Theorem 6, almost all stabilizers are compact, and the dual orbit space is standard up to a set of measure zero. Note that compactness of the stabilizer H_ξ entails that H_ξ has a type I regular ω-representation, where ω denotes an arbitrary multiplier on H_ξ. Furthermore, the orbit space is standard (outside a set of measure zero). Thus, by [15, Theorem 2.3], it follows that the regular representation of G is type I, and that the Plancherel measure of G is obtained as fibred measure with base space given by $\widehat{A}/H$, base measure given by $\overline{\lambda}$, and fibres given by the ω_ξ-duals of the H_ξ, where ω_ξ are suitably chosen multipliers on H_ξ.

Now the connection between π and the left regular representation can also be realized by observing that Mackey's construction yields a mapping

$$\widehat{A}/H \ni \xi.H \mapsto \mathrm{Ind}_{A \rtimes H_\xi}^{G}(\xi \times \mathbf{1}) \in \widehat{G}$$

identifying $\widehat{A}/H$ with a (Borel) subset of $\widehat{G}$. It then becomes apparent that the measure $\overline{\lambda}$ underlying the direct integral decomposition of π is nothing but the restriction of Plancherel measure of G to this subset. This is an alternative proof for the containment of π in the regular representation. This type of reasoning, using direct integral decompositions to study existence of inversion formulae, has been developed systematically in [10]. In particular, [10, Section 5.3] contains a rigorous investigation of the double role of the measure $\overline{\lambda}$ in decomposing both π and the regular representation. Note however that the underlying assumption of [10] is that G is type I. By contrast, we make no such initial assumption on G, and find that the regular representation is type I as a consequence of the existence of weakly admissible vectors.

REMARK 23. The results presented in this paper are satisfactory to a certain degree, since they provide a sharp characterization. However, we are not aware of an easy general procedure for the explicit verification of the criteria in concrete cases. Also, we do not know how our characterization relates to other criteria, in particular compactness of almost all ϵ-stabilizers, proved to be sufficient in [16].

To our knowledge, the first systematic and substantial investigation of regularity properties for orbit spaces was carried out by Glimm [12], who proved that standardness of the orbit space of a second countable locally compact group H acting continuously on a second countable locally compact space X is equivalent to a variety of conditions, most notably countable separatedness of X/H, or the existence of a Borel cross-section, or local compactness of the orbits in the relative topology. On the one hand, these results closely resemble our conditions (c) and (d) from Theorem 12, but also the ϵ-stabilizer condition: To see this, note that compactness of $H_{\epsilon,\xi}$, for some $\epsilon > 0$, is equivalent to (i) compactness of H_ξ, and in addition (ii) local compactness of the orbit $\xi.H$ in the relative topology (cf. the proof of [10, Proposition 5.7]). Hence, if Glimm's results were applicable to our setting, they would imply that weak admissibility of the dual action is equivalent to existence of a compact ϵ-stabilizer, for a.e. ξ.

However, a direct application of Glimm's results to our problem is impeded by the fact that, by definition, weak admissibility only concerns the behaviour of the orbits in a suitable conull subset. In particular, weak admissibility is robust under passage to a conull invariant subset, whereas the assumptions underlying Glimm's characterization can be seriously affected by this step: a conull Borel subset of a locally compact space no longer needs to be locally compact. It was mostly this obstacle that stopped previous efforts of the author to characterize weakly admissible group actions. Attempts to use more recent generalizations of Glimm's results for the study of admissibility got stuck for similar reasons.

Remark 24. Throughout this paper, all groups have been assumed to be second countable. Most of the measure-theoretic arguments in this paper strongly rely on countability assumptions, and it is currently open to what extent our results can be generalized beyond second countable groups.

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REFERENCES

[1] L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc. 62 (1966).

[2] D. Bernier and K. Taylor, *Wavelets from square-integrable representations*, SIAM J. Math. Anal. 27 (1996), 594–608.
[3] G. Bohnke, *Treillis d'ondelettes associés aux groupes de Lorentz*, Ann. Inst. H. Poincaré Phys. Théor. 54 (1991), 245–259.
[4] M. Fashandi, R. A. Kamyabi Gol, A. Niknam and M. A. Pourabdollah, *Continuous wavelet transform on a special homogeneous space*, J. Math. Phys. 44 (2003), 4260–4266.
[5] J. M. G. Fell and R. S. Doran, *Representations of ∗-Algebras, Locally Compact Groups, and Banach ∗-Algebraic Bundles, Vol. 1*, Pure Appl. Math. 125, Academic Press, 1988.
[6] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.
[7] —, *Real Analysis. Modern Techniques and Their Applications*, 2nd ed., Wiley, New York, 1999.
[8] H. Führ, *Wavelet frames and admissibility in higher dimensions*, J. Math. Phys. 37 (1996), 6353–6366.
[9] —, *Zur Konstruktion von Wavelettransformationen in höheren Dimensionen*, thesis, TU München, 1998.
[10] —, *Abstract Harmonic Analysis of Continuous Wavelet Transforms*, Lecture Notes in Math. 1863, Springer, 2005.
[11] H. Führ and M. Mayer, *Continuous wavelet transforms from semidirect products: Cyclic representations and Plancherel measure*, J. Fourier Anal. Appl. 8 (2002), 375–397.
[12] J. G. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. 101 (1961), 124–138.
[13] G. Greschonig and K. Schmidt, *Ergodic decomposition of quasi-invariant probability measures*, Colloq. Math. 84/85 (2000), 495–514.
[14] A. Grossmann, J. Morlet and T. Paul, *Transforms associated to square integrable group representations I: General results*, J. Math. Phys. 26 (1985), 2473–2479.
[15] A. Kleppner and R. L. Lipsman, *The Plancherel formula for group extensions, II*, Ann. Sci. École Norm. Sup. 6 (1973), 103–132.
[16] R. S. Laugesen, N. Weaver, G. L. Weiss and E. N. Wilson, *A characterization of the higher dimensional groups associated with continuous wavelets*, J. Geom. Anal. 12 (2002), 89–102.
[17] G. W. Mackey, *Induced representations of locally compact groups I*, Ann. of Math. 55 (1952), 101–139.
[18] R. Murenzi, *Ondelettes multidimensionnelles et application à l'analyse d'images*, thèse, Univ. Catholique de Louvain, Louvain-la-Neuve, 1990.
[19] D. W. Stroock, *Probability Theory. An Analytic View*, Cambridge Univ. Press, 1993.
[20] R. J. Zimmer, *Ergodic Theory and Semisimple Lie Groups*, Birkhäuser, Boston, 1986.

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