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DIAGONAL POINTS HAVING DENSE ORBIT

BY

T. K. SUBRAHMONIAN MOOTHATHU (Hyderabad)

Abstract. Let $f : X \to X$ be a topologically transitive continuous map of a compact metric space X. We investigate whether f can have the following stronger properties: (i) for each $m \in \mathbb{N}$, $f \times f^2 \times \cdots \times f^m : X^m \to X^m$ is transitive, (ii) for each $m \in \mathbb{N}$, there exists $x \in X$ such that the diagonal m-tuple $(x, x, \ldots, x)$ has a dense orbit in X^m under the action of $f \times f^2 \times \cdots \times f^m$. We show that (i), (ii) and weak mixing are equivalent for minimal homeomorphisms, that all mixing interval maps satisfy (ii), and that there are mixing subshifts not satisfying (ii).

1. Introduction. Using the structure theory of minimal systems, Glasner proved the following statement in [10]:

THEOREM 1. *Let $f : X \to X$ be a minimal, weakly mixing homeomorphism of a compact metric space X. Then, for each $m \in \mathbb{N}$, there exists a residual set of points $x \in X$ such that the diagonal m-tuple $(x, \ldots, x)$ has a dense orbit in X^m under the action of $f \times f^2 \times \cdots \times f^m$.*

In this article, we provide a simplified proof of this theorem without resorting to the heavy machinery of structure theory. More generally, if X is a compact metric space and $f : X \to X$ is a topologically transitive continuous map, we may ask whether f should possess the following two properties:

(i) For each $m \in \mathbb{N}$, $f \times f^2 \times \cdots \times f^m : X^m \to X^m$ is transitive.
(ii) For each $m \in \mathbb{N}$, there exists $x \in X$ such that the diagonal m-tuple $(x, \ldots, x)$ has a dense orbit in X^m under the action of $f \times f^2 \times \cdots \times f^m$.

The two properties can be thought of as formulations of the "mutual independence" in a strong sense among the actions of various powers of f. Observe that total transitivity is necessary and mixing is sufficient for f to satisfy (i). We show that weak mixing is necessary but mixing is not sufficient for f to satisfy (ii). Thus we have the interesting fact that (i) does not imply (ii). On the other hand, we establish that the properties (i), (ii), and weak mixing are all equivalent for minimal homeomorphisms. We also

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observe that all mixing interval maps and mixing subshifts of finite type satisfy (ii).

2. Preliminaries. By a *dynamical system* we mean a pair (X, f) where X is a compact metric space and $f : X \to X$ is a continuous map.

CONVENTION. *Unless otherwise specified, in the statement of our results we will assume that X is without isolated points, to avoid pathologies.*

If $x \in X$, then the f-*orbit* of x is $\{x, f(x), f^2(x), f^3(x), \ldots\}$. For $x \in X$ and $U, V \subset X$ write

$$N_f(x, U) = \{n \in \mathbb{N} : f^n(x) \in U\}, \tag{1}$$

$$N_f(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}. \tag{2}$$

A point x in a dynamical system (X, f) is called a *recurrent point* for f if $N_f(x, U) \neq \emptyset$ for every neighborhood U of x. A dynamical system (X, f) is a *minimal system* (or f is a *minimal map*) if the f-orbit of every $x \in X$ is dense in X. It is easy to see that (X, f) is a minimal system iff X has no proper, nonempty, closed f-invariant subset. An element $x \in X$ is a *minimal point* if the restriction of f to the orbit-closure of x is minimal. Also, $x \in X$ is a minimal point iff $N_f(x, U)$ is a *syndetic set* (i.e., an infinite set with bounded gaps) [8] for every neighborhood U of x.

We say f is *transitive* if $N_f(U, V) \neq \emptyset$ for any two nonempty open sets $U, V \subset X$, f is *syndetically transitive* if $N_f(U, V)$ is syndetic for any two nonempty open sets $U, V \subset X$, f is *weakly mixing* if $f \times f : X^2 \to X^2$ is transitive, and f is *mixing* if $N_f(U, V)$ is cofinite in $\mathbb{N}$ for any two nonempty open sets $U, V \subset X$. We remark that f is weakly mixing iff $N_f(U, V)$ is *thick* (i.e., contains arbitrarily large blocks of consecutive integers) for any two nonempty open sets $U, V \subset X$, and consequently, when f is weakly mixing, any finite product $f \times \cdots \times f$ is transitive [8]. Now we introduce the following notions. We say f is *multi-transitive* if for each $m \in \mathbb{N}$, $f \times f^2 \times \cdots \times f^m : X^m \to X^m$ is transitive, Δ-*transitive* if for each $m \in \mathbb{N}$, there exists a dense G_δ set $Y \subset X$ such that for every $x \in Y$, $\{(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) : n \in \mathbb{N}\}$ is dense in X^m, and Δ-*mixing* if for each $m \in \mathbb{N}$ and infinite subset $A \subset \mathbb{N}$, there exists a dense G_δ set $Y \subset X$ such that for every $x \in Y$, $\{(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) : n \in A\}$ is dense in X^m.

PROPOSITION 1. *Let $m \in \mathbb{N}$ and $A \subset \mathbb{N}$ be infinite. Then the following are equivalent for (X, f):*

(i) *If $U_0, U_1, \ldots, U_m \subset X$ are nonempty open sets, there exists $n \in A$ such that $\bigcap_{i=0}^{m} f^{-in}(U_i) \neq \emptyset$.*
(ii) *There exists a dense G_δ subset $Y \subset X$ such that for every $x \in Y$, $\{(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) : n \in A\}$ is dense in X^m.*

Moreover, when $A = \mathbb{N}$*, statements* (i) *and* (ii) *are equivalent to*

(iii) *There exists* $x \in X$ *such that* $\{(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) : n \in \mathbb{N}\}$ *is dense in* X^m*.*

Proof. To prove (i) implies (ii), consider a countable base of open balls $\{B_k : k \in \mathbb{N}\}$ of X. Put

$$Y = \bigcap_{(k_1,\ldots,k_m)\in\mathbb{N}^m} \bigcup_{n\in A} \bigcap_{i=1}^{m} f^{-in}(B_{k_i}). \tag{3}$$

The set $\bigcup_{n\in A} \bigcap_{i=1}^{m} f^{-in}(B_{k_i})$ is clearly open, and it is dense by (i). Thus by the Baire category theorem, Y is a dense G_δ subset of X. By construction, for any $x \in Y$, $\{(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) : n \in A\}$ is dense in X^m. To prove (ii) implies (i), choose $x \in Y \cap U_0$, and if $n \in A$ is such that $(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) \in U_1 \times \cdots \times U_m$, then $x \in \bigcap_{i=0}^{m} f^{-in}(U_i)$.

Now, suppose $A = \mathbb{N}$. Clearly (ii) implies (iii). We prove that (iii) implies (i). Choose $k \in \mathbb{N}$ such that $y = f^k(x) \in U_0$. Since $f \times f^2 \times \cdots \times f^m$ commutes with $f \times \cdots \times f$, the set $\{(f^n(y), f^{2n}(y), \ldots, f^{mn}(y)) : n \in \mathbb{N}\}$ is also dense in X^m. Hence $(f^n(y), f^{2n}(y), \ldots, f^{mn}(y)) \in U_1 \times \cdots \times U_m$ for some $n \in \mathbb{N}$. Thus, $y \in \bigcap_{i=0}^{m} f^{-in}(U_i)$. ∎

Let (X, f), (Y, g) be two dynamical systems. Then (X, f) is an *extension* of (Y, g), or (Y, g) is a *factor* of (X, f), if there is a continuous surjection $h : X \to Y$ (called a *factor map*) such that $h \circ f = g \circ h$. If further $\{x \in X : h^{-1}(h(x)) = \{x\}\}$ is residual in X, (X, f) is said to be an *almost one-one extension* of (Y, g). If (X, f) is an almost one-one extension of (Y, g) via a factor map $h : X \to Y$, then $\mathrm{int}[h(U)] \neq \emptyset$ for every nonempty open $U \subset X$ and the set $\{y \in Y : |h^{-1}(y)| = 1\}$ is residual in Y.

A point x in a dynamical system (X, f) is a *point of equicontinuity* for f if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{N}$ and every $y \in X$. We denote the set of equicontinuity points by $\mathrm{Eq}(f)$. If $\mathrm{Eq}(f) = X$, we say the system (X, f) is *equicontinuous* (note that then, for a given $\epsilon > 0$, using the compactness of X we can find a uniform $\delta > 0$ which works for all points in X). It is a well-known fact that if (X, f) is a dynamical system, then it has a *maximal equicontinuous factor* (Y, g) in the sense that any equicontinuous factor (Y_1, g_1) of (X, g) is a factor of (Y, g) (see [7] for instance).

It is evident that multi-transitivity, Δ-transitivity and Δ-mixing are preserved under factors, and that multi-transitivity is preserved under almost one-one extensions.

PROPOSITION 2. *Let* (Y, g) *be a dynamical system with* g *semi-open (that is,* $\mathrm{int}[g(V)] \neq \emptyset$ *for every nonempty open* $V \subset Y$*) and let* (X, f) *be an* *almost one-one extension of* (Y, g)*. If* g *is* Δ*-transitive or* Δ*-mixing, then so is* f*.*

Proof. We show only that f is Δ-mixing if g is. Let $h : X \to Y$ be an almost one-one factor map with $h \circ f = g \circ h$. Given $m \in \mathbb{N}$, an infinite subset $A \subset \mathbb{N}$ and nonempty open sets $U_0, \ldots, U_m \subset X$, let $V_i = \mathrm{int}[h(U_i)]$ for $i = 0, 1, \ldots, m$. Since g is Δ-mixing, there exists $n \in A$ with $\bigcap_{i=0}^{m} g^{-in}(V_i) \neq \emptyset$ by Proposition 1. As g is semi-open, the g-preimage of any dense open subset of Y must be dense and open in Y. Therefore, if we consider the residual set $Z = \{y \in Y : |h^{-1}(y)| = 1\}$, then $g^{-k}(Z)$ is residual in Y for any $k \geq 0$. Consequently, we can find a point y in the intersection of the residual set $\bigcap_{i=0}^{m} g^{-in}(Z)$ and the nonempty open set $\bigcap_{i=0}^{m} g^{-in}(V_i)$. If $x \in X$ is the unique point with $h(x) = y$, then $x \in \bigcap_{i=0}^{m} f^{-in}(U_i)$. Thus by Proposition 1, f is Δ-mixing. ∎

We remark that if g is either a minimal map or a transitive interval map, then g is semi-open (see [15] for the case of a minimal map). We do not know whether the assumption of semi-openness of g can be removed from Proposition 2.

3. Relation to other forms of transitivity. There is a rich collection of (mutually inequivalent) stronger versions of transitivity in the literature; the reader may have a look at [12, 13] for instance. In this section, we try to clarify the relations among multi-transitivity, Δ-transitivity, Δ-mixing and a few other prominent notions of stronger forms of transitivity.

PROPOSITION 3.

(i) *Mixing* $\Rightarrow$ *multi-transitivity* $\Rightarrow$ *total transitivity.*
(ii) Δ*-transitivity* $\Rightarrow$ *weak mixing.*
(iii) Δ*-mixing* $\Rightarrow$ *mixing.*
(iv) *There is a mixing dynamical system that is not* Δ*-transitive.*

Proof. (i) is easy.

(ii) Let (X, f) be a dynamical system and let $U, V \subset X$ be nonempty open sets. To show (X, f) is weakly mixing, it suffices to show that $N_f(U, U) \cap N_f(U, V) \neq \emptyset$ (see [4]). Now, by Δ-transitivity and Proposition 1, there exists $n \in \mathbb{N}$ with $U \cap f^{-n}(U) \cap f^{-2n}(V) \neq \emptyset$. If x belongs to this intersection, then $x \in U \cap f^{-n}(U)$ and $f^n(x) \in U \cap f^{-n}(V)$ so that $n \in N_f(U, U) \cap N_f(U, V)$.

(iii) If (X, f) is not mixing, then there exist nonempty open sets $U, V \subset X$ and an infinite $A \subset \mathbb{N}$ such that $U \cap f^{-n}(V) = \emptyset$ for every $n \in A$. Then there cannot be any $x \in U$ satisfying $(f^n(x), f^{2n}(x)) \in U \times V$ for some $n \in A$, for otherwise $f^n(x) \in U \cap f^{-n}(V)$, a contradiction.

(iv) Let $\Sigma_2 = \{x = (x_n)_{n=-\infty}^{\infty} : x_n \in \{0,1\}\}$. With respect to the product topology, Σ_2 is homeomorphic to the Cantor set and the left shift $\sigma : \Sigma_2 \to \Sigma_2$ is a homeomorphism. Let $X \subset \Sigma_2$ be the collection of all $x \in \Sigma_2$ satisfying the following two conditions: the word 11 does not appear in x, and if u, v are nonempty words over $\{0,1\}$ with $1u1v1$ appearing in x, then u and v have different lengths. Then X is closed, nonempty (as $0^\infty \in X$) and σ-invariant so that (X,σ) is a dynamical system. It is also not difficult to see that X has no isolated points.

Claim 1. *(X,σ) is mixing.*

Proof of Claim 1. Let $x, y \in X$ and $k \in \mathbb{N}$. Let $u = x_{-k}\cdots x_0 \cdots x_k$, $v = y_{-k}\cdots y_0\cdots y_k$, $U = \{z \in X : z_{-k}\cdots z_0\cdots z_k = u\}$ and $V = \{z \in X : z_{-k}\cdots z_0 \cdots z_k = v\}$. In the product topology, U, V are basic neighborhoods of x, y respectively. Now, $z(n) \in \Sigma_2$ defined as $z(n) = 0^\infty u 0^n v 0^\infty$ has the property that $z(n) \in U \cap \sigma^{-(n+2k+1)}(V)$ for all $n \in \mathbb{N}$, and $z(n) \in X$ for all large n.

Claim 2. *There does not exist $x \in X$ such that $\{(\sigma^n(x), \sigma^{2n}(x)) : n \in \mathbb{N}\}$ is dense in X^2.*

Proof of Claim 2. Suppose there is such an $x \in X$. Let $W = \{y \in X : y_0 = 1\}$, and choose $k \in \mathbb{N}$ with $y = \sigma^k(x) \in W$. Now, (y, y) must also have a dense orbit under $\sigma \times \sigma^2$, so there is $n \in \mathbb{N}$ such that $(\sigma^n(y), \sigma^{2n}(y)) \in W \times W$. Hence $y, \sigma^n(y), \sigma^{2n}(y) \in W$, and therefore $y_0\cdots y_n \cdots y_{2n}$ is of the form $1u1v1$ with u, v having the same length, a contradiction. ∎

The rest of this section is aimed at proving that for a minimal system, weak mixing and multi-transitivity coincide. For one of the implications, we look at the properties of the "set of visiting times" $N_f(U,V)$. The other implication is deduced by studying the maximal equicontinuous factor. The following result is deduced as a corollary to the Weiss–Akin–Glasner Theorem, in [12]. We provide a direct proof for the sake of completeness.

Proposition 4. *Let (X, f), (Y, g) be dynamical systems. If f, g are both syndetically transitive and weakly mixing, then $f \times g : X \times Y \to X \times Y$ is syndetically transitive and weakly mixing.*

Lemma 1. *Let $f : X \to X$ be weakly mixing and syndetically transitive. Then for every $k \in \mathbb{N}$ and for any two nonempty open sets $U, V \subset X$, there exists a syndetic set $A \subset \mathbb{N}$ such that $A + \{1,\ldots,k\} \subset N_f(U,V)$.*

Proof. By weak mixing, choose $m \in \mathbb{N}$ such that for $1 \le j \le k$, $U_j := U \cap f^{-(m+j)}(V) \neq \emptyset$. Let $r_1, \ldots, r_k \in \mathbb{N}$ be such that $W := \bigcap_{j=1}^{k} f^{-r_j}(U_j) \neq \emptyset$. If $n \in N_f(W,W)$ and $x \in W \cap f^{-n}(W)$, put $y_j = f^{r_j}(x)$ for $j = 1,\ldots,k$. Then we see that $y_j \in U_j$ and $f^{n+m+j}(y_j) \in V$. Thus $n+m+j \in N_f(U,V)$.

This proves that $[N_f(W,W)+m]+\{1,\dots,k\}\subset N_f(U,V)$; and $N_f(W,W)$ is syndetic. ∎

COROLLARY 1. *Let (X,f) be a dynamical system. If f is weakly mixing and syndetically transitive, then f^k is weakly mixing and syndetically transitive for every $k\in\mathbb{N}$.*

Proof of Proposition 4. Let $U_1,U_2\subset X$ and $V_1,V_2\subset Y$ be nonempty open sets, and let $A=N_f(U_1,U_2)$, $B=N_g(V_1,V_2)$. We should show that $A\cap B$ is both syndetic and thick. Now, if $k_1\in\mathbb{N}$ is a bound for the gaps in the syndetic set A, then applying Lemma 1 to g, we find a syndetic set $C\subset\mathbb{N}$ such that $C+\{1,\dots,k_1\}\subset B$. If k_2 is a bound for the gaps in C, then $A\cap(C+\{1,\dots,k_1\})$ is a syndetic set with gaps bounded by k_1+k_2. Also $A\cap(C+\{1,\dots,k_1\})\subset A\cap B$, and thus $A\cap B$ is syndetic. Now, to show $A\cap B$ is thick, fix $k\in\mathbb{N}$, and let $D\subset\mathbb{N}$ be a syndetic set with $D+\{1,\dots,k\}\subset B$. Since A is thick, $A\cap(D+\{1,\dots,k\})$ contains a block of k consecutive integers. Thus $A\cap B$ contains a block of k consecutive integers. ∎

COROLLARY 2. *Let (X,f) be a dynamical system. If f is weakly mixing and syndetically transitive, then $f\times\cdots\times f^m : X^m\to X^m$ is weakly mixing and syndetically transitive for every $m\in\mathbb{N}$. In particular, f is multi-transitive.*

COROLLARY 3. *Let (X,f) be a dynamical system. If f is weakly mixing and syndetically transitive, then for every $m,k\in\mathbb{N}$ and nonempty open sets $U,V\subset X$, there exists a syndetic set $A\subset\mathbb{N}$ such that $\bigcup_{i=1}^m i[A+\{1,\dots,k\}]=\{i(n+j): 1\le i\le m,\ n\in A,\ 1\le j\le k\}\subset N_f(U,V)$.*

Proof. By Corollary 2, we know that $f\times\cdots\times f^m$ is weakly mixing and syndetically transitive. Now apply Lemma 1 to $f\times\cdots\times f^m : X^m\to X^m$ and open subsets $U\times\cdots\times U$, $V\times\cdots\times V$ of X^m. ∎

THEOREM 2. *Let (X,f) be a dynamical system with X having at least two points. If $f\times f^2 : X^2\to X^2$ is transitive, then* $\mathrm{Eq}(f)=\emptyset$.

Proof. Let if possible $x\in\mathrm{Eq}(f)$. We claim that $\{(f^n(x),f^{2n}(x)) : n=0,1,2,\dots\}$ cannot be dense in X^2. For otherwise, we get a contradiction as follows. Let $U,V\subset X$ be nonempty open sets and let $k\in\mathbb{N}$ be such that $y=f^k(x)\in U$. Then $\{(f^n(y),f^{2n}(y)) : n=0,1,2,\dots\}$ must also be dense in X^2 since $f\times f^2$ commutes with $f\times f$. If $(f^n(y),f^{2n}(y))\in U\times V$, then $y\in U\cap f^{-n}(U)$ and $f^n(y)\in U\cap f^{-n}(V)$ so that $n\in N_f(U,U)\cap N_f(U,V)$. This implies that f is weakly mixing, and therefore $\mathrm{Eq}(f)$ must be empty since X is not a singleton, a contradiction.

Now, by the claim, we can choose $\epsilon>0$ such that $\bigcup_{n=0}^\infty[B(f^n(x),\epsilon)\times B(f^{2n}(x),\epsilon)]$ is not dense in X^2. For this ϵ, choose $\delta>0$ by using the fact that $x \in \mathrm{Eq}(f)$. Then the $(f^n \times f^{2n})$-image of $B(x,\delta) \times B(x,\delta)$ is contained in $B(f^n(x),\epsilon) \times B(f^{2n}(x),\epsilon)$ for each $n \in \mathbb{N}$, and therefore $f \times f^2$ cannot be transitive. ∎

This implies, for instance that irrational rotations of the circle are not multi-transitive, and thus multi-transitivity is strictly stronger than total transitivity. Also, we have:

COROLLARY 4. *Let (X,f) be a dynamical system. If $f \times f^2 : X^2 \to X^2$ is transitive, then the maximal equicontinuous factor of (X,f) must be trivial.*

Proof. Let (Y,g) be the maximal equicontinuous factor. Since (Y,g) is a factor, $g \times g^2 : Y^2 \to Y^2$ is transitive. If Y has an isolated point, then Y must be a single periodic orbit by the transitivity of g and then Y must be a singleton by the transitivity of $g \times g^2$. If Y has no isolated points, then Y must be a singleton by the above theorem. ∎

COROLLARY 5. *If (X,f) is a minimal system, then the following are equivalent:*

(i) *$f \times f^2 : X^2 \to X^2$ is transitive.*
(ii) *f is multi-transitive.*
(iii) *f is weakly mixing.*

Proof. To see (i) implies (iii), use the well-known fact that a minimal system is weakly mixing iff its maximal equicontinuous factor is trivial (see Theorem 4.31 of [11]). And (iii) implies (ii) by Corollary 2. ∎

4. Δ-mixing for interval maps and subshifts of finite type. Interval maps and subshifts of finite type are two of the most studied classes of dynamical systems. For systems in both these classes, total transitivity is equivalent to mixing. Below we observe that mixing implies Δ-mixing for interval maps and subshifts of finite type.

THEOREM 3. *Let (X,f) be a dynamical system. Suppose that for any two nonempty open sets $U,V \subset X$, there exist a nonempty open set $W \subset V$ and $n_0 \in \mathbb{N}$ such that $W \subset f^n(U)$ for every $n \geq n_0$. Then f is Δ-mixing.*

Proof. We use induction on m, where the mth statement of induction is that for nonempty open sets $U_0, U_1, \ldots, U_m \subset X$, and infinite $A \subset \mathbb{N}$, we have $\bigcap_{i=0}^{m} f^{-in}(U_i) \neq \emptyset$ for some $n \in A$. The given hypothesis implies that f is mixing and hence the induction statement is true for $m = 1$. Now, assume the statement up to m and consider nonempty open sets $U_0, U_1, \ldots, U_{m+1} \subset X$, and infinite $A \subset \mathbb{N}$. Choose a nonempty open set $W \subset U_1$ and $n_0 \in \mathbb{N}$ such that $W \subset f^n(U_0)$ for every $n \geq n_0$. By induction assumption, there exists $n \in A$ with $n \geq n_0$ so that $W' := W \cap f^{-n}(U_2) \cap f^{-2n}(U_3) \cap \cdots \cap f^{-mn}(U_{m+1}) \neq \emptyset$. Since $W \subset U_1$ and

$W' \subset f^n(U_0)$, we get $\bigcap_{i=0}^{m+1} f^{-in}(U_i) \neq \emptyset$. This completes the induction step for $m+1$, and now by Proposition 1, f is Δ-mixing. ∎

COROLLARY 6. *Let $f : [0,1] \to [0,1]$ be continuous. If f is mixing, then f is Δ-mixing.*

Proof. Mixing interval maps are known to satisfy the hypothesis of Theorem 3. We give a quick sketch for this. If $U, V \subset [0,1]$ are open intervals, choose $\delta \in (0, 1/2)$ such that $W := (\delta, 1-\delta) \cap V \neq \emptyset$. By mixing, there exists $n_0 \in \mathbb{N}$ such that $f^n(U)$ intersects $[0,\delta)$ and $(1-\delta, 1]$ for every $n \geq n_0$. Since $f^n(U)$ is connected, it follows that $W \subset f^n(U)$ for every $n \geq n_0$. ∎

See Sections 3.4 and 3.7.3 of [5], or Chapter 2 and Section 6.3 of [16] for the theory of *subshifts of finite type*. We skip the details.

PROPOSITION 5. *Let (X, f) be a mixing subshift of finite type. Then f is Δ-mixing.*

Proof. We only outline a proof. Let $m \in \mathbb{N}$ and $U_0, U_1, \ldots, U_m \subset X$ be nonempty open sets. It may be assumed that U_i's are basic open sets (sometimes called 'cylinders') represented by words $w(0), \ldots, w(m)$ respectively, and that all the words $w(i)$'s have equal length, say p. Since (X, f) is a mixing subshift of finite type, there is an associated square matrix M such that $M^k > 0$ for some $k \in \mathbb{N}$. This implies that for any $n \geq k$, there are words $u(1), \ldots, u(m)$ of length n such that the word $w(0)u(1)w(1)u(2)w(2)\cdots$ $\cdots u(m)w(m)$ appears in some $x \in X$. Consequently, $\bigcap_{i=0}^{m} f^{-i(n+p)}(U_i) \neq \emptyset$ for all $n \geq k$, and therefore f is Δ-mixing by Proposition 1. ∎

5. Strong transitivity and Δ-transitivity. A dynamical system (X, f) is said to be *strongly transitive* if for any nonempty open set $U \subset X$, we have $\bigcup_{i=1}^{m} f^i(U) = X$ for some $m \in \mathbb{N}$. It is not difficult to see that minimal systems are strongly transitive, and it is also known [14] that *positively expansive*, transitive open maps are strongly transitive.

THEOREM 4. *Let (X, f) be a dynamical system. If f is a weakly mixing and strongly transitive homeomorphism, f is Δ-transitive.*

Proof. We will borrow the elementary aspects of Glasner's proof [10] to go half-way and then will invoke Corollary 2. Let P_m denote the statement that there exists a dense G_δ subset $Y \subset X$ such that for every $x \in Y$, $\{(f^n(x), f^{2n}(x), \ldots, f^{mn}(x)) : n \in \mathbb{N}\}$ is dense in X^m. We prove P_m by induction on m. First, P_1 is clearly true. Now, assume that $P_1, \ldots, P_m$ are true. For a subset $Z \subset X$, let us write $Z^* = \{(f^n(x), \ldots, f^{(m+1)n}(x)) : x \in Z, n \in \mathbb{N}\} \subset X^{m+1}$. To prove P_{m+1}, in view of Proposition 1 it suffices to show that $\overline{U^*} = X^{m+1}$ for every nonempty open set $U \subset X$. Now, since strong transitivity implies syndetical transitivity, f is multi-transitive by

Corollary 2, and in particular $f \times \cdots \times f^{m+1}$ is transitive. Hence, as $\overline{U^*}$ is $(f \times \cdots \times f^{m+1})$-invariant, to establish P_{m+1} it is enough to show that $\overline{U^*}$ has nonempty interior in X^{m+1} for every nonempty open set $U \subset X$.

Let $\phi = f \times \cdots \times f$, the $(m+1)$-fold product. By the strong transitivity of f, $X = \bigcup_{j=0}^{k} f^j(U)$ for some $k \in \mathbb{N}$ and hence $X^* = \bigcup_{j=0}^{k} \phi^j(U^*)$ since ϕ commutes with $f \times \cdots \times f^{m+1}$. Thus $\overline{X^*} = \bigcup_{j=0}^{k} \phi^j(\overline{U^*})$, since ϕ is a continuous map in the compact metric space X^{m+1}. To show that $\mathrm{int}[\overline{U^*}] \neq \emptyset$, it suffices to show $\mathrm{int}[\phi^j(\overline{U^*})] \neq \emptyset$ for some j because ϕ is a homeomorphism (here we use the fact that f is a homeomorphism). Now, it is sufficient to show that $\mathrm{int}[\overline{X^*}] \neq \emptyset$. We proceed to do this.

Consider nonempty open sets $V_1, \ldots, V_{m+1} \subset X$. By the induction assumption, P_m is true and hence, by Proposition 1, $\bigcap_{i=0}^{m} f^{-in}(V_{i+1}) \neq \emptyset$ for some $n \in \mathbb{N}$. Applying f^{-n}, we have $\bigcap_{i=1}^{m+1} f^{-in}(V_i) \neq \emptyset$, as f is surjective. If $x \in \bigcap_{i=1}^{m+1} f^{-in}(V_i)$, then $(f^n(x), \ldots, f^{(m+1)n}(x)) \in X^* \cap (V_1 \times \cdots \times V_{m+1})$. This shows that $\overline{X^*} = X^{m+1}$, completing the proof. ∎

Combining Theorem 4 with Corollary 5, we arrive at a neat result for the class of minimal homeomorphisms.

Corollary 7. *Let (X, f) be a minimal homeomorphism. Then the following are equivalent:*

(i) $f \times f^2 : X^2 \to X^2$ *is transitive.*
(ii) f *is multi-transitive.*
(iii) f *is weakly mixing.*
(iv) f *is Δ-transitive.*

Remarks. The implication (iii)⇒(iv) is Glasner's result [10]. We have obtained it through a simpler proof without resorting to the structure theory of minimal systems. In view of the above corollary, any weakly mixing, minimal, non-mixing homeomorphism (for example, the *Chacon map*, cf. p. 27 of [11]) serves as an example of a Δ-transitive system that is not Δ-mixing. It is tempting to think that we may be able to show that any mixing, strongly transitive system is Δ-mixing, but surprisingly this is not true. A couterexample can be found in [6], where the authors produce (for a slightly different purpose) a mixing, minimal *substitution* dynamical system (X, f) having the property that there exist nonempty open sets $U_0, U_1, U_2 \subset X$ and infinite $A \subset \mathbb{N}$ such that $\bigcap_{i=0}^{2} f^{-in}(U_i) = \emptyset$ for every $n \in A$.

Question. Does Theorem 4 remain true for non-homeomorphic continuous maps?

For a dynamical system (X, f) (not necessarily minimal), we summarize below our knowledge of the implications and non-implications among various stronger forms of transitivity discussed in this article. The implications are:

$$\begin{tikzcd}
 & \text{Mixing} \arrow[r] \arrow[dr] & \text{Weak mixing} \arrow[dr] & \\
\Delta\text{-mixing} \arrow[ur] \arrow[dr] & & & \text{Total transitivity} \\
 & \Delta\text{-transitivity} \arrow[r] \arrow[ur] & \text{Multi-transitivity} \arrow[ur] &
\end{tikzcd}$$

There are no implications between mixing and Δ-transitivity, and for each implication shown in the diagram, the reverse implication is false. Here, each non-implication is based on one of the four examples: irrational rotation of the unit circle, the example from [6] alluded to earlier, Chacon map (cf. [11]) and the example constructed for Proposition 3(iv). The following question is open.

QUESTION. Are there any implications between weak mixing and multi-transitivity (if the system is not syndetically transitive)?

6. Recurrence properties of $f \times f^2 \times \cdots \times f^m$. We conclude with a few observations about the nature of recurrent points of $f \times f^2 \times \cdots \times f^m$. If $(x_1, \ldots, x_m) \in X^m$ is a minimal point of $f \times f^2 \times \cdots \times f^m$ (there is at least one minimal point, by an application of Zorn's lemma), then for any $n_1, \ldots, n_m$ in $\mathbb{N}$, $(f^{n_1}(x_1), \ldots, f^{n_m}(x_m))$ is a minimal point for $f \times f^2 \times \cdots \times f^m$ since $f^{n_1} \times \cdots \times f^{n_m}$ commutes with $f \times f^2 \times \cdots \times f^m$. Thus, if (X, f) is a minimal dynamical system, then $f \times f^2 \times \cdots \times f^m$ has a dense set of minimal points in X^m for every $m \in \mathbb{N}$.

PROPOSITION 6. *Let (X, f) be a dynamical system and suppose f has an invariant Borel probability measure of full support. Then for every $m \in \mathbb{N}$, there exists a dense G_δ set $Y \subset X$ such that for every $x \in Y$, the diagonal m-tuple $(x, \ldots, x)$ is a recurrent point for $f \times f^2 \times \cdots \times f^m$.*

Proof. Applying the multiple recurrence theorem [9] to the commuting maps $f, f^2, \ldots, f^m$, we find that for any nonempty open set $U \subset X$, $\bigcap_{i=0}^{m} f^{-in}(U) \neq \emptyset$ for some $n \in \mathbb{N}$, as U has positive measure. Now, let $Y = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} Y(k, n)$, where $Y(k, n) = \{x \in X : d(x, f^{in}(x)) < 1/k \text{ for } i = 1, \ldots, m\}$. Then $Y(k, n)$'s are open, and $\bigcup_{n=1}^{\infty} Y(k, n)$ is dense in X by the first sentence of the proof. Thus Y is a dense G_δ, and every $x \in Y$ clearly has the required property. ∎

In a dynamical system (X, f), a point x is said to be *regularly recurrent* if for any neighborhood U of x, there exists $l \in \mathbb{N}$ such that $l\mathbb{N} \subset N_f(x, U)$. For instance, an *odometer* is a minimal system in which all points are regularly recurrent (see [7]). It is easy to observe that if $x \in X$ is a regularly recurrent point, then for every $m \in \mathbb{N}$, the diagonal m-tuple $(x, \ldots, x)$ is a regularly recurrent and hence minimal point for $f \times f^2 \times \cdots \times f^m$.

For points x, y in a dynamical system (X, f), we say (x, y) is a *proximal pair* for f if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, where d is the metric on X. From the theory of *enveloping semigoups* (see p. 28 of [11]), we know that for every $x \in X$, there is a minimal point y in the orbit-closure of x such that (x, y) is a proximal pair for f. Using this fact, we can say the following about the possible minimality of the diagonal m-tuple $(x, \dots, x)$.

PROPOSITION 7. *Let (X, f) be a dynamical system, let $x \in X$ be a minimal point, and let Y be the closure of the f-orbit of x. If (x, y) is not a proximal pair for f for any $y \in Y \setminus \{x\}$, then for every $m \in \mathbb{N}$, the diagonal m-tuple $(x, \dots, x)$ is a minimal point of $f \times f^2 \times \dots \times f^m : X^m \to X^m$.*

Proof. By the remark above, there is a minimal point $(y_1, \dots, y_m) \in Y^m$ for $f \times f^2 \times \dots \times f^m$ such that $((x, \dots, x), (y_1, \dots, y_m))$ is a proximal pair for $f \times f^2 \times \dots \times f^m$. It follows that for $1 \le i \le m$, (x, y_i) is a proximal pair for f^i and hence for f. Hence $y_i = x$ for $1 \le i \le m$. ∎

COROLLARY 8. *Let (X, f) be a dynamical system, let $x \in X$ be a minimal point, and let Y be the closure of the f-orbit of x. If x is a point of equicontinuity for $f|_Y : Y \to Y$, then for every $m \in \mathbb{N}$, the diagonal m-tuple $(x, \dots, x)$ is a minimal point of $f \times f^2 \times \dots \times f^m : X^m \to X^m$.*

Proof. A minimal system containing at least one point of equicontinuity must be equicontinuous (see [2]), and thus $(Y, f|_Y)$ is equicontinuous. It is also known (see Theorem 3.4 of [3]) that two distinct points cannot form a proximal pair in a surjective equicontinuous system. ∎

In particular, Corollary 8 is applicable for minimal equicontinuous systems such as irrational rotations. Two related questions have eluded our attempt.

QUESTIONS. (i) In Proposition 7, can we drop the assumption that (x, y) is not a proximal pair for any $y \in Y \setminus \{x\}$? (ii) Can $f \times f^2 \times \dots \times f^m : X^m \to X^m$ be minimal if $m \ge 2$ and X has at least two elements?

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T. K. Subrahmonian Moothathu
Department of Mathematics and Statistics
University of Hyderabad
Hyderabad 500 046, India
E-mail: tksubru@gmail.com

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