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# *ALMOST ff-UNIVERSAL AND Q-UNIVERSAL VARIETIES OF MODULAR* 0*-LATTICES*

BY

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*To Professor Věra Trnková on her 70th birthday*

**Abstract.** A variety $\mathbb{V}$ of algebras of a finite type is almost $ff$-universal if there is a finiteness-preserving faithful functor $F : \mathbb{G} \to \mathbb{V}$ from the category $\mathbb{G}$ of all graphs and their compatible maps such that $F\gamma$ is nonconstant for every $\gamma$ and every nonconstant homomorphism $h : FG \to FG'$ has the form $h = F\gamma$ for some $\gamma : G \to G'$. A variety $\mathbb{V}$ is $Q$-universal if its lattice of subquasivarieties has the lattice of subquasivarieties of any quasivariety of algebras of a finite type as the quotient of its sublattice. For a variety $\mathbb{V}$ of modular 0-lattices it is shown that $\mathbb{V}$ is almost $ff$-universal if and only if $\mathbb{V}$ is $Q$-universal, and that this is also equivalent to the non-distributivity of $\mathbb{V}$.

A concrete category $\mathbb{K}$ is (algebraically) *universal* if the category $\mathbb{G}$ of all graphs and all their compatible mappings has a full embedding into $\mathbb{K}$. When such a full embedding sends every finite graph to a $\mathbb{K}$-object whose underlying set is finite, we say that $\mathbb{K}$ is *finite-to-finite universal* (*ff-universal*). All universal categories have quite a rich structure: for instance, for every monoid $M$ they contain a proper class of pairwise non-isomorphic objects whose endomorphism monoids are isomorphic to $M$ (see [8]). An $ff$-universal category relevant to our considerations is formed by all $(0,1)$-homomorphisms between $(0,1)$-lattices from the variety $\mathrm{Var}_{0,1}(M_3)$ generated by the five-element modular nondistributive lattice $M_3$ (this fact and the fact that $\mathrm{Var}_{0,1}(M_3)$ is a minimal universal variety follow from the classification of universal varieties of $(0,1)$-lattices given in [5] and from [10]). On the other hand, the category of all lattices and all their homomorphisms is not universal because of the existence of constant homomorphisms, and neither is the category of all 0-lattices and their 0-preserving homomorphisms.

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Yet both these categories are *almost ff-universal*, that is, each contains a class of objects determining a full subcategory whose nonconstant morphisms are closed under composition and form an $ff$-universal category. In fact, already the varieties $\mathrm{Var}(M_{3,3})$ and $\mathrm{Var}_0(M_{3,3})$ generated by the modular eight-element lattice $M_{3,3}$ given by $0 < a, b, c < d$ and $c < d, e, f < 1$ are almost $ff$-universal, and the variety $\mathrm{Var}(M_{3,3})$ is also minimal in this respect (see [6]). For an overview of universality, we refer the reader to [8].

According to Sapir [9], a quasivariety $\mathbb{Q}$ of algebras of a finite similarity type is *Q-universal* if the inclusion-ordered lattice $L(\mathbb{Q})$ of its subquasivarieties has the property that for any quasivariety $\mathbb{R}$ of algebras of a finite type, the lattice $L(\mathbb{R})$ is a quotient lattice of a sublattice of $L(\mathbb{Q})$. Just as for categorical universality, numerous instances of $Q$-universal varieties exist and are documented by Adams and Dziobiak in [1, 2], for instance. Of particular interest here is the result by Dziobiak [4] characterizing the $Q$-universal varieties of modular lattices as those which contain the variety $\mathrm{Var}(M_{3,3})$.

The two types of universality are linked through the remarkable Adams–Dziobiak Theorem [3] saying that any $ff$-universal quasivariety of algebras of a finite type must be $Q$-universal (the converse implication is known to be false, see [3]). To further improve their result, Adams and Dziobiak asked whether a weaker form of categorical universality (such as almost $ff$-universality) would still imply $Q$-universality. Motivated by this question, in [7] we found an example showing that the categorical hypothesis cannot be weakened to its natural extreme.

The above discussion of known facts indicates the reasons for asking whether the variety $\mathrm{Var}_0(M_3)$ is almost $ff$-universal or $Q$-universal. In the two sections below we show that $\mathrm{Var}_0(M_3)$—and hence also $\mathrm{Var}_1(M_3)$—have both these properties.

**1. Categorical universality.** In this section we show that the variety $\mathrm{Var}_0(M_3)$ is finite-to-finite almost universal, by means of embedding an $ff$-universal full subcategory of the variety $\mathrm{Var}_{0,1}(M_3)$ of $(0,1)$-lattices (see [5]) into $\mathrm{Var}_0(M_3)$ via an almost full functor preserving finiteness. First we present a general form of the construction (to be also used elsewhere), and then its specific application.

Throughout the paper, we identify any natural number $n$ with the set $\{0, 1, \dots, n-1\}$. For a poset $P$ and any $p \in P$ we write $[p) = \{x \in P \mid p \le x\}$, $(p] = \{x \in P \mid x \le p\}$ and, for any $p, q \in P$ with $p \le q$ we write $[p, q] = \{x \in P \mid p \le x \le q\}$. Given lattices $A$ and $B$, we say that a sublattice $C \subseteq A \times B$ is *subdirect in* $A \times B$ if the restriction of both projections to $C$ is surjective. A family $\Sigma \subseteq \hom_{0,1}(A, B)$ of lattice $(0,1)$-homomorphisms is *separating* if for any distinct $x, y \in A$ there exists an $f \in \Sigma$ with $f(x) \neq f(y)$.

Thus $\hom_{0,1}(A, B)$ contains a separating family exactly when $A$ is a sublattice of some Cartesian power of $B$.

Next we present the basic step of the general lattice construction.

CONSTRUCTION. Let $A$ and $Q$ be $(0,1)$-lattices, let $a \in A \setminus \{0,1\}$, and let $c, d \in Q$ satisfy $0 < c < d < 1$ and $Q = [c) \cup (d]$. For fixed $c, d \in Q$, we write $A *_a Q = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \subseteq A \times Q$ with the (not necessarily disjoint) sets $S_0 = \{0\} \times (d]$, $S_1 = (a] \times [c, d]$, $S_2 = \{a\} \times [c)$, $S_3 = [a) \times [d)$ and $S_4 = A \times \{d\}$.

In what follows, we also assume that $(d] \setminus [c)$ is not a singleton.

LEMMA 1.1. *For any $(0,1)$-lattice $A$ and $a \in A \setminus \{0,1\}$, the set $A *_a Q = A * Q$ is a $(0,1)$-sublattice subdirect in $A \times Q$, and*

(1) $[(0, d), (1, d)] = A \times \{d\} \subseteq A * Q$;
(2) *if $h : A \to A'$ is a lattice $(0,1)$-homomorphism (or a 0-homomorphism) satisfying $h(a) = a'$ then the domain-range restriction $h * 1$ of $h \times 1_Q$ to $A * Q$ and $A' *_{a'} Q$ is a lattice $(0,1)$-homomorphism (or a 0-homomorphism) such that $(h * 1)(z, q) = (h(z), q)$ for all $(z, q) \in A * Q$;*
(3) *if $h : A \to A'$ is a lattice 0-homomorphism with $h(a) = a'$ then $(h * 1)^{-1}\{(0, q)\} = \{(0, q)\}$ for all $q \in (d] \setminus [c)$;*
(4) *if $h : A \to A'$ is a lattice 0-homomorphism with $h(a) = a'$ then $(h * 1)^{-1}\{(a', q)\} = \{(a, q)\}$ for all $q \in Q$ incomparable with $d$.*

*Proof.* First we show that $A * Q$ is a sublattice of $A \times Q$. It is easy to see that $S_i \subseteq A \times Q$ is a sublattice for each $i \in 5$. We proceed by exhausting the remaining possibilities. To make the verification easier, we use the explicit list below.

- $s_0 = (a_0, q_0) \in S_0$ iff $a_0 = 0$ and $q_0 \le d$;
- $s_1 = (a_1, q_1) \in S_1$ iff $a_1 \le a$ and $c \le q_1 \le d$;
- $s_2 = (a_2, q_2) \in S_2$ iff $a_2 = a$ and $c \le q_2$;
- $s_3 = (a_3, q_3) \in S_3$ iff $a \le a_3$ and $d \le q_3$;
- $s_4 = (a_4, q_4) \in S_4$ iff $a_4 \in A$ and $q_4 = d$.

$\{0, i\}$ for $i = 1, 2, 3, 4$. Let $s_0 \in S_0$ and $s_i \in S_i$. Then $s_0 \wedge s_i = (0, q_0 \wedge q_i) \in S_0$ because $q_0 \wedge q_i \le q_0 \le d$. Further $s_0 \vee s_i = (a_i, q_0 \vee q_i)$ for any $i \in 5$. If $i = 1$ then $c \le q_1 \le q_1 \vee q_0 \le d$ because $q_0, q_1 \le d$ and $s_0 \vee s_1 \in S_1$. If $i = 2$ then $s_0 \vee s_i = (a, q_0 \vee q_2) \in S_2$ because $c \le q_2 \le q_0 \vee q_2$. If $i = 3$ or $i = 4$ then $s_0 \vee s_i = (a_i, q_i) = s_i \in S_i$ because $q_0 \le d = q_4 \le q_3$.

$\{3, i\}$ for $i = 1, 2, 4$. Let $s_3 \in S_3$ and $s_i \in S_i$. Then $a_3 \ge a$ and $q_3 \ge d$ and hence $s_3 \vee s_i = (a_3 \vee a_i, q_3 \vee q_i) \in S_3$ because $a \le a_3 \le a_3 \vee a_i$ and $d \le q_3 \le q_3 \vee q_i$. If $i = 1$ then $s_3 \wedge s_1 = s_1 \in S_1$ because $a_1 \le a \le a_3$ and

$q_1 \leq d \leq q_3$. If $i = 2$ then $s_3 \wedge s_2 = (a, q_2 \wedge q_3) \in S_2$ because $q_3 \geq d \geq c$. If $i = 4$ then $s_3 \wedge s_4 = (a_3 \wedge a_4, d) \in S_4$ because $q_3 \geq d = q_4$.

$\{1, 2\}$. Let $s_1 \in S_1$ and $s_2 \in S_2$. Then $s_1 \vee s_2 = (a, q_1 \vee q_2) \in S_2$ because $a_1 \leq a = a_2$ and $c \leq q_2 \leq q_1 \vee q_2$, and $s_1 \wedge s_2 = (a_1, q_1 \wedge q_2) \in S_1$ because $c \leq q_1, q_2$ and $q_1 \wedge q_2 \leq q_1 \leq d$.

$\{1, 4\}$. Let $s_1 \in S_1$ and $s_4 \in S_4$. Then $s_1 \vee s_4 = (a_1 \vee a_4, d) \in S_4$, and $s_1 \wedge s_4 = (a_1 \wedge a_4, q_1) \in S_1$ because $a_1 \wedge a_4 \leq a_1 \leq a$ and $c \leq q_1 \leq d$.

$\{2, 4\}$. Let $s_2 \in S_2$ and $s_4 \in S_4$. Then $s_2 \vee s_4 = (a_2 \vee a_4, q_2 \vee q_4) \in S_3$ because $a_2 \vee a_4 \geq a_2 = a$ and $q_2 \vee q_4 \geq q_4 = d$, and $s_2 \wedge s_4 = (a_2 \wedge a_4, q_2 \wedge q_4) \in S_1$ because $a_2 \wedge a_4 \leq a_2 = a$ and $c = c \wedge d \leq q_2 \wedge q_4 \leq q_4 = d$.

Altogether $A * Q$ is a $(0, 1)$-sublattice of $A \times Q$ because $(0, 0) \in S_0$ and $(1, 1) \in S_3$. We have $A \times \{d\} = S_4 \subseteq A * Q$, and $\big(\{a\} \times [c)\big) \cup \big(\{0\} \times (d]\big) = S_3 \cup S_1 \subseteq A * Q$ and $Q = [c) \cup (d]$, and hence the lattice $A * Q$ is subdirect in $A \times Q$.

Claim (1) holds because $[(0, d), (1, d)] = S_4$ in $A \times Q$. Since $h(a) = a'$ and $h(0) = 0$, the homomorphism $h \times 1_Q$ maps the set $S_i \subseteq A * Q$ into the corresponding set $S'_i \subseteq A' *_{a'} Q$ for each $i \in 5$, and (2) follows. To prove (3) observe that $(z, q) \in A * Q$ for some $q \in (d] \setminus [c)$ only when $z = 0$. Since $(h * 1)(z, q) = (0, q')$ implies $q = q'$, we obtain (3). For (4), suppose that $q$ is incomparable with $d$ and $(h * 1)(z, p) = (a', q)$. Then $p = q$ is incomparable to $d$, and hence $z = a$ by the definition of $A * Q$. ■

Let $\mathbb{K}$ be a concrete category with a forgetful functor $U : \mathbb{K} \to \mathrm{Set}$. We say that a functor $F : \mathbb{C} \to \mathbb{K}$ is *pointed* if for every $\mathbb{C}$-object $C$ there exists an element $a_C \in (U \circ F)C$ of the underlying set of its image $FC$ such that $(U \circ F)f(a_C) = a_{C'}$ for all $\mathbb{C}$-morphisms $f : C \to C'$.

Let $\mathbb{L}$ be the variety of all $(0, 1)$-lattices, let $F : \mathbb{C} \to \mathbb{L}$ be a pointed faithful functor such that $a_C \neq 0, 1$ for all $\mathbb{C}$-objects $C$, and let $Q \in \mathbb{L}$. Lemma 1.1 shows that setting $(F * Q)C = FC *_{a_C} Q$ for every $\mathbb{C}$-object $C$ and $(F * Q)h = Fh * 1$ for every $\mathbb{C}$-morphism $h$ defines a faithful functor

$$F * Q : \mathbb{C} \to \mathbb{L}.$$

Let $C_2 = \{0 < a < 1\}$ be the chain of length two. For any $\mathbb{C}$-object $C$, let $\xi_C : C_2 \to FC$ denote the lattice $(0, 1)$-homomorphism with $\xi_C(a) = a_C$.

For a pointed functor $F : \mathbb{C} \to \mathbb{L}$, a $(0, 1)$-lattice $A \in \mathbb{L}$ and for a category $\mathbb{K}$ of $(0, 1)$-lattices that includes all $(0, 1)$-homomorphisms between any two of its objects, we shall consider these conditions:

(c0) for every $\mathbb{C}$-object $C$ there is a separating family

$$\Sigma_C \subseteq \hom_{0,1}(FC, A)$$

such that $f(a_C) \neq 0, 1$ for all $f \in \Sigma_C$;

(c1) $FC *_{a_C} Q$ is a $\mathbb{K}$-object for every $\mathbb{C}$-object $C$.

Define $B = \{f(a_C) \in A \mid C$ is a $\mathbb{C}$-object and $f \in \Sigma_C\}$. For each $b \in B$, let

$$\omega_b : C_2 \to A$$

be the lattice $(0,1)$-homomorphism with $\omega_b(a) = b$.

(c2) $C_2 *_a Q$ and $A *_b Q$ are $\mathbb{K}$-objects for all $b \in B$;

(c3) if $b \in B$ and a $\mathbb{K}$-morphism $k : C_2 *_a Q \to A *_b Q$ are such that there is a $\mathbb{C}$-object $C$ and a lattice $(0,1)$-homomorphism from $FC$ into the interval $[k(0,d), k(1,d)]$ of $A *_b Q$ then either $k$ is constant or $k = \omega_b * 1$.

Observe that condition (c0) implies that $a_C \neq 0, 1$ for every $\mathbb{C}$-object $C$, and that condition (c1) and Lemma 1.1(2) imply that $F * Q$ is a functor from $\mathbb{C}$ to $\mathbb{K}$.

Lemma 1.2. *Let $F : \mathbb{C} \to \mathbb{L}$ be a pointed full embedding, let $A \in \mathbb{L}$ be a $(0,1)$-lattice and let $\mathbb{K}$ be a category satisfying conditions* (c0)–(c3). *Then $F * Q : \mathbb{C} \to \mathbb{K}$ is an almost full embedding. If, moreover, no constant $\mathbb{K}$-morphism from $C_2 *_a Q$ to $A *_b Q$ exists for any $b \in B$, then $F * Q$ is a full embedding of $\mathbb{C}$ into $\mathbb{K}$.*

*Proof.* By Lemma 1.1(1), the mapping $\iota_C : FC \to FC *_{a_C} Q$ given by $\iota_C(y) = (y, d)$ for all $y \in FC$ is a lattice isomorphism of $FC$ onto the interval $FC \times \{d\} = [(0,d), (1,d)]$ of $FC *_{a_C} Q$. We know that the functor $F * Q$ is faithful and that $(F*Q)h$ is a $(0,1)$-homomorphism for every $\mathbb{C}$-morphism $h$. We thus need only show that it is almost full.

Let $C$ and $C'$ be $\mathbb{C}$-objects and let $g : FC *_{a_C} Q \to FC' *_{a_{C'}} Q$ be a $\mathbb{K}$-morphism. For any given $f \in \Sigma_{C'}$ we define $g_f = (f * 1) \circ g \circ (\xi_C * 1)$ and $b = f(a_{C'})$, as shown in the diagram below.

$$\begin{tikzcd} FC \arrow[r, "\iota_C"] & FC *_{a_C} Q \arrow[r, "g"] & FC' *_{a_{C'}} Q \arrow[d, "f*1"] \\ & C_2 *_a Q \arrow[u, "\xi_C*1"] \arrow[r, "g_f"] & A *_b Q \end{tikzcd} \qquad \text{for } b = f(a_{C'}) \in B$$

Choose any $f \in \Sigma_{C'}$. Then $(f * 1) \circ g \circ \iota_C$ is a lattice $(0,1)$-homomorphism of $FC$ into the interval $[((f * 1) \circ g)(0,d), ((f * 1) \circ g)(1,d)]$ of $A *_b Q$. Since $(\xi_C * 1)(0,d) = (0,d)$ and $(\xi_C * 1)(1,d) = (1,d)$, we have $g_f(0,d) = ((f * 1) \circ g)(0,d)$ and $g_f(1,d) = ((f * 1) \circ g)(1,d)$, and, by condition (c3), the $\mathbb{K}$-morphism $g_f$ is either constant or else $g_f = \omega_b * 1$.

Suppose that $f \in \Sigma_{C'}$ is such that $g_f = \omega_b * 1$. We aim to prove that $g_{f'} = \omega_{b'} * 1$ for all $f' \in \Sigma_{C'}$, where $b' = f'(a_{C'})$. First we note that $g_f(0,y) = (0,y)$ for any $y \in (d] \setminus [c)$ and, by Lemma 1.1(3), $(f*1)^{-1}\{(0,y)\} = \{(0,y)\}$. But $(\xi_C * 1)(0,y) = (0,y)$, and hence $g(0,y) = (0,y)$. Since $(d] \setminus [c)$ is not a singleton, for distinct $y, z \in (d] \setminus [c)$ we obtain $g(0,y) = (0,y)$ and $g(0,z) = (0,z)$, so that $g_{f'}(0,y) = (0,y)$ and $g_{f'}(0,z) = (0,z)$ for each

$f' \in \Sigma_{C'}$. It follows that $g_{f'} = (\omega_{b'} * 1)$ for all $f' \in \Sigma_{C'}$ (where $b' = f'(a_{C'})$). Therefore

(a) $g_f$ is either constant for all $f \in \Sigma_{C'}$ or $g_f = \omega_b * 1$ for all $f \in \Sigma_{C'}$, where $b = f(a_{C'})$.

Next we show that

(b) $g$ is constant if and only if $g_f$ is constant for all $f \in \Sigma_{C'}$.

Clearly, if $g$ is constant then $g_f$ is constant for every $f \in \Sigma_{C'}$. So assume that $g$ is nonconstant, and let $y, z \in FC *_{a_C} Q$ be such that $v = g(y) < g(z) = w$ in $FC' *_{a_{C'}} Q$. Thus $\pi_Q(v) < \pi_Q(w)$ for the projection $\pi_Q : FC' \times Q \to Q$ or $\pi_{FC'}(v) < \pi_{FC'}(w)$ for the projection $\pi_{FC'} : FC' \times Q \to FC'$. In the first case $(f * 1)(v) < (f * 1)(w)$ for all $f \in \Sigma_{C'}$. In the second, there exists $f \in \Sigma_{C'}$ with $f(\pi_{FC'}(v)) < f(\pi_{FC'}(w))$ because $\Sigma_{C'}$ is separating and hence $(f * 1)(v) < (f * 1)(w)$. Thus in either case there exists an $f \in \Sigma_{C'}$ for which $g_f$ is not constant, and (b) holds.

Assume that $g$ is not constant, that is, let $g_f = \omega_b * 1$ for all $f \in \Sigma_{C'}$ and $b = f(a_{C'})$. Then $g_f(0, d) = (0, d)$ and $g_f(1, d) = (1, d)$ for all $f \in \Sigma_{C'}$. Since $\Sigma_{C'}$ is separating, for every $y \in FC' \setminus \{0, 1\}$ there exist $f', f'' \in \Sigma_{C'}$ with $g_{f'}(y, d) \neq (0, d)$ and $g_{f''}(y, d) \neq (1, d)$, and since $(f * 1)^{-1}(A \times \{d\}) \subseteq FC' \times \{d\}$ for every $f \in \Sigma_{C'}$, it follows that $g(0, d) = (0, d)$ and $g(1, d) = (1, d)$. Thus the domain-range restriction $h : FC \to FC'$ of $g$ to the respective intervals $[(0, d), (1, d)]$ of $FC *_{a_C} Q$ and of $FC' *_{a_{C'}} Q$ is a lattice $(0, 1)$-homomorphism. Since $F$ is a pointed full embedding, we also have $h(a_C) = a_{C'}$. Thus

(c) there exists a unique $(0, 1)$-homomorphism $h : FC \to FC'$ such that $h(a_C) = h(a_{C'})$ and $g(z, d) = (h(z), d)$ for all $z \in FC$.

Next we aim to show that $g = h * 1$.

Let $q \in (d]$ first. We begin by showing that $g(0, q) = (0, q)$. We have $g(0, q) \leq g(0, d) = (0, d)$ by (c), and hence $g(0, q) = (0, p)$ for some $p \leq d$. Since $g_f = \omega_b * 1$ for any $f \in \Sigma_{C'}$ and from the definition of $g_f$ it follows that $(0, q) = ((f * 1) \circ g)(0, q) = (f * 1)(0, p) = (0, p)$. Thus $g(0, q) = (0, q)$ for every $q \in (d]$. Now let $q \in (d]$ and $(z, q) \in FC *_{a_C} Q$. From $(0, d) \wedge (z, q) = (0, q)$ and $(0, d) \vee (z, q) = (z, d)$ we obtain $(0, d) \wedge g(z, q) = (0, q)$ and $(0, d) \vee g(z, q) = (h(z), d)$. It is easy to see that the last pair of equations has a unique solution $g(z, q) = (h(z), q)$. This completes the case of $q \in (d]$.

Analogously we find that $g(z, q) = (h(z), q)$ for all $(z, q) \in FC *_{a_C} Q$ with $q \in [d)$.

It remains to consider the elements $(z, q) \in FC *_{a_C} Q$ with $q \in Q$ incomparable to $d$. Such elements have the form $(z, q) = (a_C, q)$ with $q \geq c$. Let $f \in \Sigma_{C'}$ be arbitrary and let $b = f(a_{C'})$. From the definition of $g_f$ and the fact that $g_f = \omega_b * 1$ it follows that $(f * 1)(g(a_C, q)) = g_f(a, q) = (b, q)$, and hence $g(a_C, q) = (a_{C'}, q) = (h(a_C), q)$ by Lemma 1.1(4).

Altogether, $g = h * 1$ for any nonconstant $g$, and hence $F * Q : \mathbb{C} \to \mathbb{K}$ is an almost full embedding. If every $g$ is nonconstant, then $F * Q$ is a full embedding. ■

Now we apply this general construction to the full embedding $F : \mathbb{C} \to \mathrm{Var}_{0,1}(M_3)$ of an *ff*-universal category $\mathbb{C}$ constructed in [5] into the full subcategory $\mathbb{K}$ of $\mathrm{Var}_0(M_3)$ determined by its $(0,1)$-lattices, and to the lattice $Q \in \mathrm{Var}(M_3)$ of Figure 1. We note that $Q = (d] \cup [c)$ and that $(d] \setminus [c)$ is not a singleton.

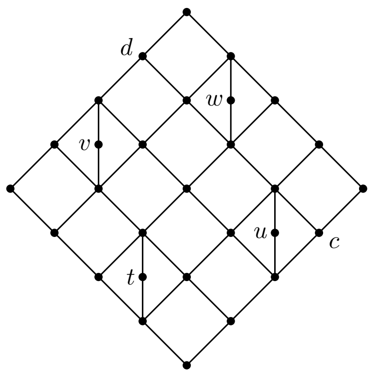


Fig. 1. The lattice $Q$

Figure 2 shows the lattice $L_1 = C_2 *_a Q$, where the interval $[z, x]$ is $C_2 \times \{d\}$, and $y = (a, d)$ for the nonextremal element $a \in C_2$.

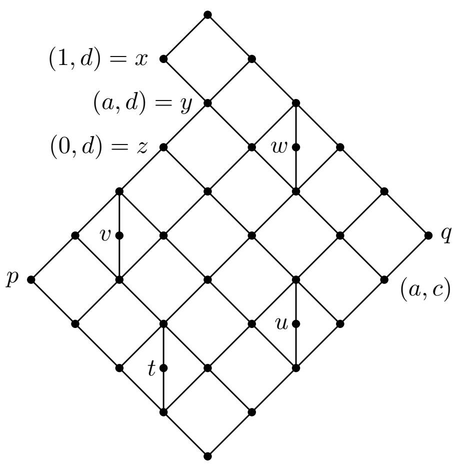


Fig. 2. The lattice $L_1 = C_2 * Q$

Similarly, in the lattice $L_0 = M_3 *_b Q$ of Figure 3, the interval $[z, x]$ is $M_3 \times \{d\}$, and $y = (b, d)$ for an arbitrary nonextremal element $b \in M_3$. In Figures 2 and 3, the letter $r \in \{t, u, v\}$ denotes the element $(0, r)$ and $w$ denotes the element $(a, w)$.

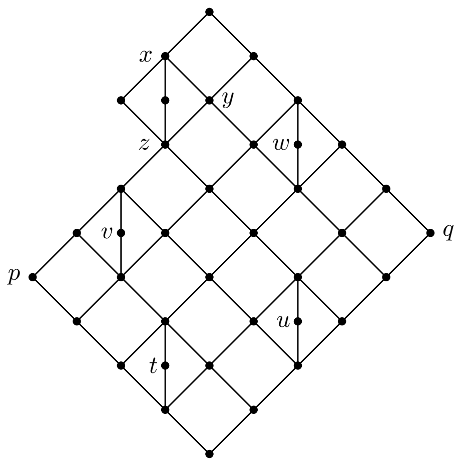


Fig. 3. The lattice $L_0 = M_3 * Q$

Next we describe the lattice 0-homomorphisms from $L_1$ to $L_0$.

LEMMA 1.3. *Any* 0*-homomorphism* $f : L_1 \to L_0$ *has one of these properties*:

(1) $f$ *is the inclusion* 0*-homomorphism*, *or*
(2) $f$ *is the constant map with the value* 0, *or*
(3) $f(z) < f(x)$ *and* $L_0$ *has no copy of* $M_3$ *with the bounds* $f(z)$ *and* $f(x)$.

*Proof.* For $r \in \{t, u, v, w\}$, let $M^{(r)}$ denote the copy of $M_3$ in $L_1$ containing $r$, and let $0_r, 1_r \in M^{(r)}$ denote its respective bounds. The congruence lattice of $L_1$ is Boolean and its atoms are the four congruences $\alpha_r$ collapsing $M^{(r)}$ for $r \in \{t, u, v, w\}$ and the two principal congruences $\theta(z, y)$ and $\theta(y, x)$.

We begin with an easy observation about $L_0$.

(a) If $A$ is a sublattice of $L_0$ isomorphic to $M_3$ then $0 \notin A$, and if $B \neq A$ is a sublattice of $L_0$ isomorphic to $M_3$ then $A \cap B = \emptyset$.

Next we investigate properties of the kernel $\mathrm{Ker}(f)$ of $f$.

First, since $0_t \wedge 0_u = 0$ and $0_t \leq 1_u$ in $L_1$, by (a) it follows that

(b) $\alpha_u \subseteq \mathrm{Ker}(f)$ implies $\alpha_t \subseteq \mathrm{Ker}(f)$.

Next suppose that $\alpha_v \subseteq \mathrm{Ker}(f)$. Then the elements $m_u = 1_u \wedge 1_v \in M^{(u)}$ and $m_t = 1_u \wedge 0_v \in M^{(t)}$ satisfy $f(m_u) = f(m_t)$. Thus, by (a), either

$\alpha_t \subseteq \mathrm{Ker}(f)$ or $\alpha_u \subseteq \mathrm{Ker}(f)$. If $\alpha_t \subseteq \mathrm{Ker}(f)$ then from $f(1_t) \geq f(0_u)$ and $0_t \wedge 0_u = 0$ we get $f(0_u) = f(1_t) \wedge f(0_u) = f(0_t) \wedge f(0_u) = 0$ and, by (a), $\alpha_u \subseteq \mathrm{Ker}(f)$. Using (a) for the case when $\alpha_u \subseteq \mathrm{Ker}(f)$, we conclude that

(c) $\alpha_v \subseteq \mathrm{Ker}(f)$ implies $\alpha_u \vee \alpha_t \subseteq \mathrm{Ker}(f)$.

Next we show that

(d) $\alpha_w \subseteq \mathrm{Ker}(f)$ implies $\alpha_v \vee \alpha_u \vee \alpha_t \subseteq \mathrm{Ker}(f)$.

Indeed, if $\alpha_w \subseteq \mathrm{Ker}(f)$, then from $0_v \leq 1_w$ and $1_t = 0_v \wedge 0_w$ it follows that $f(0_v) = f(1_t)$ and, by (a), either $\alpha_v \subseteq \mathrm{Ker}(f)$ or $\alpha_t \subseteq \mathrm{Ker}(f)$. In the first case the conclusion of (d) follows from (c), so let us assume that $\alpha_w \vee \alpha_t \subseteq \mathrm{Ker}(f)$. We have $m_v = 1_w \wedge 1_v \in M^{(v)}$ and $m_u = 0_t \vee 0_u \in M^{(u)}$. Since $0_w \wedge 1_v = 1_t \vee 0_u$ in $L_1$ we obtain $f(m_v) = f(0_w \wedge 1_v) = f(1_t \vee 0_u) = f(m_u)$ and, by (a), it follows that $\alpha_v \subseteq \mathrm{Ker}(f)$ or $\alpha_u \subseteq \mathrm{Ker}(f)$. The first case is covered by (c), in the second case from $0 = 0_t \wedge 0_u$, $0_t \leq 1_u$ and $\alpha_u \subseteq \mathrm{Ker}(f)$ it follows that $f(0_t) = f(0) = 0$ and from $0_v = v \wedge 1_w$, $1_t = v \wedge 0_w$ and $\alpha_t \vee \alpha_w \subseteq \mathrm{Ker}(f)$ it follows that $f(0_v) = f(1_t) = f(0_t) = 0$, and (a) completes the proof of (d).

Now we apply these four properties as follows.

If $\alpha_w \subseteq \mathrm{Ker}(f)$, then $f(z) = 0$ follows by (d), and hence $f$ satisfies (2) or (3). In the remainder of the proof we thus assume that $f$ is one-to-one on $M^{(w)}$.

Case 1: *$f$ does not collapse $M^{(t)}$.* Then $f$ is one-to-one on all sublattices $M^{(r)}$ of $L_1$ with $r \in \{t, u, v, w\}$ (see (b) and (c)). Since $0_t \wedge 0_u = 0$ and $0_t \leq 1_u$ and $t$, $u$ are the only elements of $L_0$ with these properties, it follows that $f(0_t) = 0_t$, $f(0_u) = 0_u$ and $f(1_t) = 1_t$, $f(1_u) = 1_u$. From $0_v \wedge 1_u \leq 1_t$ and $0_u \leq 1_v$ we then obtain $f(0_v) \wedge 1_u \leq 1_t$ and $0_u \leq f(1_v)$, and $f(0_v) = 0_v$ and $f(1_v) = 1_v$ follow. But then $f(z) = f(1_u) \vee f(1_v) = z$. Next we note that $0_w \wedge 0_v = 1_t$ implies that $f(0_w) \wedge 0_v = 1_t$, so that $f(0_w) = 0_w$ and $f(1_w) = 1_w$. Thus $f(y) = f(1_v \vee 0_w) = y$ and $f(y \vee 1_w) = y \vee 1_w$, and thus $f$ is the inclusion on the sublattice $B \subseteq L_1$ generated by $y$, $z$ and the extremal elements of the four copies of $M_3$ in $L_1$. In both $L_1$ and $L_0$, the doubly irreducible element $p$ is the unique complement of $1_u \wedge 1_v \in B$ in the interval $[0_t, 1_v]$, and hence $f(p) = p$. Similarly, $q$ is the unique complement of $1_v \wedge 1_w \in B$ in the interval $[0_u, 1_w]$, and hence $f(q) = q$. But then $f$ is the inclusion map on the distributive sublattice $(y \vee 1_w] \setminus \{t, u, v, w\}$ of $L_1$ generated by $B \cup \{p, q\}$, and it follows that the restriction of $f$ to $(y \vee 1_w]$ is the inclusion map. For the element $x$ we have $f(x) \geq f(y) = y$, and from $x \wedge w = 0_w$ it follows that $f(x) \wedge w = 0_w$. Hence $f(x) \in \{x, y\}$. If $f(x) = y$, then the interval $[f(z), f(x)] = [z, y]$ has two elements, and hence (3) holds. If $f(x) = x$, then $f$ is the inclusion map, that is, (1) holds.

Case 2: *$f$ collapses $M^{(t)}$ (but not $M^{(w)}$).* Thus $f$ satisfies neither (1) nor (2). Arguing indirectly, we suppose that there is a copy $M(f)$ of $M_3$ isomorphic to a $(0,1)$-sublattice of the interval $[f(z), f(x)] \subseteq L_0$. We also note that any interval $[a,b] \subseteq L_0$ containing a $(0,1)$-copy of $M_3$ is, in fact, isomorphic to $M_3$.

Case 2.1: $f(z) = f(y)$. Noting that $z \wedge 0_w = 1_t \vee 1_u$, $y \geq 0_w$ and $0_t \leq 1_u$ we obtain $f(0_w) = f(y) \wedge f(0_w) = f(0_t) \vee f(1_u) = f(1_u)$ because $f(0_t) = f(1_t)$. Since $f$ is one-to-one on $M^{(w)}$, (a) implies that $\alpha_u \subseteq \mathrm{Ker}(f)$. Thus $f(0_w) = f(0_u)$ and $f(1_t) = f(0_t) = f(0_t \wedge 1_u) = f(0_t) \wedge f(0_u) = f(0) = 0$. Also, since $f(y) = f(z)$, from $1_u \vee 1_v = z$ and $0_u \leq 1_v$ we obtain $f(y) = f(1_v)$. But then $f(y \wedge 1_w) = f(1_v \wedge 1_w) \in f(M^{(w)}) \cap f(M^{(v)})$, and hence $f(0_v) = f(1_v)$, by (a). From $0_u \leq 1_v$ and $0_u \wedge 0_v \leq 1_t$ it then follows that $f(0_w) = f(0_u) = f(0_u \wedge 0_v) \leq f(1_t) = 0$. This is a contradiction to (a). Therefore this case cannot occur.

Case 2.2: $f(z) < f(y)$. First we show that $\mathrm{Ker}(f) \subseteq \alpha_t \vee \theta(x,y)$. Indeed, should $\alpha_v \subseteq \mathrm{Ker}(f)$, then $f(0_v \vee 0_w) = f(1_v \vee 0_w) \in M(f) \cap f(M^{(w)})$, contrary to (a). Thus $f$ is one-to-one also on $M^{(v)}$. Similarly, if $\alpha_u \subseteq \mathrm{Ker}(f)$, then the contradictory $M(f) \cap f(M^{(v)}) \neq \emptyset$ results. Therefore $f$ is one-to-one on each $M^{(r)}$ with $r \in \{u,v,w\}$ and hence $\mathrm{Ker}(f) \subseteq \alpha_t \vee \theta(x,y)$, as claimed. Next, from $z = 1_u \vee 1_v$ it follows that $f(z) = f(1_u) \vee f(1_v)$, that is, the zero of $M(f)$ is the join of the units $f(1_u)$ and $f(1_v)$ of the lattices $f(M^{(u)})$ and $f(M^{(v)})$ isomorphic to $M_3$. But this occurs in $L_0$ only when $f(z) = z$, and from $1_v \geq 0_u$ and $1_u \not\geq 0_v$ it follows that $f(1_u) = 1_u$ and $f(1_v) = 1_v$. Thus $f(0_u) = 0_u$ and $f(0_v) = 0_v$ as well. And $f(1_w) = 1_w$ and $f(0_w) = 0_w$ because $0_v \vee 1_u \leq 1_w$. But then $f(0_t) = f(1_t) = f(0_v \wedge 0_w) = 0_v \wedge 0_w = 1_t$ and hence $f(0) = f(0_t \wedge 0_u) = 1_t \wedge 0_u > 0$, a contradiction. Therefore any lattice 0-homomorphism $f : L_1 \to L_0$ collapsing $M^{(t)}$ but not $M^{(w)}$ satisfies (3). ∎

Now let $L_2$ be the $(0,1)$-lattice in Figure 4 and let $L_3$ be the $(0,1)$-sublattice of $L_2 \times L_2$ consisting of all $(x,y) \in L_2 \times L_2$ such that $x = 0$ or $y = 1$. Thus both the ideal $((0,1)]$ and the filter $[(0,1))$ of $L_3$ are isomorphic to $L_2$ and $L_3 = ((0,1)] \cup [(0,1))$.

Lemma 1.4. *Let $f : L_i \to L_0$ be a lattice homomorphism for $i = 2,3$. Then* $\mathrm{Im}(f)$ *is either a singleton or an interval of $L_0$ isomorphic to $M_3$.*

*Proof.* Consider a lattice homomorphism $f : L_2 \to L_0$. Observe that the interval $[u_0, u_1]$ in $L_2$ is subdirect in $(M_3)^6$, that $[u_0,u_1]/\varrho$ is isomorphic to $M_3$ for any coatom congruence $\varrho$ of the interval $[u_0,u_1]$, and that if $\sigma$ is a congruence of the interval $[u_0,u_1]$ other than the universal congruence or any coatom congruence, then $[u_0,u_1]/\sigma$ contains two distinct copies of $M_3$ that intersect. Any two distinct sublattices of $L_0$ isomorphic to $M_3$ are

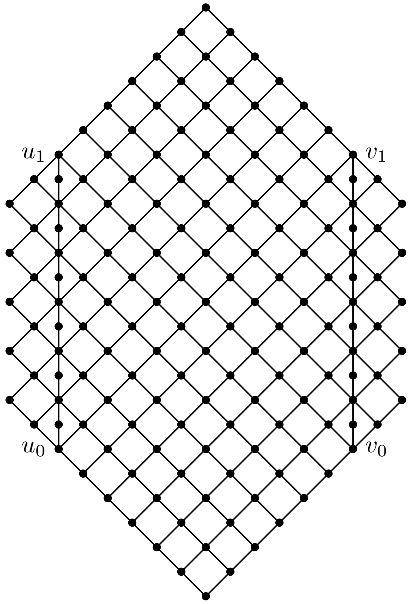


Fig 4. The lattice $L_2$

disjoint, however, and it follows that the restriction of $\mathrm{Ker}(f)$ to the interval $[u_0, u_1]$ is either its coatom congruence (with $f[u_0, u_1] \cong M_3$) or the universal congruence. By symmetry, the same conclusion holds for the interval $[v_0, v_1]$ of $L_2$. Since $[u_0, u_1] \cap [v_0, v_1] = \{u_1 \wedge v_1\}$, and because distinct copies of $M_3$ are disjoint in $L_0$, it follows that $\mathrm{Ker}(f)$ is the universal congruence on at least one of these intervals. If $f(u_0) = f(u_1)$ then $\mathrm{Im}(f) = f([v_0, v_1])$, and if $f(v_0) = f(v_1)$ then $\mathrm{Im}(f) = f([u_0, u_1])$. Hence $\mathrm{Im}(f)$ is either a sublattice of $L_0$ isomorphic to $M_3$ or a singleton, and the claim holds for $L_2$.

Now let $f : L_3 \to L_0$ be a lattice homomorphism. Since $L_3 = (\{0\} \times L_2) \cup (L_2 \times \{1\})$, the claim for $L_2$ implies that $f(\{0\} \times L_2)$ and $f(L_2 \times \{1\})$ are either singletons or sublattices isomorphic to $M_3$. But $(0, 1) \in (\{0\} \times L_2) \cap (L_2 \times \{1\})$, and hence $f(\{0\} \times L_2)$ or $f(L_2 \times \{1\})$ is a singleton, and the claim holds for $L_3$ as well. ■

Now we show how the present and certain earlier results combine to give the almost $ff$-universality of $\mathrm{Var}_0(M_3)$.

In [5], Goralčík *et al.* presented an $ff$-universal category $\mathbb{C}$ and a finite-to-finite full embedding $F : \mathbb{C} \to \mathbb{L}$ such that

(1) for every $\mathbb{C}$-object $C$ there is a separating family

$$\Sigma_C \subseteq \hom_{0,1}(FC, M_3)$$

consisting of surjective homomorphisms;

(2) there exists an injective lattice $(0,1)$-homomorphism $\lambda_C : L_3 \to FC$ for every $\mathbb{C}$-object $C$.

Property (1) just says that every $FC$ is a subdirect power of the lattice $M_3$. For the sake of completeness, we recall that the lattice $L_2$ is denoted as $L_{\emptyset,\emptyset}$ in [5], that Statement 4.6 in [5] gives an injective homomorphism from $L_{\emptyset,\emptyset}$ into $L_{\delta,\varepsilon}$ for all $\delta,\varepsilon \subseteq 4$, and that Lemma 5.1 in [5] gives an injective lattice $(0,1)$-homomorphism from $(\{0\} \times L_{\delta,\varepsilon}) \cup (L_{\delta,\varepsilon} \times \{1\})$ into $FC$. This establishes (2).

In [6], where the functor $F : \mathbb{C} \to \mathrm{Var}_{0,1}(M_3)$ was also used, it was shown that

(3) for every $\mathbb{C}$-object $C$ there exists an element $a_C \in FC$ such that $Fh(a_C) = a_{C'}$ for every $\mathbb{C}$-morphism $h : C \to C'$, and $f(a_C) = b \in M_3 \setminus \{0,1\}$ for every $f \in \Sigma_C$.

By (3), the functor $F$ is pointed. Choose $A = M_3$ and $B = \{b\}$ as in (3). Then condition (c0) follows from (1) and (3). For the full subcategory $\mathbb{K}$ of $\mathrm{Var}_0(M_3)$ determined by its $(0,1)$-lattices, conditions (c1) and (c2) are satisfied by Lemma 1.1 and (3). To prove (c3), let $k : L_1 \to L_0$ be a 0-homomorphism, and let $h$ be a $(0,1)$-homomorphism from $FC$ to the interval $[k(0,d), k(1,d)]$ of $L_0$. For the $(0,1)$-homomorphism $\lambda_C$ from (2), the composite $\gamma = h \circ \lambda_C$ is a $(0,1)$-homomorphism from $L_3$ to $[k(0,d), k(1,d)]$ whose image $\mathrm{Im}(\gamma)$ is either a singleton or it is isomorphic to $M_3$, by Lemma 1.4. But Lemma 1.3 then implies that $k$ is either a constant or the inclusion map $\omega_b * 1$. This proves (c3). By Lemma 1.2, the functor $F * Q : \mathbb{C} \to \mathbb{K}$ is an almost full embedding, and since $Q$ is finite, $F * Q$ preserves finiteness. Since $\mathbb{C}$ is $ff$-universal, this completes the proof of the theorem below.

THEOREM 1.5. *The variety* $\mathrm{Var}_0(M_3)$ *is almost ff-universal.* ∎

REMARK 1.6. Since the variety $\mathbb{D}_0$ of distributive 0-lattices is the only nontrivial variety of modular 0-lattices not containing $\mathrm{Var}_0(M_3)$ and because for any $D \in \mathbb{D}_0$ and any $x \in D \setminus \{0\}$ there is an endomorphism $f_x$ of $D$ with $\mathrm{Im}(f_x) = \{0, x\}$, the variety $\mathbb{D}_0$ is not almost universal. Thus, in fact, Theorem 1.5 characterizes almost universal varieties of modular 0-lattices.

**2. $Q$-universality.** For a set $S$ of algebras of the same similarity type, let $\mathbf{Q}S$ denote the smallest quasivariety containing $S$.

For a collection $\mathcal{A} = \{A_W \mid W \subseteq \mathbb{N} \text{ finite}\}$ of finite algebras of a given finite similarity type, we consider the following four conditions, in which $X$, $Y$ and $Z$ denote finite subsets of $\mathbb{N}$.

(P1) $A_\emptyset$ is a singleton algebra;
(P2) if $X = Y \cup Z$, then $A_X \in \mathbf{Q}\{A_Y, A_Z\}$;
(P3) if $X \neq \emptyset$ and $A_X \in \mathbf{Q}\{A_Y\}$, then $X = Y$;

(P4) if $B, C \in \mathbf{Q}\mathcal{A}$ are finite algebras and if $A_X$ is a subalgebra of $B \times C$, then there exist $Y$ and $Z$ such that $A_Y \in \mathbf{Q}\{B\}$, $A_Z \in \mathbf{Q}\{C\}$ and $X = Y \cup Z$.

In [4] and in [2] it was shown that any quasivariety $\mathbf{K}$ of a finite type containing a collection $\mathcal{A}$ of finite algebras satisfying (P1)–(P4) has various other properties that imply $Q$-universality. The reader is referred to [2] for a review of these properties. We aim to prove the $Q$-universality of $\mathrm{Var}_0(M_3)$ by constructing an infinite set $\mathcal{A}$ of its finite members satisfying conditions (P1)–(P4).

For a positive integer $n$, let $C_n$ denote the chain $0 < 1 < \ldots < n$ of length $n$, and recall that $n = \{0, 1, \ldots, n-1\}$. We say that $A \subseteq n \times n$ is a *permutation set* if $A = \{(i, \phi(i)) \mid i \in n\}$ for some permutation $\phi : n \to n$. In other words, for every $i \in n$ there is a unique $j \in n$ such that $(i, j) \in A$, and for every $j \in n$ there is a unique $i \in n$ such that $(i, j) \in A$.

For a permutation set $A \subseteq n \times n$, let $L(n, A)$ be the disjoint extension of the lattice $C_n \times C_n$ by the set $\{u_{i,j} \mid (i, j) \in A\}$, with the least partial order in which

(d) $(i, j) < u_{i,j} < (i + 1, j + 1)$ for every $(i, j) \in A$.

Then $L(n, A) \in \mathrm{Var}(M_3)$ is a lattice, and we call it a *permutation lattice* (for an example of such a lattice, see Figure 1). It is clear that each interval

$$M(i, j) = \{(i, j), (i + 1, j), u_{i,j}, (i, j + 1), (i + 1, j + 1)\}$$

of $L(n, A)$ with $(i, j) \in A$ is isomorphic to $M_3$ and that $L(n, A)$ contains no other copies of $M_3$. For the permutation set $A^{-1}$ given by the permutation inverse to that defining $A$, it is clear that the map $(i, j) \mapsto (j, i)$ determines a unique isomorphism of $L(n, A)$ onto $L(n, A^{-1})$.

For $(p, q) \in A$, let $\alpha(p, q)$ denote the equivalence on $L(n, A)$ whose non-singleton classes are all doubletons $\{(i, q), (i, q + 1)\}$ with $i \notin \{p, p + 1\}$, all doubletons $\{(p, j), (p + 1, j)\}$ with $j \notin \{q, q + 1\}$ and the interval $[(p, q), (p + 1, q + 1)]$ isomorphic to $M_3$. The restriction of $\alpha(p, q)$ to the $(0, 1)$-sublattice $C_n \times C_n$ of $L(n, A)$ is thus the congruence $\theta(p, p + 1) \times \theta(q, q + 1)$ of $C_n \times C_n$. Since all elements $u_{i,j}$ with $(i, j) \in A$ are doubly irreducible, the equivalence $\alpha(p, q)$ is a congruence of $L(n, A)$.

Further, for $(p, q) \in A$, let $\pi(p, q)$ denote the equivalence on $L(n, A)$ whose classes are the intervals $(p] \times (q]$, $(p] \times [q + 1)$, $[p + 1) \times (q]$, $[p + 1) \times [q + 1)$ of $L(n, A)$ and the singleton $\{u_{p,q}\}$. It is easily seen that $\pi(p, q)$ is a congruence and that $L(n, A)/\pi(p, q) \cong M_3$.

LEMMA 2.1. *The congruence lattice of $L(n, A)$ is Boolean. Its atoms are the $n$ congruences $\alpha(p, q)$ associated with the elements $(p, q) \in A$. The congruence $\pi(p, q)$ is complementary to $\alpha(p, q)$ for each $(p, q) \in A$.*

*Proof.* The congruence $\pi(p,q)$ is a coatom because $L(n,A)/\pi(p,q) \cong M_3$ is simple. It is easy to see that $\alpha(p,q)$ is the complement of $\pi(p,q)$, so that $\alpha(p,q)$ is an atom for every $(p,q) \in A$. If $(p,q),(p',q') \in A$ are distinct then $\alpha(p,q) \neq \alpha(p',q')$ and hence $\alpha(p,q) \wedge \alpha(p',q')$ is the diagonal congruence. Since the join of all $\alpha(p,q)$ with $(p,q) \in A$ is the total congruence, no other atoms exist. ∎

LEMMA 2.2. *Let $L(n,A)$ and $L(m,B)$ be permutation lattices. Then*

(1) *for any congruence $\theta$, the quotient $L(n,A)/\theta$ is isomorphic to a permutation lattice $L(k,C)$ with $k \leq n$; there are surjective homomorphisms $g,h : C_n \to C_k$ and a surjective homomorphism $f : L(n,A) \to L(k,C)$ with* $\operatorname{Ker}(f) = \theta$ *such that $f(i,j) = (g(i),h(j))$ for all $(i,j) \in C_n^2$, and for any $(p,q) \in A$ either $\alpha(p,q) \subseteq \theta$ (and hence $f(M(p,q)) = \{f(p,q)\}$, $g(p+1) = g(p)$ and $h(q+1) = h(q)$), or else $\theta \subseteq \pi(p,q)$ and $g(p+1) = g(p)+1$, $h(q+1) = h(q)+1$, $(g(p),h(q)) \in C$ and $f(u_{p,q}) = u_{g(p),h(q)}$; furthermore,*

(1a) *for each $(p',q') \in C$ there is a unique $(p,q) \in A$ such that*

$$f(M(p,q)) = M(p',q');$$

(2) *if $L(k,C)$ is a permutation lattice with $k > 1$ and if $e : L(k,C) \to L(m,B)$ is an injective* $0$*-homomorphism, then $k \leq m$ and* $\operatorname{Im}(e) = ((k,k)]$; *there is an injective* $0$*-homomorphism*

$$\tilde{e} : L(k,C) \to L(m,B)$$

*such that* $\operatorname{Im}(\tilde{e}) = \operatorname{Im}(e)$, *and either $\tilde{e}(i,j) = (i,j)$ for all $(i,j) \in C_k^2$ and $C \subseteq B$, or else $\tilde{e}(i,j) = (j,i)$ for all $(i,j) \in C_k^2$ and $C^{-1} \subseteq B$.*

*Proof.* First we prove (1). Let $t : L(n,A) \to L(n,A)/\theta$ be a surjective homomorphism with $\operatorname{Ker}(t) = \theta$. According to Lemma 2.1, there is a subset $A' = \{(p,q) \mid \theta \subseteq \pi(p,q)\} = \{(p,q) \mid \alpha(p,q) \not\subseteq \theta\}$ of $A$ for which

$$\theta = \bigwedge\{\pi(p,q) \mid (p,q) \in A'\} = \bigvee\{\alpha(p,q) \mid (p,q) \in A \setminus A'\}.$$

For any $(p,q) \in A \setminus A'$, the restriction of $\alpha(p,q)$ to the sublattice $C_n^2$ of $L(n,A)$ is the product congruence $\theta(p,p+1) \times \theta(q,q+1)$. For the congruences $\sigma = \bigvee\{\theta(p,p+1) \mid (p,q) \in A \setminus A'\}$ and $\tau = \bigvee\{\theta(q,q+1) \mid (p,q) \in A \setminus A'\}$ on $C_n$ let $g : C_n \to C_n/\sigma$ and $h : C_n \to C_n/\tau$ be the corresponding surjective homomorphisms. Then $C_n/\sigma \cong C_n/\tau \cong C_k$ for $k = |A'|$. Writing $C_n/\sigma = C_n/\tau = C_k$, we then conclude that $g(i+1) \in \{g(i), g(i)+1\}$ and $h(j+1) \in \{h(j), h(j)+1\}$ for any $i,j \in C_n$, and that $g(i+1) = g(i)+1$ and $h(j+1) = h(j)+1$ if and only if $(i,j) \in A'$. It also follows that there is an injective homomorphism $d : C_k^2 \to L(n,A)/\theta$ such that $t(i,j) = d(g(i),h(j))$ for all $(i,j) \in C_n^2 \subseteq L(n,A)$. Now if $(i,j) \in A'$, then $\theta \subseteq \pi(i,j)$, and hence $t$ is injective on $M(i,j)$. Since $t$ is surjective, the copy $t(M(i,j))$ of $M_3$ is an interval in $L(n,A)/\theta$, and $g(i+1) = g(i)+1$ and $h(j+1) = h(j)+1$. Define $C = \{(g(i),h(j)) \mid (i,j) \in A'\}$. If $(i,j),(i',j') \in A'$ are distinct then $g(i) \neq g(i')$ and $h(j) \neq h(j')$, and hence $C$ is a permutation set. For each $(i,j) \in A'$, add a new element $u_{g(i),h(j)}$ satisfying $(g(i),h(j)) < u_{g(i),h(j)} < (g(i)+1,h(j)+1)$ to the lattice $C_k^2 = (g \times h)(C_n^2)$, thereby obtaining a permutation lattice $L(k,C)$. Extending $d$ to all of $L(k,C)$ by setting $d(u_{g(i),h(j)}) = t(u_{i,j})$ for each $(i,j) \in A'$ gives rise to an isomorphism $d : L(k,C) \to L(n,A)/\theta$. To complete the proof of (1), we set $f = d^{-1} \circ t$.

Claim (1a) follows from the fact that, for each $(p,q) \in A'$, the singleton $\{u_{p,q}\}$ is a class of the coatom congruence $\pi(p,q)$.

We turn to (2). First we observe that nonzero elements of $L(m,B)$ meet the zero element $(0,0)$ only when one of them lies in $((m,0)] \cup \{u_{p,0}\}$ and the other in $((0,m)] \cup \{u_{0,q}\}$ for some $(p,0),(0,q) \in B$. And we have $e(k,0) \wedge e(0,k) = (0,0)$, of course.

Case A. Suppose that $e(k,0) \in ((m,0)] \cup \{u_{p,0}\}$ and $e(0,k) \in ((0,m)] \cup \{u_{0,q}\}$. Then $k \leq m$, and $e(k-1,0) \leq (m-1,0)$, $e(0,k-1) \leq (0,m-1)$ because $e$ is injective and $(p,0)$ (resp. $(0,q)$) is the only element of $L(m,B)$ covered by $u_{p,0}$ (resp. by $u_{0,q}$). For any $i \leq k-1$, define $g$ and $h$ by $e(i,0) = (g(i),0)$ and $e(0,i) = (0,h(i))$. The maps $g$ and $h$ defined, so far, for $i \leq k-1$ are injective, and $e(i,j) = (g(i),h(j))$ for $i,j \leq k-1$.

Let $i \leq k-2$. Then $(i,j) \in C$ for some $j \leq k-1$. Since $e$ is injective, the sublattice $e(M(i,j))$ of $L(m,B)$ isomorphic to $M_3$ is the interval $[e(i,j), e(i+1,j+1)]$. Thus $(g(i),h(j)) = e(i,j) \in B$ and hence $g(i+1) = g(i)+1$. From $g(0) = 0$ it now follows that $g(i) = i$ for each $i \leq k-1$. Together with a similar argument for the other component, this shows that

(1,1) $$e(i,j) = (i,j) \quad \text{for all } i,j \leq k-1.$$

A.1. Suppose that $e(k,0) \leq (m,0)$. We have $(k-1,q) \in C$ for some $q \leq k-1$ and hence $e(k-1,q) = (k-1,q)$, by (1,1). Thus $(k-1,q) \in B$, and the sublattice $e(M(k-1,q))$ of $L(m,B)$ isomorphic to $M_3$ is the interval $[(k-1,q),(k,q+1)]$, so that $e(k,q+1) = (k,q+1)$. But then $e(k,0) = (k,0)$ and, from (1,1),

(0,1) $$e(i,j) = (i,j) \quad \text{for all } i \leq k \text{ and } j \leq k-1.$$

A.2. Similarly we find that $e(0,k) \leq (0,m)$ implies that

(1,0) $$e(i,j) = (i,j) \quad \text{for all } i \leq k-1 \text{ and } j \leq k.$$

A.3. Suppose that $e(k,0) \not\leq (m,0)$, that is, let $e(k,0) = u_{k-1,0}$. By (1,1), for the element $(i,0) \in C$ we have $(i,0) = e(i,0) \in B$, and hence $i = k-1$. We have $e(M(k-1,0)) = M(k-1,0)$ and thus $e(k,1) = (k,1)$, and

$e(k-1,1)=(k-1,1)$, $e(k-1,0)=(k-1,0)$ by (1,1). Since $e(k,0)=u_{k-1,0}$, it follows that $e(u_{k-1,0})$ must be the remaining element $(k,0)$ of the sublattice $e(M(k-1,0)) = M(k-1,0)$ of $L(m,B)$. The mapping $\alpha_1 : L(k,C) \to L(k,C)$ exchanging $(k,0)$ and $u_{k-1,0}$ and leaving all other elements fixed is an automorphism of $L(k,C)$, and the composite $e_1 = e \circ \alpha_1$ satisfies (0,1).

A.4. Suppose that $e(0,k) \not\leq (0,m)$. Then $e(0,k) = u_{0,k-1}$. Similarly to A.3, for the automorphism $\alpha_2$ of $L(k,C)$ exchanging $(0,k)$ and $u_{0,k-1}$, the composite $e \circ \alpha_2$ satisfies (1,0).

We have $\alpha_2 \circ \alpha_1 = \alpha_1 \circ \alpha_2$ because $k > 1$. Applying these automorphisms when needed, we obtain an embedding $\tilde{e}$ with $\mathrm{Im}(\tilde{e}) = \mathrm{Im}(e)$ and $\tilde{e}(i,j) = (i,j)$ for all $i,j \leq k$.

Case B. If $e(k,0) \in ((0,m)] \cup \{u_{0,q}\}$ and $e(0,k) \in ((m,0)] \cup \{u_{p,0}\}$, we apply the previous argument to the map $e^*$ given by $e^*(x,y) = e(y,x)$. ∎

Let $m \geq 1$. An interval $[(i,j),(i+m,j+m)]$ of a lattice $L$ is called its $(i,j,m)$-*block* if it is isomorphic to some permutation lattice $L(m,B)$. Thus the interval $[(i,j),(i+m,j+m)]$ of a permutation lattice $L(n,A)$ is its $(i,j,m)$-block if and only if for any $p \in \{i,\dots,i+m-1\}$ there is $q \in \{j,\dots,j+m-1\}$ with $(p,q) \in A$ and vice versa. Thus the $(i,j,1)$-blocks of $L(n,A)$ are exactly its intervals $M(i,j)$ with $(i,j) \in A$.

We say that a 0-homomorphism $s : L(n,A) \to L(m,B)$ is *standard* if $s(C_n^2) \subseteq C_m^2$. By Lemma 2.2, the restriction of $s$ to $C_n^2 \subset L(n,A)$ has the form $s(i,j) = (g(i),h(j))$ or $s(i,j) = (h(j),g(i))$ for some surjective maps $g,h : C_n \to C_k$ with $k \leq m,n$.

Corollary 2.3. *Let $f : L(n,A) \to L(m,B)$ be a nonconstant 0-homomorphism. Then*

(1) $\mathrm{Im}(f)$ *is a $(0,0,k)$-block for some $k \leq m,n$;*
(2) *if $L(m,B)$ has no $(0,0,k)$-block with $k < m$ then $f$ is surjective;*
(3) *there is a standard 0-homomorphism $s : L(n,A) \to L(m,B)$ such that* $\mathrm{Ker}(s) = \mathrm{Ker}(f)$ *and* $\mathrm{Im}(s) = \mathrm{Im}(f)$; *if $s(i,j) = (g(i),h(j))$ for all $(i,j) \in C_n^2$ we say that $f$ is* direct *and if $s(i,j) = (h(j),g(i))$ we say that $f$ is* reversing;
(4) *for any $(i,j,q)$-block $Q$, if $f$ is direct then $f(Q) = \{(g(i),h(j))\}$ or $f(Q)$ is a $(g(i),h(j),k)$-block for $k = g(i+q)-g(i) = h(j+q)-h(j) \leq q$; and if $f$ is reversing then $f(Q) = \{(h(j),g(i))\}$ or $f(Q)$ is an $(h(j),g(i),k)$-block for $k = g(i+q)-g(i) = h(j+q)-h(j) \leq q$.* ∎

Thus if $f : L(n,A) \to L(m,B)$ is a 0-homomorphism, then $\mathrm{Im}(f) = ((k,k)]$ for some $k \leq m,n$ and $f$ is standard whenever $(0,k-1),(k-1,0) \notin B$.

Next we define specific permutation lattices $L(i) = L(n(i),A(i))$ with $i = 0,1,\dots$.

We set $n(i) = 3i + 9$ for every $i \geq 0$, and let $A(i)$ consist of the pairs

(1) $(3k, 3k+2)$ and $(3k+2, 3k)$ with $k \in \{0, \dots, i+2\}$,
(2) $(3k-2, 3k+1)$ with $k \in \{1, \dots, i+2\}$,
(3) $(n(i)-2, 1)$.

LEMMA 2.4. *If $i, j \geq 0$ and $f : L(i) \to L(j)$ is a nonconstant 0-homomorphism, then $i = j$ and $f$ is the identity mapping of $L(i)$.*

*Proof.* First we show that the lattice $L(j)$ has no $(0, 0, l)$-block with $l < n(j)$. There is no such block for $l \leq 2$ because $(0, 2), (2, 0) \in A(j)$. Since $(n(j)-2, 1) \in A(j)$, there is no $(0, 0, l)$-block with $3 \leq l \leq n(j)-2$. For $l = n(j)-1$, we have $l = 3j+8$ and $(3j+6, 3j+8) \in A(j)$—and since $3j+6 < l$, this completes the proof that $L(j)$ has no proper $(0, 0, l)$-blocks. Therefore $f : L(i) \to L(j)$ is surjective and standard, and $n(i) \geq n(j)$, by Corollary 2.3.

In this paragraph only, we say that sublattices $A, B \subseteq L(k)$ isomorphic to $M_3$ form an *independent pair* if no element of $A$ is comparable to any element of $B$. It is clear that sublattices $f(A), f(B) \subseteq L(j)$ form an independent pair only when $A, B \subseteq L(i)$ do. It is routine to verify that for any $(p, q) \neq (n(j)-2, 1)$ the sublattice $M(p, q) \subseteq L(j)$ belongs to at most two independent pairs, while $M(n(j)-2, 1)$ forms an independent pair with every $M(r, s)$ other than those with $(r, s) \in \{(0, 2), (2, 0), (n(j)-3, n(j)-1), (n(j)-1, n(j)-3)\}$. Since $j \geq 9$, there are at least four independent pairs containing $M(n(j)-2, 1) \subseteq L(j) = \mathrm{Im}(f)$. Each $M(p, q) \subseteq L(i)$ with $(p, q) \neq (n(i)-2, 1)$ belongs to at most two independent pairs, so that from Lemma 2.2 it follows that $f(M(n(i)-2, 1)) = M(n(j)-2, 1)$, and since $n(j)-2 > 1$, the surjective homomorphism $f$ is direct, that is, there are surjective maps $g, h : C_{n(i)} \to C_{n(j)}$ such that $f(p, q) = (g(p), h(q))$ for all $p, q \in C_{n(i)}$. Clearly $g(n(i)-2) = n(j)-2$ and $h(q) = q$ for $q \in \{0, 1, 2\}$.

If $M(r, s)$ is the sublattice of $L(i)$ for which $f(M(r, s)) = M(2, 0) \subseteq L(j)$ then $h(s) = 0$, and $s = 0$ follows because $h(1) = 1$ and $h$ preserves order. Thus $g(2) = 2$ and $g(3) = 3$, and hence $g(p) = p$ for $p \in \{0, 1, 2, 3\}$. If $f(M(r, s)) = M(0, 2)$ then $r = 0$ because $g(1) = 1$ and $g$ preserves order, and hence $h(q) = q$ for $q \in \{0, 1, 2, 3\}$. Altogether $g(x) = h(x) = x$ for all $x \leq 3$.

Proceeding inductively from the initial claim that $g(x) = h(x) = x$ for all $x \leq 3$, we next suppose that $1 \leq k \leq j+2$ is such that $g(x) = h(x) = x$ for every $x \leq 3k$. First we note that the sublattice $f(M(3k-2, 3k+1))$ of $L(j)$ cannot be a singleton because $g(3k-2) = 3k-2 < 3k-1 = g(3k-1)$. Since $L(j)$ is a permutation lattice, we must have $f(M(3k-2, 3k+1)) = M(3k-2, 3k+1)$ and hence $h(3k+1) = 3k+1$ and $h(3k+2) = 3k+2$. Then $f$ cannot collapse the sublattice $M(3k+2, 3k) \subseteq L(i)$ and hence $g(3k+2) = 3k+2$ and $g(3k+3) = 3k+3$, that is, $g(x) = x$ for every

$x \le 3(k+1)$. We thus have $f(M(r,s)) = M(3k, 3k+2) \subseteq L(j)$ only for $(r,s) = (3k, 3k+2)$, and hence also $h(x) = x$ for all $x \le 3(k+1)$. This induction shows that $g(x) = h(x) = x$ for all $x \le 3k$ with $1 \le k \le j+3$, that is, for all $x \le n(j)$. Now if $n(j) < n(i)$ then $n(j) < n(i) - 2$ and hence $n(j) = g(n(j)) \le g(n(i)-2)$; but this contradicts the earlier found fact that $g(n(i)-2) = n(j)-2$. Therefore $i = j$ and $g = h$ is the identity map of $C_n$, and hence $f$ is the identity endomorphism of $L(i)$, as was to be shown. ∎

Next we use the lattices $L(j) = L(n(j), A(j))$ from Lemma 2.4 to build permutation lattices representing finite sets of natural numbers. Let $Y = \{y_0, \dots, y_{k-1}\}$ be a nonvoid subset of $\mathbb{N} = \{0, 1, \dots\}$ indexed in the ascending order, that is, let $y_0 < y_1 < \cdots < y_{k-1}$.

We define $m_Y^0 = 0$ and $m_Y^p = \sum_{i=0}^{p-1} n(y_i)$ for $p \in \{1, \dots, k\}$, and write $m_Y = m_Y^k$. In the first step, a lattice $L(m_Y, C_Y)$ is defined as the permutation lattice whose interval $J_p = [(m_Y^p, m_Y^p), (m_Y^{p+1}, m_Y^{p+1})]$ is isomorphic to the lattice $L(y_p) = L(n(y_p), A(y_p))$ for each $p \in k$. Described formally, the set $C_Y$ consists of all $(q,r) \in m_Y \times m_Y$ for which there exists $p \in k$ such that $m_Y^p \le q, r < m_Y^{p+1}$ and $(q - m_Y^p, r - m_Y^p) \in L(y_p)$.

It is then clear that $(0,0,s)$-blocks of $L(m_Y, C_Y)$ are exactly those with $s = m_Y^i$ for some $i \le k$, and the intervals $J_p = [(m_Y^p, m_Y^p), (m_Y^{p+1}, m_Y^{p+1})]$ with $p \in k$ isomorphic to $L(y_p)$ are also blocks of $L(m_Y, C_Y)$. For each $p \in k$, define $\pi_p = \bigwedge\{\pi(q,r) \mid (q,r) \in J_p \cap C_Y\}$, and let $\alpha_p$ be the congruence of $L(m_Y, C_Y)$ complementary to $\pi_p$. Thus $\alpha_p$ is the least congruence collapsing the interval $J_p$ for each $p \in k$. The lattice $L(m_Y, C_Y)/\pi_p$ is thus isomorphic to $L(y_p)$ for each $p \in k$, and $L(m_Y, C_Y)$ is a subdirect product of the lattices $L(y_p)$ with $p \in k$.

In the second step, we extend $L(m_Y, C_Y)$ to a permutation lattice $L[B_Y] = L(m_Y + 1, B_Y)$ by the requirement that $(q,r) \in B_Y$ iff either $(q-1, r) \in C_Y$ or $(q,r) = (0, m_Y)$. It is clear that $L[B_Y]$ is a permutation lattice which is subdirect in the product of $L(m_Y, C_Y)$ and a single copy of $M_3$.

LEMMA 2.5. *If $Y \subset \mathbb{N}$ is finite and nonvoid then*

(1) *$L[B_Y]$ has no proper $(0,0,q)$-block;*
(2) *$L[B_Y]$ has no $(0,1,q)$-block at all;*
(3) *the $(1,0,q)$-blocks of $L[B_Y]$ and the $(0,0,q)$-blocks of $L(m_Y, C_Y)$ are the same.* ∎

LEMMA 2.6. *For any $i$ and $Y$, the only $0$-homomorphism $f : L(i) \to L[B_Y]$ is constant.*

*Proof.* If $f : L(i) \to L[B_Y]$ is nonconstant, then it is surjective, by Corollary 2.3(2) and Lemma 2.5(1). Let $h_p : L[B_Y] \to L(y_p)$ be the surjective homomorphism with $\operatorname{Ker} h_p = \pi_p \vee \alpha(0, m_Y)$ for some $p \in k$. Then $h_p \circ f : L(i) \to L(y_p)$ is surjective, and hence $i = y_p$ and $h_p \circ f$ is the identity, by Lemma 2.4. But then $f$ is also injective, and it maps a proper subinterval of $L[B_Y]$ isomorphically onto $L[B_Y]$—a contradiction. ■

For a nonvoid subset $Z$ of a finite $Y \subset \mathbb{N}$ define

$$\pi_Z = \bigwedge\{\pi_p \mid y_p \in Z\} = \alpha(0, m_Y) \vee \bigvee\{\alpha_q \mid y_q \in Y \setminus Z\},$$

where $\pi_p$ and $\alpha_q$ are respectively the largest and the least extensions of the identically named congruences from the interval $L(m_Y, C_Y)$ to all of $L[B_Y]$. Thus $\pi_p \geq \alpha(0, m_Y)$ and $\alpha_p \wedge \alpha(0, m_Y)$ is the diagonal congruence for every $y_p \in Y$.

PROPOSITION 2.7. *If* $Y = \{y_0, \ldots, y_{k-1}\}$ *and* $Z = \{z_0, \ldots, z_{l-1}\}$ *are nonvoid subsets of* $\mathbb{N}$, *then*

(1) *there exists a nonconstant* 0*-homomorphism* $L[B_Y] \to L[B_Z]$ *only when* $Z \subseteq Y$;
(2) *if* $Z \subseteq Y$ *and* $f : L[B_Y] \to L[B_Z]$ *is a nonconstant* 0*-homomorphism then* $f$ *is direct and surjective*, *and* $\operatorname{Ker}(f) = \pi_Z \wedge \pi(0, m_Y)$;
(3) *if* $Z, Z' \subseteq Y$ *are nonvoid*, *then* $L[B_Y]$ *is isomorphic to a sublattice of* $L[B_Z] \times L[B_{Z'}]$ *if and only if* $Y = Z \cup Z'$.

*Proof.* Let $f : L[B_Y] \to L[B_Z]$ be a nonconstant 0-homomorphism. Then $f$ is surjective, by Corollary 2.3(2) and Lemma 2.5(1). Since $f$ is surjective and because only $(0,1)$ and $(1,0)$ are the atoms in $L[B_Z]$, we must have $f(1,0) \in \{(0,0),(1,0),\{0,1\}\}$. If $f(1,0) = (0,1)$ then Corollary 2.3 and Lemma 2.5(2)(3) imply that $f(m_Y + 1, m_Y) = (0,1)$, and thus $f(1, m_Y) = (0,1)$. But then $f(0, m_Y) \leq (0,1)$, and from $f(1,0) \wedge f(0, m_Y) = (0,0)$ it follows that $f(0, m_Y) = (0,0)$. Since $(0, m_Y) \in B_Y$ and $(0,0) \notin B_Z$ we get the contradictory $(0,1) = f(1,0) \leq f(1, m_Y + 1) = (0,0)$. Thus $f(1,0) \neq (0,1)$. Suppose that $f(1,0) = (0,0)$. Then $f$ maps the $(1,0,m_Y)$-block of $L[B_Y]$ isomorphic to $L(m_Y, C_Y)$ onto $L[B_Z]$, by Lemma 2.5(1) and Corollary 2.3(2); in particular, $f(m_Y + 1, m_Y) = (m_Z + 1, m_Z + 1)$. On the other hand, by Lemma 2.6, the restriction of $f$ to the $(1, 0, m_Y^1)$-block of $L[B_Y]$ isomorphic to $L(y_0)$ must be constant, that is, $f(m_Y^1, m_Y^1 + 1) = (0,0)$. Then the restriction of $f$ to the $(m_Y^1, m_Y^1 + 1, n(y_1))$-block of $L(m_Y, C_Y)$ isomorphic to $L(y_1)$ preserves the zero, and hence must be constant by Lemma 2.6 again; and a simple inductive argument along these lines shows that $f(m_Y + 1, m_Y) = (0,0)$, a contradiction. The only remaining possibility is that $f(1,0) = (1,0)$. Therefore $f$ is direct.

There exists a unique $(r,s) \in B_Y$ such that $f(r,s) = (0, m_Z)$, and we cannot have $r > 0$ because $f(1,0) = (1,0)$. Thus $f(0, m_Y) = (0, m_Z)$ and $f(1, m_Y + 1) = (1, m_Z + 1)$, and there are surjective $g, h : C_{m_Y+1} \to C_{m_Z+1}$ such that $f(i,j) = (g(i), h(j))$ for $i, j \in C_{m_Y+1}$. In particular, $h(m_Y) = m_Z$ and $g(m_Y + 1) = h(m_Y + 1) = m_Z + 1$. Therefore $\operatorname{Ker}(f) \subseteq \pi(0, m_Y)$, and $f$ is a direct 0-homomorphism that maps the interval $[(1,0),(m_Y + 1, m_Y)]$

of $L[B_Y]$ isomorphic to $L(m_Y, C_Y)$ onto the interval $[(1,0),(m_Z+1,m_Z)]$ isomorphic to $L(m_Z, C_Z)$. We shall now investigate the surjective domain-range restriction $f'$ of $f$ to these intervals, temporarily setting $f'(i,j) = f(i+1,j)$ to simplify the notation.

Since the $(0,0,s)$-blocks of the lattice $L(m_Y, C_Y)$ are exactly those with $s = m_Y^{i+1}$ for some $i \in k$ and because $L(m_Z, C_Z)$ has a similar property, there is an order-preserving surjective mapping $\phi : (k+1) \to (l+1)$ such that $\phi(0) = 0$, $\phi(k) = l$ and $f'(m_Y^i, m_Y^i) = (m_Z^{\phi(i)}, m_Z^{\phi(i)})$ for every $i \in k$.

Choose $z_j \in Z$ and select $(q', r') \in C_Z$ with $m_Z^j < q', r' < m_Z^{j+1}$. By Lemma 2.2(1a) and the definitions of $L(z_j)$ and of $L(m_Y, C_Y)$, there is a unique $(q,r) \in C_Y$ such that $f'(M(q,r)) = M(q',r')$, and a unique $y_i \in Y$ such that $m_Y^i < q, r < m_Y^{i+1}$. Let $e_i : L(y_i) \to L(m_Y, C_Y)$ denote the isomorphism from $L(y_i)$ onto the interval $[(m_Y^i, m_Y^i),(m_Y^{i+1}, m_Y^{i+1})]$ of $L(m_Y, C_Y)$, and let $p_j : L(m_Z, C_Z) \to L(z_j)$ be the surjective homomorphism with $\mathrm{Ker}(p_j) = \pi_j$. Since $f'$ is nonconstant on the image of $e_i$ and $\pi_j$ is the diagonal congruence on the interval $[(m_Z^j, m_Z^j),(m_Z^{j+1}, m_Z^{j+1})]$ of $L(m_Z, C_Z)$, the composite $\gamma_{i,j} = p_j \circ f' \circ e_i$ is nonconstant. In addition, $(m_Z^{\phi(i)}, m_Z^{\phi(i)}) = f'(m_Y^i, m_Y^i) \le f'(q,r) = (q', r')$, so that $p_j(m_Z^{\phi(i)}, m_Z^{\phi(i)})$ is the zero of $L(z_j)$. Thus $\gamma_{i,j} : L(y_i) \to L(z_j)$ is a nonconstant 0-homomorphism, and hence $y_i = z_j$ and $\gamma_{i,j}$ is the identity map, by Lemma 2.4. But then $Z \subseteq Y$, and (1) is proved.

Now if $\phi(i) < j$, then $(m_Z^j, m_Z^j) = f'(u,v)$ for some $(u,v)$ satisfying $(m_Y^i, m_Y^i) < (u,v) < (q,r)$, and hence $p_j(f'(u,v))$ is the zero of $L(z_j)$, contradicting the fact that $\gamma_{i,j}$ is the identity map. Therefore $\phi(i) = j$. We also know that $p_j(m_Z^{\phi(i+1)}, m_Z^{\phi(i+1)}) = p_j(f'(m_Y^{i+1}, m_Y^{i+1}))$ is the unit of $L(z_j)$. If $\phi(i+1) > j+1$ then there must be some $(s,t)$ satisfying $(q,r) < (s,t) < (m_Y^{i+1}, m_Y^{i+1})$ such that $f'(s,t) = (m_Z^{j+1}, m_Z^{j+1})$. But then $p_j(f'(s,t))$ is the unit of $L(z_j)$ and hence $\gamma_{i,j}$ is not the identity. Therefore $\phi(i+1) = j+1 = \phi(i)+1$ as well as $\phi(i) = j$, and hence $\mathrm{Ker}(f') \subseteq \pi_j$ for every $z_j \in Z$. Therefore $\mathrm{Ker}(f) \subseteq \pi_Z$.

If $y_i \in Y \setminus Z$, then $\gamma_{i,j} : L(y_i) \to L(z_j)$ is the constant map for every $z_j \in Z$ in view of Lemma 2.4. Since $L(m_Z, C_Z)$ is a subdirect product of the lattices $L(z_j)$ with $z_j \in Z$, it follows that $\alpha_i \subseteq \mathrm{Ker}(f')$. Altogether, $\mathrm{Ker}(f) = \pi_Z \wedge \pi(0, m_Y)$, and hence (2) holds.

For (3), let $f$ and $f' : L[B_Y] \to L[B_{Z'}]$ be 0-homomorphisms as in (2). If $Y = Z \cup Z'$ then $\mathrm{Ker}(f) \wedge \mathrm{Ker}(f')$ is the diagonal congruence. If $y_p \in Y \setminus (Z \cup Z')$, then $\alpha_p \subseteq \mathrm{Ker}(f) \wedge \mathrm{Ker}(f')$, and hence no homomorphism $L[B_Y] \to L[B_Z] \times L[B_{Z'}]$ can be injective. ∎

THE DEFINITION OF $\mathcal{A}$. We let $\mathcal{A}$ consist of the singleton lattice $A_\emptyset$ and all lattices $A_W = L[B_W]$ with finite nonvoid $W \subset \mathbb{N}$.

THEOREM 2.8. *The variety* $\mathrm{Var}_0(M_3)$ *is Q-universal.*

*Proof.* We show that the set $\mathcal{A}$ just defined satisfies conditions (P1)–(P4).

Condition (P1) obviously holds. For (P2), let $X = Y \cup Z$ be finite. Then $A_X$ is isomorphic to a 0-sublattice of $A_Y \times A_Z$ by Proposition 2.7(3), and hence $A_X \in \mathbf{Q}\{A_Y, A_Z\}$. For (P3), suppose that $X \neq \emptyset$ and $A_X \in \mathbf{Q}\{A_Y\}$. Then $A_X$ is a sublattice of some Cartesian power $A_Y^k$. The restriction of a product projection $A_Y^k \to A_Y$ to $A_X$ is a nonconstant 0-homomorphism $A_X \to A_Y$ only when $Y \subseteq X$ is nonvoid, and all of these restrictions have the same kernel $\theta = \pi_Y \wedge \pi(0, m_X)$, by Proposition 2.7. But $\theta$ is the diagonal congruence only when $Y = X$, and hence (P3) holds.

To prove (P4), suppose that $B, C \in \mathbf{Q}\mathcal{A}$ are finite and $A_X$ is a 0-sublattice of $B \times C$. It suffices to consider the case of $X \neq \emptyset$. Let $r_B : A_X \to B$ and $r_C : A_X \to C$ denote the domain restrictions of the two product projections. If $r_B$ is constant, then $A_X$ is isomorphic to a 0-sublattice of $C$ and hence (P4) holds for $Y = \emptyset$ and $Z = X$. We may thus assume that both $r_B$ and $r_C$ are nonconstant. It is also clear that $A_X$ is a 0-sublattice of $\mathrm{Im}(r_B) \times \mathrm{Im}(r_C)$. Since $B \in \mathbf{Q}\mathcal{A}$ is finite, the lattice $B$ is a 0-sublattice of some finite product $P = \prod\{A_{Y_i} \mid i \in I'\}$; let $p_i : P \to A_{Y_i}$ denote the product projection, and let $I$ be the set of all $i \in I'$ for which the composite $f_i = p_i \circ r_B : A_X \to A_{Y_i}$ is nonconstant, and hence also $Y_i \neq \emptyset$. For each $i \in I$ we obtain $Y_i \subseteq X$ by Proposition 2.7(1) and $\mathrm{Ker}(f_i) = \pi_{Y_i} \wedge \pi(0, m_X)$ by Proposition 2.7.(2). For the subset $Y = \bigcup\{Y_i \mid i \in I\}$ of $X$ we then have $\pi_Y \wedge \pi(0, m_X) = \mathrm{Ker}(r_B)$ because the projections $p_i$ with $i \in I'$ separate points of $\mathrm{Im}(r_B)$, and Proposition 2.7(2) then implies that $\mathrm{Im}(r_B) \subseteq B$ is isomorphic to $A_Y$ (with nonvoid $Y$). Therefore $A_Y \in \mathbf{Q}\{B\}$. The same argument shows that $\mathrm{Im}(r_C) \cong A_Z \in \mathbf{Q}\{C\}$ for some nonvoid $Z \subseteq X$. But then $X = Y \cup Z$, by Proposition 2.7(3), and hence (P4) holds. ∎

REMARK 2.9. The only nontrivial variety of modular 0-lattices not containing $\mathrm{Var}_0(M_3)$ is the variety $\mathbb{D}_0$ of distributive 0-lattices, and the only nontrivial critical algebra in $\mathbb{D}_0$ is the 2-element lattice. Theorem 2.8 thus gives a complete characterization of $Q$-universal varieties of modular 0-lattices. Together with Remark 1.6, this observation justifies the claim made in the abstract.

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