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HYPERSURFACES IN SPACES OF CONSTANT CURVATURE SATISFYING SOME RICCI-TYPE EQUATIONS

BY

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Abstract. We investigate hypersurfaces M in semi-Riemannian spaces of constant curvature satisfying some Ricci-type equations and for which the tensor H^3 is a linear combination of the tensor H^2, the second fundamental tensor H of M and the metric tensor g of M.

1. Introduction. Let (M,g), $n = \dim M \geq 4$, be a semi-Riemannian manifold and let ∇, R, S, C and κ be the Levi–Civita connection, the curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor and the scalar curvature of (M,g), respectively. Let $\mathcal{U}_R, \mathcal{U}_S$ and $\mathcal{U}_C$ be to subsets of M defined by

$$\mathcal{U}_R = \left\{ x \in M \;\middle|\; R - \frac{\kappa}{(n-1)n}\, G \neq 0 \text{ at } x \right\},$$
$$\mathcal{U}_S = \left\{ x \in M \;\middle|\; S - \frac{\kappa}{n}\, g \neq 0 \text{ at } x \right\},$$
$$\mathcal{U}_C = \{ x \in M \mid C \neq 0 \text{ at } x \}.$$

Evidently, $\mathcal{U}_S \subset \mathcal{U}_R$ and $\mathcal{U}_C \subset \mathcal{U}_R$. Further, let the $(0,4)$-tensor B be a generalized curvature tensor on M. According to [24] the generalized curvature tensor B on M satisfies the *Ricci-type equation* if on M we have $R\cdot B = B\cdot B$. If either $B = C$ or $B = R - C$ satisfies the Ricci-type equation then

$$R \cdot C = C \cdot C, \tag{1}$$
$$C \cdot R = C \cdot C, \tag{2}$$

respectively. We extend the above notion. Namely, the equation $C\cdot B = B\cdot B$ will also be called a Ricci-type equation. If the tensors $B = R$ or $B = C - R$ satisfy the latter equation then

$$R \cdot C = R \cdot R, \tag{3}$$

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$$C \cdot R = R \cdot R, \tag{4}$$

respectively. Clearly, we consider (1)–(4) on $\mathcal{U}_C \cap \mathcal{U}_S \subset M$. For a more general extension of the notion of the Ricci-type equation we refer to [2].

Let M be a hypersurface in a semi-Riemannian space $N_s^{n+1}(c)$ of constant curvature with signature $(s, n+1-s)$, $n \geq 4$, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space and $c = \widetilde{\kappa}/n(n+1)$. We denote by $\mathcal{U}_H$ the set of all points of M at which the tensor H^2, the square of the second fundamental tensor H of M, is not a linear combination of the metric tensor g and H. It is known that $\mathcal{U}_H \subset \mathcal{U}_C \cap \mathcal{U}_S \subset M$. For precise definitions of the symbols used we refer to Sections 2 and 3. We investigate hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ one of the Ricci-type equations (1)–(4). In the case of (1) or (2), in addition, we assume that on $\mathcal{U}_H$ we have

$$H^3 = \phi H^2 + \psi H + \varrho g, \tag{5}$$

where ϕ, ψ and ϱ are some functions on $\mathcal{U}_H$. In the case of (3) or (4) we do not need this additional assumption (see Remark 3.2). We prove (see Theorem 4.1) that if at every point of $\mathcal{U}_H$ either (1), or (2), or (3) and (5), or (4) and (5) is satisfied then on this set we have

$$H^3 = \operatorname{tr}(H) H^2 + \psi H + \varrho g. \tag{6}$$

Further, in Section 5 we prove that if at every point of $\mathcal{U}_H$ one of the equations (1)–(4) is satisfied then on this set we have

$$\operatorname{rank} H = 2, \tag{7}$$

which is equivalent on $\mathcal{U}_H$ to (see Theorem 3.2)

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g, R). \tag{8}$$

Thus the hypersurface M is pseudosymmetric ([7, Theorem 3.1]). We note that (7) implies (cf. Proposition 3.1)

$$H^3 = \operatorname{tr}(H) H^2 + \psi H, \tag{9}$$

i.e. on $\mathcal{U}_H$ we have $\varrho = 0$.

We recall that a semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *pseudosymmetric* ([8, Section 3.1]), resp., a *manifold with pseudosymmetric Weyl tensor* ([8, Section 12.6]), if at every point of M the tensors $R \cdot R$ and $Q(g, R)$, resp. $C \cdot C$ and $Q(g, C)$, are linearly dependent. The first condition is equivalent on $\mathcal{U}_R \subset M$ to

$$R \cdot R = L_R Q(g, R), \tag{10}$$

where L_R is some function on $\mathcal{U}_R$. The second condition is equivalent on $\mathcal{U}_C \subset M$ to

$$C \cdot C = L_C Q(g, C), \tag{11}$$

where L_C is some function on $\mathcal{U}_C$. If on M we have

$$R \cdot R = 0 \tag{12}$$

then the manifold (M, g) is called *semisymmetric*. Theorem 4.3 of [22] and Lemma 4.1 and Theorem 4.1 of [5] imply

THEOREM 1.1. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, with pseudosymmetric Weyl tensor, satisfying* (9) *on $\mathcal{U}_H \subset M$, then* (8) *holds on this set.*

Theorem 5.2 of [27] shows that the above theorem remains true if we replace (9) by (6). We note that on the subset $M - \mathcal{U}_H$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, (10) and (11) are always satisfied (see e.g. [22, Theorem 3.1]). Hypersurfaces satisfying (10), resp. (11), were investigated in [3], [5], [6], [7], [15] and [23], resp. in [20], [21] and [22]. We say that (10) and (11) are conditions of pseudosymmetry type. For a recent review of results on manifolds satisfying such conditions we refer to [4] (see also references therein).

Hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (9) on $\mathcal{U}_H \subset M$ were investigated in many papers: [1], [3], [5]–[7], [9]–[13], [15], [17], [19]–[23] and [26]. These papers are also related to the *P. J. Ryan problem* (see e.g. [10] and [11]).

Hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (6) on $\mathcal{U}_H \subset M$ were investigated in [2], [18] and [27]. In the present paper we continue the investigation of hypersurfaces satisfying (6). We will impose no restrictions on the signature of the ambient space. Thus in particular, the ambient space can be an $(n + 1)$-dimensional, $n \geq 4$, Lorentzian space of constant curvature or in particular an $(n + 1)$-dimensional, $n \geq 4$, Minkowski space. We mention that semisymmetric and conformally flat Lorentzian hypersurfaces in Minkowski spaces were investigated in [28] and [29], respectively. We also refer to [30] for results related to semisymmetric hypersurfaces in anti-de Sitter space.

Our main results are given in Theorems 5.1–5.4. We prove that if at every point of $\mathcal{U}_H \subset M$ one of the equations (1)–(4) is satisfied then on this set we have

$$\frac{\widetilde{\kappa}}{n+1} = \frac{\kappa}{n-1}. \tag{13}$$

This together with Proposition 4.3 reduces (1), (3) and (4) to $R \cdot C = C \cdot C = 0$, $R \cdot C = R \cdot R = 0$ and $C \cdot R = R \cdot R = 0$, respectively. Moreover, on $\mathcal{U}_H$ we have $\kappa = \widetilde{\kappa} = 0$. If (2) holds on U_H then $C \cdot R = C \cdot C = 0$ and (8) hold on this set. However, the scalar curvatures κ and $\widetilde{\kappa}$ are not necessarily equal to zero. At the end of Section 5 we give examples of hypersurfaces related to our main results.

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2. Preliminaries. Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞. Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian manifold and let ∇ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X,Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$
$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where A is a symmetric $(0,2)$-tensor on M and $X,Y,Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$ and the scalar curvature κ of (M,g) are defined by $S(X,Y) = \operatorname{tr}\{Z \mapsto \mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y) = S(X,Y)$ and $\kappa = \operatorname{tr} \mathcal{S}$. The endomorphism $\mathcal{C}(X,Y)$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right)Z.$$

Now the $(0,4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$
$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3, X_4),$$
$$C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3, X_4),$$

where $X_i \in \Xi(M)$. Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be the $(0,4)$-tensor associated with $\mathcal{B}(X,Y)$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3, X_4). \tag{14}$$

The tensor B is said to be a *generalized curvature tensor* if

$$B(X_1,X_2,X_3,X_4) + B(X_2,X_3,X_1,X_4) + B(X_3,X_1,X_2,X_4) = 0,$$
$$B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2).$$

Let $\mathcal{B}(X,Y)$ be a skew-symmetric endomorphism of $\Xi(M)$ and let B be defined by (14). We extend $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, by assuming that it commutes with contractions and $\mathcal{B}(X,Y) \cdot f = 0$ for any smooth function f on M. Now for a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0,k+2)$-tensor $B \cdot T$ by

$$\begin{aligned}(B \cdot T)(X_1,\ldots,X_k;X,Y) &= (\mathcal{B}(X,Y) \cdot T)(X_1,\ldots,X_k;X,Y)\\ &= -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k).\end{aligned}$$

In addition, if A is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$Q(A,T)(X_1,\dots,X_k;X,Y) = (X\wedge_A Y\cdot T)(X_1,\dots,X_k;X,Y)$$
$$= -T((X\wedge_A Y)X_1, X_2,\dots,X_k) - \dots - T(X_1,\dots,X_{k-1},(X\wedge_A Y)X_k).$$

In this manner we obtain the $(0,6)$-tensors $B\cdot B$ and $Q(A,B)$. Setting in the above formulas $\mathcal{B}=\mathcal{R}$ or $\mathcal{B}=\mathcal{C}$, $T=R$ or $T=C$ or $T=S$, $A=g$ or $A=S$, we get the tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $C\cdot C$, $R\cdot S$, $C\cdot S$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(g,S)$.

For symmetric $(0,2)$-tensors E and F we define their *Kulkarni–Nomizu product* $E\wedge F$ by

$$(E\wedge F)(X_1,X_2,X_3,X_4) = E(X_1,X_4)F(X_2,X_3) + E(X_2,X_3)F(X_1,X_4)$$
$$- E(X_1,X_3)F(X_2,X_4) - E(X_2,X_4)F(X_1,X_3).$$

Clearly, the tensors R, C, G and $E\wedge F$ are generalized curvature tensors. For a symmetric $(0,2)$-tensor E we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E}=\frac{1}{2}E\wedge E$. We have $\overline{g}=G=\frac{1}{2}g\wedge g$. We note that the Weyl tensor C can be represented in the form

$$C = R - \frac{1}{n-2}\, g\wedge S + \frac{\kappa}{(n-2)(n-1)}\, G. \tag{15}$$

We also have (see e.g. [12, Section 3])

$$Q(E, E\wedge F) = -Q(F,\overline{E}). \tag{16}$$

Now (15) and (16) yield

$$Q(g,C) = Q(g,R) + \frac{1}{n-2}\, Q(S,G). \tag{17}$$

For a symmetric $(0,2)$-tensor E and a $(0,k)$-tensor T, $k\geq 2$, we define their Kulkarni–Nomizu product $E\wedge T$ by ([10])

$$(E\wedge T)(X_1,X_2,X_3,X_4;Y_3,\dots,Y_k)$$
$$= E(X_1,X_4)T(X_2,X_3,Y_3,\dots,Y_k) + E(X_2,X_3)T(X_1,X_4,Y_3,\dots,Y_k)$$
$$- E(X_1,X_3)T(X_2,X_4,Y_3,\dots,Y_k) - E(X_2,X_4)T(X_1,X_3,Y_3,\dots,Y_k).$$

Using the above definitions we can prove

LEMMA 2.1 ([10], [23]). *Let E_1, E_2 and F be symmetric $(0,2)$-tensors at a point x of a semi-Riemannian manifold (M,g), $n\geq 3$. Then at x we have*

$$E_1\wedge Q(E_2,F) + E_2\wedge Q(E_1,F) = -Q(F, E_1\wedge E_2).$$

If $E=E_1=E_2$ then

$$E\wedge Q(E,F) = -Q(F,\overline{E}). \tag{18}$$

3. Hypersurfaces in spaces of constant curvature. Let M, $n\geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold (N,g^N). We denote by g the metric tensor induced on M from g^N. Further, we denote by ∇ and ∇^N the Levi-Civita connections corresponding to the metric tensors g and g^N, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = g^N(\xi, \xi) = \pm 1$. We can write the *Gauss formula* and the *Weingarten formula* of (M, g) in (N, g^N) in the following form: $\nabla^N_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\nabla^N_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M, H is the *second fundamental tensor* of (M, g) in (N, g^N), $\mathcal{A}$ is the *shape operator* and $H^k(X, Y) = g(\mathcal{A}^k X, Y)$, $k \geq 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and R^N the Riemann–Christoffel curvature tensors of (M, g) and (N, g^N), respectively.

Let $x^r = x^r(y^k)$ be the local parametric expression of (M, g) in (N, g^N), where y^k and x^r are local coordinates of M and N, respectively, and $h, i, j, k \in \{1, \ldots, n\}$ and $p, r, t, u \in \{1, \ldots, n+1\}$. The *Gauss equation* of (M, g) in (N, g^N) has the form

$$\begin{aligned} R_{hijk} &= R^N_{prtu} B_h{}^p B_i{}^r B_j{}^t B_k{}^u + \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}), \\ B_k{}^r &= \frac{\partial x^r}{\partial y^k}, \end{aligned} \tag{19}$$

where R^N_{prtu}, R_{hijk} and H_{hk} are the local components of the tensors R^N, R and H, respectively.

If (N, g^N) is a conformally flat space then we have ([15, Section 4])

$$C_{hijk} = \mu G_{hijk} + \varepsilon \overline{H}_{hijk} + \frac{\varepsilon}{n-2}\,(g \wedge (H^2 - \operatorname{tr}(H)H))_{hijk}, \tag{20}$$

$$\mu = \frac{1}{(n-2)(n-1)}\,(\kappa - 2\widetilde{S}_{rt} B^r_h B^t_k g^{hk} + \widetilde{\kappa}), \tag{21}$$

where $\widetilde{S}_{rt}$ are the local components of the Ricci tensor $\widetilde{S}$ of the ambient space, G_{hijk} are the local components of the tensor G, and $\widetilde{\kappa}$ and κ are the scalar curvatures of (N, g^N) and (M, g), respectively. From (20) we get

$$\begin{aligned} C \cdot H = \frac{\varepsilon}{n-2}\,&(Q(g, H^3) + (n-3)Q(H, H^2) \\ &- \operatorname{tr}(H)Q(g, H^2)) + \mu Q(g, H), \end{aligned} \tag{22}$$

$$\begin{aligned} C \cdot H^2 = \varepsilon(Q(H, H^3) + \frac{1}{n-2}\,&(Q(g, H^4) - \operatorname{tr}(H)Q(g, H^3) \\ &- \operatorname{tr}(H)Q(H, H^2))) + \mu Q(g, H^2). \end{aligned} \tag{23}$$

We have

THEOREM 3.1 ([21, Theorem 3.1]). *Every 2-quasi-umbilical hypersurface M, $\dim M \geq 4$, in a conformally flat semi-Riemannian manifold is a manifold with pseudosymmetric Weyl tensor.*

Let now M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$. Clearly, (19) and (21) read

$$R_{hijk} = \varepsilon \overline{H}_{hijk} + \frac{\widetilde{\kappa}}{n(n+1)}\, G_{hijk}, \tag{24}$$

$$\mu = \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right), \tag{25}$$

respectively. Contracting (24) with g^{ij} and g^{kh}, we obtain

$$S_{hk} = \varepsilon(\mathrm{tr}(H)H_{hk} - H^2_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g_{hk}, \tag{26}$$

$$\kappa = \varepsilon((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1}, \tag{27}$$

where $\mathrm{tr}(H) = g^{hk}H_{hk}$, $\mathrm{tr}(H^2) = g^{hk}H^2_{hk}$ and S_{hk} are the local components of the Ricci tensor S of M. Further, on every hypersurface M in $N^{n+1}_s(c)$, $n \geq 4$, we have ([19])

$$R \cdot R - Q(S,R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,C), \tag{28}$$

which by making use of (17) and (18) turns into

$$R \cdot R = Q(S,R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,R) - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(S,G). \tag{29}$$

If (8) holds on $\mathcal{U}_H$ then (29) yields

$$Q(S,R) = \frac{\widetilde{\kappa}}{n(n+1)}\,((n-1)Q(g,R) + Q(S,G)). \tag{30}$$

Let A be the $(0,2)$-tensor on M defined by ([13])

$$A = H^3 - \mathrm{tr}(H)H^2 + \frac{\varepsilon\kappa}{n-1}\, H. \tag{31}$$

From Theorem 5.1 of [17] it follows that A, defined by (31), vanishes on the subset $\mathcal{U}_H$ of any quasi-Einstein Ricci-semisymmetric hypersurface M in $\mathbb{E}^{n+1}_s$, $n \geq 4$. It is also known ([13, Theorem 5.1]) that $A = 0$ on the subset $\mathcal{U}_H$ of a hypersurface M in $N^{n+1}_s(c)$, $n \geq 4$, if and only if on $\mathcal{U}_H$ we have

$$R \cdot C - C \cdot R = \frac{1}{n-2}\, Q(S,R) + \frac{(n-1)\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(g,R).$$

Examples of hypersurfaces with nonzero tensor A are given in [13].

On any hypersurface M in $N^{n+1}_s(c)$, $n \geq 4$, we have the following identities ([13, Theorem 3.1]):

$$\begin{aligned} R \cdot C = {} & Q(S,R) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\, Q(g,R) \\ & - \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(S,G) + \frac{1}{n-2}\, g \wedge Q(H,A), \end{aligned} \tag{32}$$

$$\begin{aligned} C \cdot R = {} & \frac{n-3}{n-2}\, Q(S,R) - \frac{(n^2-3n+3)\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(g,R) \\ & - \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)}\, Q(S,G) + \frac{1}{n-2}\, H \wedge Q(g,A). \end{aligned} \tag{33}$$

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (5) on $\mathcal{U}_H \subset M$. We set

$$
\begin{aligned}
&\beta_1 = \varepsilon(\phi - \operatorname{tr}(H)),\\
&\beta_2 = -\frac{\varepsilon}{n-2}\,(\phi(2\operatorname{tr}(H) - \phi) - (\operatorname{tr}(H))^2 - \psi - (n-2)\varepsilon\mu),\\
&\beta_3 = \varepsilon\mu\operatorname{tr}(H) + \frac{1}{n-2}\,(\psi(2\operatorname{tr}(H) - \phi) + (n-3)\varrho),\\
&\beta_4 = \beta_3 - \varepsilon\beta_2\operatorname{tr}(H) + \frac{(n-1)\widetilde{\kappa}\beta_1}{n(n+1)},\\
&\beta_5 = \frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)} + \beta_1\operatorname{tr}(H),\\
&\beta_6 = \beta_2 - \frac{(n-3)\widetilde{\kappa}}{n(n+1)},
\end{aligned}
\tag{34}
$$

where the functions ϕ, ψ and ϱ are defined by (5). If (9) holds on $\mathcal{U}_H$ then (34) yields

$$
\begin{aligned}
&\beta_1 = 0,\\
&\beta_2 = \frac{1}{n-2}\left(\varepsilon\psi + \frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right),\\
&\beta_3 = \varepsilon\operatorname{tr}(H)\beta_2,\\
&\beta_4 = 0,\\
&\beta_5 = \frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)},\\
&\beta_6 = \beta_2 - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}.
\end{aligned}
\tag{35}
$$

To end this section we present some results from [3], [5], [7] and [23] which we apply in the next section.

THEOREM 3.2. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*

(i) ([3, Theorem 3.1]) *The conditions* (10) *and* $R \cdot C = L_R Q(g, C)$ *are equivalent on* $\mathcal{U}_C \subset M$.

(ii) ([5, Lemma 4.1 and Theorem 4.1]) *If M is pseudosymmetric then* (8) *holds on* $\mathcal{U}_H \subset M$.

(iii) ([7, Theorem 3.1]) *If* (7) *holds on* $\mathcal{U}_H \subset M$ *then so does* (8).

(iv) ([7, Theorem 5.1]) *If M is pseudosymmetric then* (7) *holds on* $\mathcal{U}_H \subset M$.

REMARK 3.1. Examples of hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (7) are given in [15].

Proposition 3.1. *If M is a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then on $\mathcal{U}_H \subset M$ we have* (9) *with*

$$\psi = \frac{1}{2}(\operatorname{tr}(H^2) - (\operatorname{tr}(H))^2) = \frac{\varepsilon(n-1)}{2}\left(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}\right). \tag{36}$$

Proof. Since M is pseudosymmetric, from Theorem 3.2(iv) it follows that (7) holds on $\mathcal{U}_H$. Now, using Lemma 2.1(i) of [9] and (27) we get our assertion.

Remark 3.2. (i) It is well known that any semi-Riemannian manifold (M, g), $n \geq 4$, satisfies the *Walker identity*

$$\begin{aligned}(R \cdot R)(X_1, X_2, X_3, X_4; X_5, X_6) + (R \cdot R)(X_3, X_4, X_5, X_6; X_1, X_2)\\ + (R \cdot R)(X_5, X_6, X_1, X_2; X_3, X_4) = 0,\end{aligned} \tag{37}$$

where $X_1, \ldots, X_6$ are vector fields on M.

(ii) Let now M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Clearly, if (3) or (4) is satisfied on $\mathcal{U}_H \subset M$ then on this set we have

$$\begin{aligned}(R \cdot C)(X_1, X_2, X_3, X_4; X_5, X_6) + (R \cdot C)(X_3, X_4, X_5, X_6; X_1, X_2)\\ + (R \cdot C)(X_5, X_6, X_1, X_2; X_3, X_4) = 0,\end{aligned} \tag{38}$$

$$\begin{aligned}(C \cdot R)(X_1, X_2, X_3, X_4; X_5, X_6) + (C \cdot R)(X_3, X_4, X_5, X_6; X_1, X_2)\\ + (C \cdot R)(X_5, X_6, X_1, X_2; X_3, X_4) = 0,\end{aligned} \tag{39}$$

respectively, where $X_1, \ldots, X_6$ are vector fields on $\mathcal{U}_H$. In [13, Proposition 4.1] it is shown that (38), (39) and

$$\begin{aligned}(R \cdot C - C \cdot R)&(X_1, X_2, X_3, X_4; X_5, X_6)\\ &+ (R \cdot C - C \cdot R)(X_3, X_4, X_5, X_6; X_1, X_2)\\ &+ (R \cdot C - C \cdot R)(X_5, X_6, X_1, X_2; X_3, X_4) = 0\end{aligned} \tag{40}$$

are equivalent on any semi-Riemannian manifold (M, g), $n \geq 4$. Proposition 5.2 of [23] implies that if on $\mathcal{U}_H \subset M$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, (3) or (4) is satisfied then (6) holds on this set.

(iii) In the next section we prove (see Theorem 4.2) that if (5) holds on $\mathcal{U}_H \subset M$ of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, then on this set we have

$$\begin{aligned}(C \cdot C)(X_1, X_2, X_3, X_4; X_5, X_6) + (C \cdot C)(X_3, X_4, X_5, X_6; X_1, X_2)\\ + (C \cdot C)(X_5, X_6, X_1, X_2; X_3, X_4) = 0.\end{aligned} \tag{41}$$

(iv) The relations (38)–(41) are called the *Walker-type identities.*

Theorem 3.3. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$,*

$$R \cdot C = LQ(g, C), \tag{42}$$

where L is some function on $\mathcal{U}_H$, then (7) *and* (9) *hold on $\mathcal{U}_H$, i.e.* (6) *holds with $\varrho = 0$.*

Proof. Clearly, from (42) it follows that (38) holds on $\mathcal{U}_H$. Now, in view of Proposition 5.1 of [23], on $\mathcal{U}_H$ we have (6) and

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g,S) + \varrho Q(g,H). \tag{43}$$

On the other hand, from (42), in view of Theorem 3.2(i), (ii), it follows that (8) holds on $\mathcal{U}_H$, which in view of Theorem 3.2(iv) implies (7) on $\mathcal{U}_H$. Further, from (8), by contraction, we get $R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g,S)$. This together with (43) gives $\varrho = 0$. Our theorem is thus proved.

4. Hypersurfaces satisfying $H^3 = \phi H^2 + \psi H + \varrho g$. In this section we consider hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (5) on $\mathcal{U}_H \subset M$. Applying (5) to (31) we obtain

$$A = (\phi - \operatorname{tr}(H))H^2 + \left(\psi + \frac{\varepsilon\kappa}{n-1}\right)H + \varrho g. \tag{44}$$

Proposition 4.1. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (5) *on $\mathcal{U}_H \subset M$ for some functions ϕ, ψ and ϱ on $\mathcal{U}_H$, then on this set we have*

$$\begin{aligned}(n-2)R \cdot C = {} & (n-2)Q(S,R) - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}\, Q(g,R)\\ & - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, Q(S,G) + \varrho Q(H,G) + (\phi - \operatorname{tr}(H))g \wedge Q(H,H^2),\end{aligned} \tag{45}$$

$$\begin{aligned}(n-2)C \cdot R = {} & \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right)Q(g,R)\\ & + (n-3)Q(S,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, Q(S,G) + (\phi - \operatorname{tr}(H))H \wedge Q(g,H^2),\end{aligned} \tag{46}$$

$$\begin{aligned}(n-2)C \cdot C = {} & \beta_1 Q(S, g\wedge H) + \beta_4 Q(H,G)\\ & + (n-3)Q(S,R) + \beta_5 Q(g,R) + \beta_6 Q(S,G),\end{aligned} \tag{47}$$

where $\beta_1, \ldots, \beta_6$ are defined by (34).

Proof. (32), by making use of (18) and (44), yields (45). Applying now (5) to (33) and using (18) and (44), we get (46). Further, (5) and (31) yield

$$\begin{aligned} H^3 - \operatorname{tr}(H)H^2 &= (\phi - \operatorname{tr}(H))H^2 + \psi H + \varrho g,\\ H^4 - \operatorname{tr}(H)H^3 &= (\psi + \phi(\phi - \operatorname{tr}(H)))H^2\\ &\quad + (\varrho + \psi(\phi - \operatorname{tr}(H)))H + \varrho(\phi - \operatorname{tr}(H))g,\end{aligned} \tag{48}$$

Applying (48) to (22) and (23) we obtain

$$\begin{aligned} C \cdot H = {} & \frac{\varepsilon(\phi - \operatorname{tr}(H))}{n-2}\, Q(g,H^2) + \left(\mu + \frac{\varepsilon\psi}{n-2}\right)Q(g,H)\\ & + \frac{(n-3)\varepsilon}{n-2}\, Q(H,H^2),\end{aligned} \tag{49}$$

$$C \cdot H^2 = \varepsilon\Big(\phi - \frac{\mathrm{tr}(H)}{n-2}\Big)Q(H,H^2) + \Big(\mu + \frac{\varepsilon}{n-2}\,(\psi + \phi(\phi - \mathrm{tr}(H)))\Big)Q(g,H^2) + \frac{\varepsilon}{n-2}\,(\psi(\phi - \mathrm{tr}(H)) - (n-3)\varrho)Q(g,H). \tag{50}$$

Using (25), (26), (34), (49) and (50) we find

$$\begin{aligned} C \cdot S &= \varepsilon C \cdot (\mathrm{tr}(H)H - H^2) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,C \cdot g \\ &= -\varepsilon\beta_1 Q(H,H^2) - \varepsilon\beta_2 Q(g,H^2) + \beta_3 Q(g,H). \end{aligned} \tag{51}$$

Applying (26) and (34) in (51) we get

$$C \cdot S = \beta_1 Q(H,S) + \beta_4 Q(g,H) + \beta_2 Q(g,S). \tag{52}$$

Now (15), (18), (44), (46) and (52) imply

$$\begin{aligned} (n-2)C \cdot C &= (n-2)C \cdot R - g \wedge (C \cdot S) \\ &= (n-2)C \cdot R - g \wedge (\beta_1 Q(H,S) + \beta_4 Q(g,H) + \beta_2 Q(g,S)) \\ &= (n-3)Q(S,R) + \Big(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\Big)Q(g,R) \\ &\quad + \Big(\beta_2 - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\Big)Q(S,G) + \beta_4 Q(H,G) \\ &\quad + \beta_1(\varepsilon H \wedge Q(g,H^2) - g \wedge Q(H,S)). \end{aligned} \tag{53}$$

Further, using (24) and (26) and Lemma 2.1 we obtain

$$\begin{aligned} &H \wedge Q(g,\varepsilon H^2) - g \wedge Q(H,S) \\ &\quad = H \wedge Q(g,\varepsilon\,\mathrm{tr}(H)H) - H \wedge Q(g,S) - g \wedge Q(H,S) \\ &\quad = \mathrm{tr}(H)Q(g,R) - (H \wedge Q(g,S) + g \wedge Q(H,S)) \\ &\quad = \mathrm{tr}(H)Q(g,R) + Q(S, g \wedge H). \end{aligned} \tag{54}$$

Applying (54) and (34) in (53) we get (47). Our proposition is thus proved.

THEOREM 4.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*

(i) *If* (3) *or* (4) *hold on $\mathcal{U}_H \subset M$ then* (44) *becomes*

$$A = \Big(\psi + \frac{\varepsilon\kappa}{n-1}\Big)H + \varrho g, \tag{55}$$

i.e. (6) *holds on $\mathcal{U}_H$, where ψ and ϱ are some functions on $\mathcal{U}_H$.*

(ii) *If* (1) *or* (2) *hold on $\mathcal{U}_H \subset M$ and in addition on $\mathcal{U}_H$ we have* (5) *then* (44) *becomes* (55).

Proof. (i) Since we assume that (3) or (4) holds at a point $x \in \mathcal{U}_H$, our assertion is an immediate consequence of Corollary 4.1 of [13].

(ii) Let now (1) or (2) be satisfied at a point $x \in \mathcal{U}_H$. Applying (47) to these relations we find that at x the tensors $R \cdot C$ and $C \cdot R$ are expressed as linear combinations of finite sums of tensors of the form $Q(E, B)$, where E is a symmetric $(0,2)$-tensor and B is a generalized curvature tensor. Therefore our assertion is a consequence of Corollary 4.1 of [13].

THEOREM 4.2. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (5) *on $\mathcal{U}_H \subset M$ for some functions ϕ, ψ and ϱ on $\mathcal{U}_H$, then the Walker-type identity* (41) *holds on this set.*

Proof. This follows immediately from (47) and the fact that for any symmetric $(0,2)$-tensor A and any generalized curvature tensor T we have the identity

$$Q(A,T)(X_1,X_2,X_3,X_4;X_5,X_6) + Q(A,T)(X_3,X_4,X_5,X_6;X_1,X_2) + Q(A,T)(X_5,X_6,X_1,X_2;X_3,X_4) = 0.$$

PROPOSITION 4.2. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (6) *on $\mathcal{U}_H \subset M$ for some functions ψ and ϱ on $\mathcal{U}_H$, then on this set we have*

$$(n-2)R \cdot C = (n-2)Q(S,R) - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}\,Q(g,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G) + \varrho Q(H,G), \tag{56}$$

$$(n-2)C \cdot R = \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right)Q(g,R) + (n-3)Q(S,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G), \tag{57}$$

$$(n-2)C \cdot C = (n-3)Q(S,R) + \beta_4 Q(H,G) + \beta_5 Q(g,R) + \beta_6 Q(S,G), \tag{58}$$

where $\beta_2,\ldots,\beta_6$ are defined by (34).

Proof. This is an immediate consequence of Proposition 4.1 and Theorem 4.1.

PROPOSITION 4.3. *If M is a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then on $\mathcal{U}_H \subset M$ we have*

$$R \cdot C = \frac{\widetilde{\kappa}}{n(n+1)}\,Q(g,C), \tag{59}$$

$$C \cdot R = \frac{n-3}{2(n-2)}\left(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}\right)Q(g,R), \tag{60}$$

$$C \cdot C = \frac{n-3}{2(n-2)}\left(\frac{\widetilde{\kappa}}{n+1} - \frac{\kappa}{n-1}\right) Q(g, C). \tag{61}$$

Proof. From Proposition 3.2 it follows that (9) and (36) hold on $\mathcal{U}_H$. In addition, on $\mathcal{U}_H$ we have $\varrho = 0$. Applying this, (17), (30) and (35) in (56)–(58) we obtain (59)–(61), completing the proof.

REMARK 4.1. Proposition 2.1 of [26] states that $C \cdot R = LQ(g, R)$ implies $C \cdot C = LQ(g, C)$ on $U_C \subset M$ of any semi-Riemannian manifold (M, g), $n \geq 4$.

We finish this section with the following two lemmas.

LEMMA 4.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Suppose that at $x \in \mathcal{U}_H \subset M$ the following condition is satisfied*:

$$Q(S, R) - \alpha_1 Q(g, R) - \alpha_2 Q(S, G) + \alpha_3 Q(H, G) = 0, \tag{62}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

(i) *At x we have*

$$\alpha_2 = \frac{\widetilde{\kappa}}{n(n+1)}, \tag{63}$$

$$H \wedge H^2 - \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha_1\right) g \wedge H + \varepsilon \alpha_3 G = \lambda \overline{H}, \tag{64}$$

$$\begin{aligned} Q(H, H^3) - \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha_1\right) Q(g, H^2) + \alpha_3 Q(g, H) & \\ = \lambda Q(H, H^2), & \end{aligned} \tag{65}$$

where $\lambda \in \mathbb{R}$.

(ii) *Moreover, if* (6) *holds at x then at this point we have*

$$\alpha_1 = \frac{(n-1)\widetilde{\kappa}}{n(n+1)}, \qquad \alpha_3 = \varrho, \qquad \lambda = \operatorname{tr}(H), \tag{66}$$

$$Q\left(S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) + \alpha_3 Q(H, G) = 0. \tag{67}$$

(iii) *If the above assumptions are satisfied and, in addition, $\varrho = 0$ at x, then $\psi = 0$ at x, i.e.* (13) *holds at x.*

Proof. (i) We suppose that $\frac{\widetilde{\kappa}}{n(n+1)} - \alpha_2 \neq 0$ at x. Now (62) turns into

$$\begin{aligned} & Q(S, R - \alpha_2 G) - \alpha_1 Q(g, R - \alpha_2 G) \\ & + \alpha_3 \left(\frac{\widetilde{\kappa}}{n(n+1)} - \alpha_2\right)^{-1} \left(Q\left(H, \left(\frac{\widetilde{\kappa}}{n(n+1)} - \alpha_2\right) G\right) + Q(H, \varepsilon \overline{H})\right) = 0, \end{aligned}$$

whence

$$Q(S - \alpha_1 g, R - \alpha_2 G) + \alpha_3 \left(\frac{\widetilde{\kappa}}{n(n+1)} - \alpha_2\right)^{-1} Q(H, R - \alpha_2 G) = 0,$$

and so

$$Q\left(S-\alpha_1 g+\alpha_3\left(\frac{\widetilde{\kappa}}{n(n+1)}-\alpha_2\right)^{-1} H, R-\alpha_2 G\right)=0. \tag{68}$$

Since $x \in \mathcal{U}_H$, at this point we have

$$\begin{aligned} S-\alpha_1 g+\alpha_3\left(\frac{\widetilde{\kappa}}{n(n+1)}-\alpha_2\right)^{-1} H &= -\varepsilon H^2 \\ &+\left(\varepsilon \operatorname{tr}(H)+\alpha_3\left(\frac{\widetilde{\kappa}}{n(n+1)}-\alpha_2\right)^{-1}\right) H+\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}-\alpha_1\right) g \neq 0. \end{aligned}$$

Now from (68), in view of Lemma 3.4 of [14], it follows that

$$(R-\alpha_2 G) \cdot (R-\alpha_2 G)=Q(S-(n-1)\alpha_2 g, R-\alpha_2 G).$$

Applying the identity

$$(R-\alpha_2 G) \cdot (R-\alpha_2 G)=R \cdot R-\alpha_2 Q(g, R)$$

we get

$$R \cdot R-Q(S, R)=-(n-2)\alpha_2\left(Q(g, R)+\frac{1}{n-2} Q(S, G)\right),$$

which, in virtue of (15)–(17), turns into

$$R \cdot R-Q(S, R)=-(n-2)\alpha_2 Q(g, C).$$

Comparing the right hand sides of (28) and the last equation we obtain $\alpha_2=\widetilde{\kappa}/n(n+1)$, a contradiction. Thus (63) holds at x. Further, applying (24), (26) and (63) in (62) we find

$$Q(H^2, \overline{H})-\varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}-\alpha_1\right) Q(g, \overline{H})-\alpha_3 Q(H, G)=0.$$

This, by making use of (16), yields

$$-Q(H, H \wedge H^2)+\varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}-\alpha_1\right) Q(H, g \wedge H)-\alpha_3 Q(H, G)=0,$$

whence

$$Q\left(H, H \wedge H^2-\varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}-\alpha_1\right) g \wedge H+\alpha_3 G\right)=0.$$

Since $\operatorname{rank} H>1$ at x, Lemma 3.4 of [14] now implies (64), and the latter yields

$$\begin{aligned} H_{hk} H_{ij}^2+H_{ij} H_{hk}^2-H_{hj} H_{ik}^2-H_{ik} H_{hj}^2+\alpha_3(g_{hk} g_{ij}-g_{hj} g_{ik}) \\ -\varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}-\alpha_1\right)(g_{hk} H_{ij}+g_{ij} H_{hk}-g_{hj} H_{ik}-g_{ik} H_{hj}) \\ = \lambda(H_{hk} H_{ij}-H_{hj} H_{ik}). \end{aligned}$$

Transvecting with $H^h_l = g^{hk}H_{kl}$ we find

$$H_{ij}H^3_{kl} - H_{ik}H^3_{jl} + H^2_{ij}H^2_{kl} - H^2_{ik}H^2_{jl} + \alpha_3(g_{ij}H_{kl} - g_{ik}H_{jl}) - \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha_1\right)(H_{ij}H_{kl} - H_{jl}H_{ik} + g_{ij}H^2_{kl} - g_{ik}H^2_{jl}) = \lambda(H_{ij}H^2_{kl} - H_{ik}H^2_{jl}). \tag{69}$$

This by symmetrization in l, i leads to (65).

(ii) If (6) holds at x then (65) turns into

$$(\mathrm{tr}(H) - \lambda)Q(H, H^2) - \varepsilon\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \alpha_1\right)Q(g, H^2) + (\alpha_3 - \varrho)Q(g, H) = 0.$$

Since $x \in \mathcal{U}_H$, Lemma 2.4(ii) of [19] now implies (66). Now (62) turns into (67).

(iii) Under our assumptions, (69) reduces to

$$H_{ij}H^3_{kl} - H_{ik}H^3_{jl} + H^2_{ij}H^2_{kl} - H^2_{ik}H^2_{jl} = \mathrm{tr}(H)(H_{ij}H^2_{kl} - H_{ik}H^2_{jl}). \tag{70}$$

Applying (9) we get

$$\psi(H_{ij}H_{kl} - H_{ik}H_{jl}) + H^2_{ij}H^2_{kl} - H^2_{ik}H^2_{jl} = 0.$$

Contracting with g^{ij} we obtain $\psi(\mathrm{tr}(H)H - H^2) + \mathrm{tr}(H^2)H^2 - H^4 = 0$. We note that (9) implies $H^4 = (\psi + (\mathrm{tr}(H))^2)H^2 + \psi\,\mathrm{tr}(H)H$. Now the last two relations give $\psi H^2 = 0$, whence $\psi = 0$ at x. This, by (36), gives (13), which completes the proof.

LEMMA 4.2. *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H$ the condition*

$$Q\left(S - \alpha g, R - \frac{\widetilde{\kappa}}{n(n+1)}\,G\right) = 0. \tag{71}$$

Then (8) *holds on $\mathcal{U}_H$, where α is a function on $\mathcal{U}_H$.*

Proof. From (71), in view of Lemma 3.4 of [14], at every point $x \in \mathcal{U}_H$ we have either $\mathrm{rank}(S - \alpha g) = 1$ or

$$R - \frac{\widetilde{\kappa}}{n(n+1)}\,G = \frac{\lambda}{2}\,(S - \alpha g) \wedge (S - \alpha g),$$

where $\lambda \in \mathbb{R} - \{0\}$. In the first case, in view of Lemma 2.3 of [16], at x we have $R \cdot R = \gamma_1 Q(g, R)$, $\gamma_1 \in \mathbb{R}$. Similarly, in the second case, in view of Lemma 3.4 of [14], at x we have $R \cdot R = \gamma_2 Q(g, R)$, $\gamma_2 \in \mathbb{R}$. But from Theorem 3.2(ii) it follows that $\gamma_1 = \gamma_2 = \widetilde{\kappa}/n(n+1)$ at x. Thus (8) holds on $\mathcal{U}_H$. Our lemma is thus proved.

5. The Ricci-type equations. In this section we consider hypersurfaces M in $N^{n+1}_s(c)$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ one of the Ricci-type equations (1)–(4).

Theorem 5.1. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (1) *on $\mathcal{U}_H \subset M$ then the ambient space is a semi-Euclidean space and on this set we have $R \cdot C = C \cdot C = 0$ and $\kappa = 0$.*

Proof. From Theorem 4.1 it follows that (6) holds on $\mathcal{U}_H$. Now (1), by (34), (56) and (58), turns into

$$Q(S,R) - \left(\varepsilon\psi + \frac{\kappa}{n-1} - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) Q(g,R) - \frac{1}{n-2}\left(\varepsilon\psi + \frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right) Q(S,G) + \frac{\varrho}{n-2} Q(H,G) = 0. \tag{72}$$

Clearly, this relation is of the form (62). Therefore Lemma 4.1(ii) implies $\varrho = 0$. Thus (72) reduces to (71). Now, in view of Lemma 4.2, (8) holds on U_H. Since $\varrho = 0$, Lemma 4.1(iii) implies (13), which together with (1), (59) and (61) completes the proof.

Theorem 5.2. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (2) *on $\mathcal{U}_H \subset M$ then on this set we have* (8), (13) *and $C \cdot R = C \cdot C = 0$.*

Proof. From Theorem 4.1 it follows that (6) holds on $\mathcal{U}_H$. Now (2), by (34), (57) and (58), turns into

$$\beta_2 Q(S,G) + \frac{(n-3)\varrho}{n-2} Q(H,G) = 0. \tag{73}$$

Clearly, this is of the form (62). Therefore Lemma 4.1(ii) implies $\varrho = 0$. Thus (72) reduces to (71). Now, in view of Lemma 4.2, (8) holds on U_H. Since $\varrho = 0$, Lemma 4.1(iii) implies (13), which together with (60) and (61) completes the proof.

Theorem 5.3. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (3) *on $\mathcal{U}_H \subset M$ then the ambient space is a semi-Euclidean space and on $\mathcal{U}_H$ we have $R \cdot C = R \cdot R = 0$ and $\kappa = 0$.*

Proof. From Theorem 4.1 it follows that (6) holds on $\mathcal{U}_H$. Now (4), by making use of (26), (29) and (56), yields

$$Q\left(\left(\frac{\varepsilon\widetilde{\kappa}}{n(n+1)}\operatorname{tr}(H) + \varrho\right)H - \frac{\varepsilon\widetilde{\kappa}}{n(n+1)} H^2 + \frac{(n-1)\widetilde{\kappa}^2}{n^2(n+1)^2} g, G\right) = 0.$$

By a suitable contraction this gives

$$\left(\frac{\varepsilon\widetilde{\kappa}}{n(n+1)}\operatorname{tr}(H) + \varrho\right)H - \frac{\varepsilon\widetilde{\kappa}}{n(n+1)} H^2 + \alpha g = 0,$$

where α is some function on $\mathcal{U}_H$. It follows that $\widetilde{\kappa} = \varrho = 0$ on $\mathcal{U}_H$. Thus (6) reduces to (9). This implies (8), which reduces to (12). Evidently, (12) implies $R \cdot C = 0$. Further, (12) and (28) imply $Q(S,R) = 0$. Now in view of Lemma 4.1(iii) we have (13) and in consequence $\kappa = 0$, which completes the proof.

THEOREM 5.4. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* (4) *on $\mathcal{U}_H \subset M$ then the ambient space is a semi-Euclidean space and on $\mathcal{U}_H$ we have* (8), (13), $R \cdot R = C \cdot R = 0$ *and* $\kappa = 0$.

Proof. From Theorem 4.1 it follows that (6) holds on $\mathcal{U}_H$. Now (4), by making use of (29) and (57), yields

$$Q(S,R) - \frac{\widetilde{\kappa}}{n(n+1)}\, Q(S,G) - \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) Q(g,R) = 0,$$

whence

$$Q\left(S - \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g, R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) = 0. \tag{74}$$

Clearly, this is of the form (71). Now, in view of Lemma 4.2, (8) holds on U_H. Since $\varrho = 0$, Lemma 4.1(iii) implies (13), which together with (4) and (60) completes the proof.

EXAMPLE 5.1 (i) Examples 4.1 and 5.1 of [12] yield an example of a hypersurface M in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ among other things $R \cdot R = R \cdot C = 0$, $C \cdot S = 0$ and $\kappa = 0$. Thus (13) holds on $\mathcal{U}_H$. In addition on this set we have

$$\begin{aligned} C \cdot C &= C \cdot \left(R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G\right) \\ &= C \cdot R - \frac{1}{n-2}\, g \wedge (C \cdot S) = 0. \end{aligned}$$

(ii) In [15, Example 5.1] an example is given of a pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, $c \neq 0$, satisfying (13) on $\mathcal{U}_H \subset M$. Evidently, that example is related to our Theorem 5.2.

(iii) An example of a semisymmetric ($R \cdot R = 0$) Einstein ($S = (\kappa/n)g$) hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, $c \neq 0$, is given in [25]. Clearly, the set $\mathcal{U}_H$ of that hypersurface is empty.

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