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SEMIGROUPS GENERATED BY CERTAIN PSEUDO-DIFFERENTIAL OPERATORS ON THE HALF-SPACE $\mathbb{R}^{n+1}_{0+}$

BY

VICTORIA KNOPOVA (Kiev)

Abstract. The aim of the paper is two-fold. First, we investigate the ψ-Bessel potential spaces on $\mathbb{R}^{n+1}_{0+}$ and study some of their properties. Secondly, we consider the fractional powers of an operator of the form

$$-A_{\pm} = -\psi(D_{x'}) \pm \frac{\partial}{\partial x_{n+1}}, \qquad (x', x_{n+1}) \in \mathbb{R}^{n+1}_{0+},$$

where $\psi(D_{x'})$ is an operator with real continuous negative definite symbol ψ: $\mathbb{R}^n \to \mathbb{R}$. We define the domain of the operator $-(-A_{\pm})^{\alpha}$ and prove that with this domain it generates an L_p-sub-Markovian semigroup.

0. Introduction. Consider the operator

$$(0.1)\quad (-A_{\pm})^{\alpha} = \left(\psi(D_{x'}) \pm \frac{\partial}{\partial x_{n+1}}\right)^{\alpha}, \qquad 0<\alpha<1,\ x=(x', x_{n+1}) \in \mathbb{R}^{n+1}_{0+},$$

where $\psi(D_{x'})$ is an operator with a real-valued continuous negative definite symbol, satisfying some special conditions, and $\mathbb{R}^{n+1}_{0+} = \mathbb{R}^n \times [0,\infty)$.

It has been proved that the operator with a real continuous negative definite symbol is the generator of an L_p-sub-Markovian semigroup, and it is even possible to determine the domain of such an operator in L_p in terms of certain function spaces (see [FJS] and also [J2]). In [JK1] and [Kn2] these results were extended to some cases when the symbol of the operator considered is not necessarily real.

The aim of this paper is to show that the operator of the form (0.1) with a suitable domain is the generator of an L_p-sub-Markovian semigroup in $L_p(\mathbb{R}^{n+1}_{0+})$ and investigate these semigroups, which will depend on the condition on the boundary $x_{n+1}=0$. We note that this problem was considered in [Kr] (see also [JK2]) for the situation when the operator can be decomposed into two parts: one is the one-dimensional fractional derivative for which the boundary value problem is posed, and the other is some pseudo-differential operator, which acts "inside" the boundary.

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To handle this problem we first need to determine the domain of $(-A_\pm)^\alpha$, $0<\alpha<1$, in terms of appropriate function spaces. In view of [JK1] or [Kn1], the natural candidates for such domains are the $\Re$-Bessel potential spaces on $\mathbb{R}^{n+1}_{0+}$, where $\Re = \operatorname{Re}\operatorname{symb}(-A_\pm)^\alpha$. Note that semigroups generated by $(-A_\pm)^\alpha$ depend on the boundary conditions.

In the first section we collect fundamental results on ψ-Bessel potential spaces, fractional powers of operators, and subordination in the sense of Bochner.

In Section 2 we define the ψ-Bessel potential spaces of order s on the half-space $\mathbb{R}^n_{0+}$, i.e. $\widetilde{H}^{\psi,s}_{p,0+}$ and $H^{\psi,s}_{p,0+}$, and investigate some of their properties, namely we find some dense sets in these spaces, isomorphisms between such spaces of different order, prove the existence of retractions and coretractions, as well as interpolation theorems.

In the third section we prove that the operators $(-(-A_\pm)^\alpha, H^{\Re,2}_{p,0+})$ and $(-(-A_\pm)^\alpha, \widetilde{H}^{\Re,2}_{p,0+})$ are generators of L_p-sub-Markovian semigroups, and find explicit representations of these semigroups. Solving this problem, we find, using the Laplace transform technique, the solutions to the equation

$$(\lambda + (-A)^\alpha_\pm) f(x) = g(x), \quad x = (x', x_{n+1}) \in \mathbb{R}^{n+1}_+, \tag{0.2}$$

where $f \in H^{\Re,2}_{p,0+}$ or $\widetilde{H}^{\Re,2}_{p,0+}$, and $g \in L_p(\mathbb{R}^{n+1}_{0+})$, with the Dirichlet (zero or non-zero) and zero Neumann boundary conditions. Then the representations of the resolvents of (0.2) give us the corresponding semigroups. In [Kn3] we considered the operator (0.1) with zero Dirichlet boundary condition; now we will treat a more general situation.

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1. Preliminaries. In this section we summarize the main results from the theory of semigroups and subordination in the sense of Bochner, and recall the definition of ψ-Bessel potential spaces. We refer to [Y], [J2], and [FJS].

There is one-to-one correspondence between the Bernstein functions f and the convolution semigroups $(\eta_t)_{t\ge0}$ via the Laplace transform, i.e.

$$\int_0^\infty e^{-sx}\,\eta_t(ds) = e^{-tf(x)}. \tag{1.1}$$

We recall that the convolution semigroup of measures $\eta^\alpha_t(dx) = \sigma_\alpha(x,t)dx$ with densities $\sigma_\alpha(x,t)$, $t>0$, corresponding to the Bernstein function $f(x) = x^\alpha$ is called the *one-sided stable semigroup of order* α.

Some properties of the functions $\sigma_\alpha(x,t)$, $t>0$, will be helpful.

(1) The Laplace transform of $\sigma_\alpha(x,t)$, $x>0$, with respect to t is

$$\int_0^\infty e^{-t\mu}\sigma_\alpha(x,t)\,dt = \frac{e'_\alpha(x,\mu)}{-\mu}, \qquad \mu>0, \tag{1.2}$$

where $e'_\alpha(x,\mu)$ is the derivative of the *Mittag-Leffler type function* $e_\alpha(x,\mu)$, $\mu>0$:

$$e_\alpha(x,\mu) := E_{\alpha,1}(-\mu y^\alpha) = \sum_{k=0}^\infty \frac{(-\mu)^k y^{\alpha k}}{\Gamma(\alpha k+1)}, \quad x>0. \tag{1.3}$$

Here $E_{\alpha,\beta}(z)$ is the *Mittag-Leffler function*. For the properties of such functions see [BE, Vol. 3, §18.1].

(3) The Laplace transform of $\frac{e'_\alpha(x,\mu)}{-\mu}$ in x is

$$L_{x\to z}\left[\frac{e'_\alpha(x,\mu)}{-\mu}\right] = \frac{1}{\mu+z^\alpha}, \qquad \operatorname{Re} z>0. \tag{1.4}$$

(4) The Laplace transform of $e_\alpha(x,\mu)$ in x is

$$L_{x\to z}[e_\alpha(x,\mu)] = \frac{z^\alpha}{\mu+z^\alpha}, \qquad \operatorname{Re} z>0. \tag{1.5}$$

We will need the notion of the *subordinated semigroup*. Starting with a sub-Markovian semigroup $(T_t)_{t\ge0}$ on a Banach space X, and the convolution semigroup $(\eta_t)_{t\ge0}$ which corresponds to the Bernstein function f, we can construct the subordinated semigroup

$$T_t^f u = \int_0^\infty T_s u\,\eta_t(ds), \qquad u\in X, \tag{1.6}$$

which is again sub-Markovian (see [J1]).

Starting with a closed linear operator $(A, D(A))$ on the Banach space X we may apply the Hille–Yosida theorem to check if this operator is the generator of a strongly continuous contraction semigroup. We will quote the version of the Hille–Yosida theorem given in [J1].

Denote by $R(A)$ the range of the operator A.

THEOREM 1.1 (Hille–Yosida theorem). *A closed operator $(A, D(A))$ on a Banach space $(X, \|\cdot\|_X)$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\ge0}$ if and only if the following conditions hold*:

(1) *$D(A)\subset X$ is dense*;
(2) *A is a dissipative operator*, *i.e.* $\|(\lambda-A)u\|_X \ge \lambda\|u\|$ *for all* $u\in D(A)$ *and some* $\lambda>0$;
(3) *$R(\lambda-A)=X$ for some $\lambda>0$.*

Having the generator $(A, D(A))$ of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on a Banach space X one may construct for $0<\alpha<1$ the fractional powers of $-A$:

$$(1.7)\qquad (-A)^{\alpha}u=\frac{1}{\Gamma(-\alpha)}\int_0^{\infty} t^{-\alpha-1}(T_t u-u)\,dt,\qquad u\in D(A),$$

and

$$(1.8)\qquad (-A)^{-\alpha}u=\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}T_t u\,dt,\qquad u\in X.$$

Formulae (1.7) and (1.8) are called *Balakrishnan's formulae* (see [Y] and [S]).

We recall the definition of ψ-Bessel potential spaces on $\mathbb{R}^n$. Our references on these spaces are [FJS] and [J2].

Let $\psi:\mathbb{R}^n\to\mathbb{R}$ be a continuous negative definite function with the representation

$$(1.9)\qquad \psi(\xi)=\int_{\mathbb{R}^n\setminus\{0\}}(1-\cos(y\cdot\xi))\,\nu(dy),$$

where the Lévy measure $\nu(dy)$ is such that $\int_{\mathbb{R}^n\setminus\{0\}}(|y|^2\wedge 1)\,\nu(dy)<\infty$. Further, let A be the generator of an L_p-sub-Markovian semigroup $(T_t)_{t\geq 0}$, associated with ψ. For $s\geq 0$ define the *ψ-Bessel potential space of order s* as

$$H_p^{\psi,s}=H_p^{\psi,s}(\mathbb{R}^n)=(I-A)^{-s/2}(L_p(\mathbb{R}^n)),$$

with the norm given by

$$\|u\|_{H_p^{\psi,s}}=\|f\|_{L_p}\qquad\text{for } u=(I-A)^{-s/2}f.$$

The space $H_p^{\psi,s}$ coincides (see [FJS]) with the closure of $S(\mathbb{R}^n)$, the Schwartz space on $\mathbb{R}^n$, with respect to the norm

$$(1.10)\qquad \|F^{-1}((1+\psi(\cdot))^{s/2}\widehat{u})\|_{L_p}.$$

The spaces $H_p^{\psi,s}$ are defined for a real-valued continuous negative definite function ψ; however, in some cases we can define them for complex-valued functions. Let $\chi(\xi)=\psi(\xi')+i\xi_{n+1}$, $\xi=(\xi',\xi_{n+1})\in\mathbb{R}^{n+1}$, where ψ is a real-valued continuous negative definite function with representation (1.9). Then we can define the χ-Bessel potential space $H_p^{\chi,2,1}=H_p^{\chi,2,1}(\mathbb{R}^{n+1})$ as the closure of the tensor product $H_p^{\psi,2}\otimes H_p^1$ with respect to the graph norm of the operator with symbol $\chi(\xi)$ (see [Kn1] for details). Here H_p^1 is the classical Sobolev space of order 1.

LEMMA 1.2. *Suppose that the operators $(A, D(A))$ and $(B, D(B))$ with $D(A)\subset L_p(X, d\mu_1)$ and $D(B)\subset L_p(Y, d\mu_2)$ can be extended to generators of strongly continuous contraction semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ on*

$L_p(X, d\mu_1)$ and $L_p(Y, d\mu_2)$. Then the closure $(C, \overline{D(A) \otimes D(B)}^{\|\cdot\|_C})$ of the operator $C = A \oplus B = A \otimes I_X + I_Y \otimes B$ with domain $D(A) \otimes D(B)$ generates a strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on $L_p(X \times Y, d\mu_1 \otimes d\mu_2)$. The operator $(C, D(C))$ is sub-Markovian if $(A, D(A))$ and $(B, D(B))$ are.

Here $\|f\|_C = \|Cf\|_{p,X\times Y} + \|f\|_{p,X\times Y}$ is the graph norm of the operator C.

The proof of the strong continuity and contractivity of $(T(t))_{t\geq 0} = (T_1(t) \oplus T_2(t))_{t\geq 0}$ is standard and we omit it (see [Kn1] and [Kr]).

2. The ψ-Bessel potential spaces on $\mathbb{R}^n_{0+}$. In the previous section we gave the definition of the ψ-Bessel potential spaces on $\mathbb{R}^n$. Now we extend this definition to the case of half-spaces.

DEFINITION 2.1. Let ψ be a continuous negative definite function with representation (1.9). We define

$$H^{\psi,s}_{p,+} := \{f : \exists g \in H^{\psi,s}_p,\ f = g|_{\mathbb{R}^n_{0+}}\}$$

with the norm

$$\|f\|_{\psi,s,+} = \|f\|_{\psi,s,p,+} = \inf_{g\in H^{\psi,s}_p,\ f=g|_{\mathbb{R}^n_{0+}}} \|g\|_{\psi,s,p},$$

and

$$\widetilde{H}^{\psi,s}_{p,+} := \{f :\ f \in H^{\psi,s}_p,\ \operatorname{supp} f \subset \mathbb{R}^n_{0+}\}$$

with the norm of $H^{\psi,s}_p$.

Similarly we can define the spaces $H^{\psi,s}_{p,-}$ and $\widetilde{H}^{\psi,s}_{p,-}$.

Though we do not need a more general definition, we point out that in Definition 2.1 we can replace the space $\mathbb{R}^n_{0+}$ by an arbitrary $G \subset \mathbb{R}^n$.

We also note that this definition is also applicable to the spaces $H^{\chi,2,1}_p$, where $\chi_\pm(\xi) = \psi(\xi') \pm i\xi_{n+1}$, $\xi = (\xi', \xi_{n+1}) \in \mathbb{R}^{n+1}$, and the continuous negative definite function ψ admits representation (1.9).

One may see that

$$H^{\psi,s}_{p,+} = H^{\psi,s}_p / \widetilde{H}^{\psi,s}_{p,-}, \qquad s \in \mathbb{R}. \tag{2.1}$$

Knowing dense sets in $H^{\psi,s}_p$, we can easily find dense sets in $H^{\psi,s}_{p,0+}$ and $\widetilde{H}^{\psi,s}_{p,0+}$:

THEOREM 2.2. (a) $C^\infty_0(\mathbb{R}^n_{0+}) = C^\infty_0(\mathbb{R}^n)|_{\mathbb{R}^n_{0+}}$ *is dense in* $H^{\psi,s}_{p,+}$ *for all* $s > 0$.

(b) $C^\infty_0(\mathbb{R}^n_+) = \{f\colon f \in C^\infty_0(\mathbb{R}^n),\ \operatorname{supp} f \subset \mathbb{R}^n_{0+}\}$ *is dense in* $\widetilde{H}^{\psi,s}_{p,+}$ *for all* $s > 0$.

For the proof we refer to [Kn2].

Consider the $\Re$-Bessel potential spaces, where $\Re = \operatorname{Re}\operatorname{symb}(-A_\pm)^\alpha = (\chi_\pm)^\alpha$, $0 < \alpha < 1$. In [Kn2] it was proved that under some conditions the operators $(-A_\pm)^\alpha$ are isomorphisms between $H_p^{\Re,s}$ and $H_p^{\Re,s-2}$. These conditions are:

(A1) $\psi(\xi) = f(\phi(\xi))$, where f is a Bernstein function, and ϕ is a real continuous negative definite function such that for all i, $1 \le i \le n$, ϕ_i' exists for $|\xi_i| > 0$ and does not depend on ξ_j, $i \ne j$ (we denote by g_i' the derivative of $g(\xi_1, \dots, \xi_n)$ with respect to ξ_i);

(A2) For $\xi \in \mathbb{R}^n$ with $|\xi| > 0$,

$$\sup_{\xi \in \mathbb{R}^n,\, |\xi_j| > 0} \left| \frac{\xi_1 \cdots \xi_k \phi_1' \cdots \phi_k'}{\phi^k} \right| < \infty, \qquad k = 1, \dots, n.$$

It follows from the Paley–Wiener theorem that the operator $(-A_+)^\alpha$ is also an isomorphism between $\widetilde{H}_{p,0+}^{\Re,s}$ and $\widetilde{H}_{p,0+}^{\Re,s-2}$.

Theorem 2.3. *Let $-\infty < t < \infty$ and $1 < p < \infty$. Then*

$$(-A_+)^\alpha : \widetilde{H}_{p,+}^{\Re,t} \to \widetilde{H}_{p,+}^{\Re,t-2}$$

isomorphically, where A_+ is an operator with symbol χ_+.

Proof. We proceed similarly to [T2, Theorem 2.10.3]. It was proved in [Kn1] that $(-A_+)^\alpha$: $H_p^{\Re,t} \to H_p^{\Re,t-2}$. What we need to know is that if $f \in C_0^\infty(\mathbb{R}^{n+1})$ with $\operatorname{supp} f \subset \mathbb{R}_{0+}^{n+1}$, then $\operatorname{supp}(-A_+)^\alpha f \subset \mathbb{R}_{0+}^{n+1}$. For this we use the Paley–Wiener theorem (see [Y, pp. 226–229]).

Let $g \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} g \subset (-\infty, \varepsilon)$ for some $\varepsilon > 0$. Then we derive an estimate for the Fourier–Laplace transform $\widehat{g}(z)$ of g, where $z = \xi + i\eta$:

$$\begin{aligned}(2.2)\qquad |(1+|z|)^N \widehat{g}(z)| &= \left| \int_{-\infty}^{\varepsilon} \frac{e^{-izx}}{(2\pi)^{1/2}} (1 - (-\Delta)^{1/2})^N g(x)\, dx \right| \\ &= \left| \int_{-\infty}^{\varepsilon} \frac{e^{x\eta - ix\xi}}{(2\pi)^{1/2}} (1 - (-\Delta)^{1/2})^N g(x)\, dx \right| \le C_{g,N} e^{\eta\varepsilon}\end{aligned}$$

for all $N \in \mathbb{N}$ and some constant $C_{g,N}$.

Consider

$$\begin{aligned}&F^{-1}((i\xi_{n+1} + \psi(\xi'))^\alpha \widehat{f}(\xi)) \\ &\qquad = \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{i(x',\xi') + ix_{n+1}\xi_{n+1}} (i\xi_{n+1} + \psi(\xi'))^\alpha \widehat{f}(\xi)\, d\xi_{n+1}\, d\xi'.\end{aligned}$$

Since the function $iz + \psi(\xi')$, $z \in \mathbb{C}$, has a root z_0: $\operatorname{Re} z_0 = 0$, $\operatorname{Im} z_0 = \psi(\xi') > 0$, we extend (see [T, §3.1]) the function $(iz + \psi(\xi'))^\alpha \widehat{f}(\xi', z)$ to the lower half-plane of $\mathbb{C}$. Consider the rectangle $\{-k \le \operatorname{Re} z \le k,\ -N \le \operatorname{Im} z \le 0\}$, where $k, N \geq 0$. In view of (2.2) for $f \in C_0^\infty(\mathbb{R}^{n+1})$ with $\operatorname{supp} f \subset \mathbb{R}^n \times (-\infty, \varepsilon)$, we have

$$|(1+\psi(\xi')+|z|)^N \widehat{f}(\xi', z)| \leq C_{f,N,\varepsilon} e^{\eta\varepsilon} \tag{2.3}$$

for some constant $C_{f,N,\varepsilon}$ (uniformly in ξ', because we can make N large, and the growth in ξ' in the denominator will "kill" the growth in ξ' in the numerator).

The integrals along $\{\operatorname{Re} z = -k, \operatorname{Im} z$ from $-N$ to $0\}$ and $\{\operatorname{Re} z = k, \operatorname{Im} z$ from 0 to $-N\}$ tend to 0 as $k \to \infty$. Indeed, integrating along $\{\operatorname{Re} z = -k$, $\operatorname{Im} z$ from $-N$ to $0\}$ we obtain

$$\begin{aligned}\Big| \int_{-N}^{0} & e^{i(x',\xi')-ikx_{n+1}-x\tau}(-ik-\tau+\psi(\xi'))^\alpha \widehat{f}(\xi', -k+i\tau)\,d\tau \Big| \\ &\leq C_{f,N,\varepsilon} \int_{-N}^{0} e^{(\varepsilon-x_{n+1})\tau} \frac{(k^2+(\psi(\xi')-\tau)^2)^{\alpha/2}}{(1+\psi(\xi')+(\tau^2+k^2)^{1/2})^N}\,d\tau,\end{aligned}$$

and the right-hand side tends to 0 as $k \to \infty$ by the Lebesgue dominated convergence theorem. For the integral along $\{\operatorname{Re} z = k, \operatorname{Im} z$ from 0 to $-N\}$ the estimate is similar.

Therefore in view of the Cauchy theorem, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(\xi'))^\alpha \widehat{f}(\xi)\,dz \\ &\quad = \lim_{k\to\infty} \int_{-iN-k}^{-iN+k} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(\xi'))^\alpha \widehat{f}(\xi', z)\,dz \\ &\quad = \int_{-\infty}^{\infty} e^{i(x',\xi')+ix_{n+1}\tau-Nx_{n+1}}(i\tau+N+\psi(\xi'))^\alpha \widehat{f}(\xi', \tau-iN)\,d\tau.\end{aligned}$$

In view of (2.3), for some large N and a constant $C_{f,N,\varepsilon}$ we have

$$\begin{aligned}&|e^{i(x',\xi')+ix_{n+1}\tau-Nx_{n+1}}(i\tau+N+\psi(\xi'))^\alpha \widehat{f}(\xi', \tau-iN)| \\ &\qquad \leq \frac{C_{f,N,\varepsilon} e^{-(\varepsilon-x_{n+1})N}(\tau^2+(N+\psi)^2)^{\alpha/2}}{|(1+\psi(\xi')+(\tau^2+N^2)^{1/2})^N|},\end{aligned}$$

and thus by the Lebesgue dominated convergence theorem we get

$$\int_{-\infty}^{\infty} e^{i(x',\xi')+ix_{n+1}z}(iz+\psi(\xi'))^\alpha \widehat{f}(\xi', z)\,dz = 0.$$

Thus, if $f \in C_0^\infty(\mathbb{R}^{n+1})$ and $\operatorname{supp} f \subset \mathbb{R}^n \times (-\infty, \varepsilon)$ then

$$F^{-1}((i\xi_{n+1}+\psi(\xi'))^\alpha \widehat{f}(\xi))(x', x_{n+1}) = 0, \tag{2.4}$$

and letting $\varepsilon \to 0$ we obtain (2.4) for $f \in C_0^\infty(\mathbb{R}^{n+1})$ with $\operatorname{supp} f \subset \mathbb{R}^n \times (\infty, 0]$. By density arguments,

$$\operatorname{supp} F^{-1}((i\xi_{n+1} + \psi(\xi'))^\alpha \widehat{f}(\xi))(x', x_{n+1}) \subset \mathbb{R}^n \times [0, \infty)$$

for all $f \in \widetilde{H}_{p,+}^{\Re,2}$. ∎

From Theorem 2.3 we immediately derive, in view of (2.1),

Theorem 2.4. *Let $-\infty < t < \infty$ and $1 < p < \infty$. Then*

$$(-A_-)^\alpha : H_{p+}^{\Re,t} \to H_{p+}^{\Re,t-2} \tag{2.5}$$

isomorphically, where A_- is an operator with symbol χ_-.

Now we want to prove the existence of retractions and coretractions between the spaces $\widetilde{H}_{p,+}^{\Re,s}$ (see [T2, §1.2.4]).

Denote by $L(A, B)$ the space of continuous linear operators from A to B, where A and B are normed vector spaces.

Let A and B be two complex Banach spaces. The operator $R \in L(A, B)$ is called a *retraction* if there exists an operator $S \in L(B, A)$ such that

$$RS = I. \tag{2.6}$$

An operator S such that (2.6) holds is called a *coretraction* corresponding to R.

Theorem 2.5. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then for all s there exists a coretraction from $\widetilde{H}_{p,+}^{\Re,s}$ to $H_p^{\Re,s}$, and for all s with $|s| < 2N$ there exists a retraction from $H_p^{\Re,s}$ to $\widetilde{H}_{p,+}^{\Re,s}$.*

The idea of the proof is the following. We want to construct the retraction and the coretraction using the known result, proved in [T2, Theorem 2.10.4/2], for the case of Triebel–Lizorkin spaces. We formulate this theorem for the Bessel potential spaces H_p^s $(= F_{p2}^s)$.

Theorem 2.6. *Let $1 < p < \infty$ and $-\infty < s < \infty$. Define*

$$\widetilde{R}\varphi(x) = 1_{\{x_n \geq 0\}}(x)\Big(\varphi(x) - \sum_{j=1}^{N+1} a_j \varphi(x'; -\lambda_j x_n)\Big), \qquad \varphi \in C_0^\infty(\mathbb{R}),$$

where $1_{\{x_n \geq 0\}}$ is the characteristic function of $\mathbb{R}_{0+}^n$, $0 < \lambda_1 < \cdots < \lambda_{N+1} < \infty$ and the coefficients a_j are such that

$$\frac{\partial^k}{\partial x_n^k} \varphi(x', x_n)\Big|_{x_n = 0} = \sum_{j=1}^{N+1} a_j \frac{\partial^k}{\partial x_n^k} \varphi(x', -\lambda_j x_n)\Big|_{x_n = 0}.$$

Then $\widetilde{R}$ extends to a continuous retraction R from H_p^s to $\widetilde{H}_{p,+}^s$, $|s| < N$, with coretraction

$$Sf = \begin{cases} f, & x_n \geq 0, \\ 0 & x_n < 0. \end{cases}$$

We also refer to Theorem 2.10.3.a in [T2], where it was proved that

$$J_s f = F^{-1}(ix_n + (1+|x'|^{1/2})^s \widehat{f})$$

is an isomorphic mapping from $\widetilde{H}^{\sigma}_{p,+}$ to $\widetilde{H}^{\sigma-s}_{p,+}$.

Proof of Theorem 2.5. Since $(-A_\pm)^\alpha : H_p^{\Re,s} \to H_p^{\Re,s-2}$ is an isomorphism (see [JK1]), we can deduce, applying $(-A_+)^{\pm\alpha}$ N times, that $(-A_+)^{\alpha N}$: $H_p^{\Re,2N} \to L_p$ is an isomorphism, and then, by Theorem 2.3, that so is $(-A_+)^{\alpha N}$: $\widetilde{H}^{\Re,2N}_{p,+} \to L_p$. Then, using Theorem 2.6 and Theorem 2.10.3.a of [T2] we can construct the diagrams

$$\begin{tikzcd} H_p^{\psi,2N} \arrow[r, "(-A_+)^{\alpha N}"] \arrow[d, "R_0"'] & L_p \arrow[r, "J_s^{-1}"] & H_p^s \arrow[d, "R"] \\ \widetilde{H}^{\psi,2N}_{p,+} & L_{p,+} \arrow[l, "(-A_+)_{-\alpha N}"'] & \widetilde{H}^s_{p,+} \arrow[l, "J_s"'] \end{tikzcd}$$

and

$$\begin{tikzcd} H_p^{\psi,2N} & L_p \arrow[l, "(-A_+)^{\alpha N}"'] & H_p^s \arrow[l, "J_s"'] \\ \widetilde{H}^{\psi,2N}_{p,+} \arrow[u, "S_0"] \arrow[r, "(-A_+)^{-\alpha N}"] & L_{p,+} \arrow[r, "J_s^{-1}"] & \widetilde{H}^s_{p,+} \arrow[u, "S"'] \end{tikzcd}$$

and without loss of generality we can put $s = 2N$ in the definition of J_s.

Since here all operators are isomorphisms, it follows that

$$S_0 = (-A_+)^{-\alpha N} J_{2N} S\, J_{2N}^{-1} (-A_+)^{\alpha N}$$

is a coretraction from $\widetilde{H}^{\Re,2N}_{p,+}$ to $H_p^{\Re,2N}$ which corresponds to the retraction $R_0 = (-A_+)^{-\alpha N} J_{2N} R\, J_{2N}^{-1} (-A_+)^{\alpha N}$. The same is true for $\widetilde{H}^{\Re,-2N}_{p,+}$ and $H_p^{\Re,-2N}$. Then, by applying Theorem 1.2.4 of [T2], we conclude that $R_0 S_0 = I$, that is, S_0 and R_0 are a coretraction and retraction for the spaces $\widetilde{H}^{\Re,s}_{p,+}$ and $H_p^{\Re,s}$, $|s| < 2N$. ∎

Analogously, we obtain

THEOREM 2.7. *For all $-\infty < s < \infty$ and $1 < p < \infty$, and ψ satisfying conditions* (A1) *and* (A2), *the restriction from $H_p^{\Re,s}$ to $H_{p,+}^{\Re,s}$ is a retraction and for all N there exists a coretraction which does not depend on p and s, $|s| < N$.*

Proof. The proof is a modification of the proof of Theorem 2.5 by taking the coretraction S_1 given by

$$S_1 f = \begin{cases} f, & x_n \geq 0, \\ \sum_{j=1}^{N+1} a_j f(x'; -\lambda_j x_n), & x_n < 0, \end{cases}$$

instead of S; S_1 corresponds to the retraction R_1, restriction to the half-space $\mathbb{R}^n_{0+}$. Next, we apply the operators $I_s f = F^{-1}((1+|\xi'|^2)^{1/2} - i\xi_n)^s \widehat{f})$ and $(-A_-)^{\pm\alpha}$ to construct the desired retraction and coretraction between $H^{\Re,s}_{p,+}$ and $H^{\Re,s}_p$. ∎

The interpolation theorem for the spaces $\widetilde{H}^{\psi,s}_{p,0+}$ follows immediately from Theorem 1.17.1/1 of [T2]:

THEOREM 2.8. *Let* $1 < p_0, p_1 < \infty$, $-\infty < s_0, s_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, *and* $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$[\widetilde{H}^{\Re,s_0}_{p_0,+}, \widetilde{H}^{\Re,s_1}_{p_1,+}]_\theta = \widetilde{H}^{\Re,s}_{p,+}.$$

REMARK 2.9. Since $H^{\Re,s}_{p,+} = H^{\Re,s}_p / \widetilde{H}^{\Re,s}_{p,-}$, for the same parameters we have

$$[H^{\Re,s_0}_{p_0,+}, H^{\Re,s_1}_{p_1,+}]_\theta = H^{\Re,s}_{p,+}.$$

This follows from Theorem 1.17.2 in [T2].

3. Semigroups generated by $(-(-A_\pm)^\alpha, D((-A_\pm)^\alpha)$**.** Now we are ready to formulate our main results. We start with the case $\alpha = 1$, i.e. we want to prove that $(-A_\pm, D(A_\pm))$ with $\mathrm{symb}(A_\pm) = \chi_\pm = \psi(\xi') \pm i\xi_{n+1}$, $\xi = (\xi', \xi_{n+1}) \in \mathbb{R}^{n+1}$, are generators of L_p-sub-Markovian semigroups on $L_p(\mathbb{R}^{n+1}_{0+})$. First we note that we can define the operators $-A_\pm$ on the tensor product of $H^{\psi,2}_p$ and $H^1_{p,+}$ (or $\widetilde{H}^1_{p,+}$), and the closure of $H^{\psi,2}_p \otimes H^1_{p,+}$ (resp. $H^{\psi,2}_p \otimes \widetilde{H}^1_{p,+}$) with respect to the graph norm of $A_\pm$ gives us the domains of $A_\pm$. We show this in detail in the proof below.

In the following we assume that

$$\psi(\xi') \geq (1+|\xi'|^2)^{\delta/2}, \qquad \xi' \in \mathbb{R}^n, \tag{3.1}$$

for some δ, $0 < \delta < 2$.

THEOREM 3.1. *The operators* $(-A_+, H^{\chi,2,1}_{p,+})$ *and* $(-A_+, \widetilde{H}^{\chi,2,1}_{p,+})$ *are generators of* L_p*-sub-Markovian semigroups* $(T^{(1)}_t)_{t\geq 0}$ *and* $(T^{(2)}_t)_{t\geq 0}$ *respectively. Moreover*,

$$T^{(1)}_t f(x) = \int_{\mathbb{R}^n} \frac{f(x'-y', x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{n/2}} 1_{\{x_{n+1}\geq t\}}(x) + \int_{\mathbb{R}^n} \frac{h(x'-y')\mathcal{W}_{x_{n+1}}(y')\,dy'}{(2\pi)^{n/2}} 1_{\{x_{n+1}< t\}}(x) \tag{3.2}$$

and

$$T^{(2)}_t f(x) = \int_{\mathbb{R}^n} \frac{f(x'-y', x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{n/2}} 1_{\{x_{n+1}\geq t\}}(x), \tag{3.3}$$

where $\mathcal{W}_t = F^{-1}(e^{-\psi(\xi')t})$ *exists as a function*, ψ *satisfies* (A1), (A2) *and* (3.1), $h \in \operatorname{tr}_{\mathbb{R}^n} H^{\chi,2,1}_{p,+}$ (*the trace space of* $H^{\chi,2,1}_{p,+}$ *on* $\mathbb{R}^n$), *and* $f(x', x_{n+1}) = 0$ *for* $x_{n+1} \le 0$ *in* (3.3).

REMARK 3.2. It is possible to find the trace spaces for the operators we consider, but we will not do it now, because it requires the study of spaces of generalized smoothness, which is not the aim of this paper. Now for us it is important that the trace exists; for this we suppose below that $1/p < \alpha < 1$.

Proof. That $(-A_+, \widetilde{H}^{\chi,2,1}_{p,+})$ is the generator of the L_p-sub-Markovian semigroup (3.3) was proved in [Kn3]. Now we will prove the second part of the theorem.

First we show that $H^{\chi,2,1}_{p,+}$ and $\widetilde{H}^{\chi,2,1}_{p,+}$ are domains of A_+, i.e.

$$\overline{H^{\psi,2}_p \otimes H^1_{p,+}}^{\|\cdot\|_A} = H^{\chi,2,1}_{p,+} \quad \text{and} \quad \overline{H^{\psi,2}_p \otimes \widetilde{H}^1_{p,+}}^{\|\cdot\|_A} = \widetilde{H}^{\chi,2,1}_{p,+}.$$

Indeed,

$$H^{\psi,2}_p \otimes \widetilde{H}^1_{p,+} = \{f : f \in H^{\psi,2}_p \otimes H^1_p, \operatorname{supp} f \subset \mathbb{R}^{n+1}_{0+}\}$$

gives the second equality; the first is proved analogously.

By Lemma 1.2 the operators $(-A_+, H^{\chi,2,1}_{p,+})$ and $(-A_+, \widetilde{H}^{\chi,2,1}_{p,+})$ are generators of L_p-sub-Markovian semigroups. To find these semigroups, we consider the equation

$$\begin{aligned} (\lambda + A_+)f(x) &= g(x), \qquad g \in L_p(\mathbb{R}^{n+1}_{0+}),\ x \in \mathbb{R}^{n+1}_+, \\ \lambda f(x', 0) &= h(x'), \qquad h \in \operatorname{tr}_{\mathbb{R}^n} H^{\chi,2,1}_{p,+}. \end{aligned} \tag{3.4}$$

Denote by $\widehat{g}(\xi, \eta)$ the function $L_{x_{n+1}\to\eta} F_{x'\to\xi'}(g(x', x_{n+1}))$, where $L_{x_{n+1}\to\eta}$ is the Laplace transform, $F_{x'\to\xi'}$ the Fourier transform, and set $\widehat{g}(\xi', 0) = F_{x'\to\xi'}(g(x', 0))$ and $\widehat{g}(\xi', x_{n+1}) = F_{x'\to\xi'}(g(x', x_{n+1}))$.

Taking the Fourier transform $F_{x'\to\xi'}$ of the left-hand side of $(3.4)_1$,

$$\begin{aligned} &F_{x'\to\xi'}((\lambda + A_+)f)(\xi', x_{n+1}) \\ &\qquad = \lambda \widehat{f}(\xi', x_{n+1}) + \psi(\xi')\widehat{f}(\xi', x_{n+1}) + \frac{\partial}{\partial x_{n+1}} \widehat{f}(\xi', x_{n+1}), \end{aligned}$$

and then the Laplace transform $L_{x_{n+1}\to\eta}$,

$$L_{x_{n+1}\to\eta} F_{x'\to\xi'}((\lambda + A_+)f)(\xi', x_{n+1}) = (\lambda + \psi(\xi') + \eta)\widehat{f}(\xi, \eta) - \widehat{f}(\xi', 0),$$

we finally derive that

$$\widehat{f}(\xi, \eta) = \frac{\widehat{g}(\xi, \eta) + \widehat{f}(\xi', 0)}{(\lambda + \psi(\xi') + \eta)} \tag{3.5}$$

is the $L_{x_{n+1}\to\eta} F_{x'\to\xi'}$-transform of the solution to $(3.4)_1$ with some boundary conditions. Consider the operator

$$T_t^{(1)}g(x) = \int\limits_{\mathbb{R}^n} \frac{g(x'-y', x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{n/2}}\, 1_{\{x_{n+1}\ge t\}}(x) + \int\limits_{\mathbb{R}^n} \frac{h(x'-y')\mathcal{W}_{x_{n+1}}(y')\,dy'}{(2\pi)^{n/2}}\, 1_{\{x_{n+1}< t\}}(x),$$

where $g \in L_p(\mathbb{R}^{n+1}_{0+})$ and $h \in \mathrm{tr}_{\mathbb{R}^n} H^{\chi,2,1}_{p,+}$. It is bounded in $L_p(\mathbb{R}^{n+1}_{0+})$. Indeed, since $\mathcal{W}_t(y') = F^{-1}(e^{-t\psi(\xi')})$ is an L_p-multiplier, we have $T_t^{(1)}g(\cdot, x_{n+1}) \in L_p(\mathbb{R}^n)$ for $g \in L_p(\mathbb{R}^{n+1}_{0+})$. Further, the first term in the representation of $T_t^{(1)}g(x', \cdot)$ belongs to $L_p(\mathbb{R}_{0+})$ since $g(x', \cdot)$ does, and the second is bounded and with finite support with respect to x_{n+1}.

Let $S(\mathbb{R}^{n+1}_{0+}) = S(\mathbb{R}^{n+1})|_{\mathbb{R}^{n+1}_{0+}}$, where $S(\mathbb{R}^{n+1})$ is the Schwartz space. If we show that for $g \in S(\mathbb{R}^{n+1}_{0+})$ and $h \in S(\mathbb{R}^n)$ the resolvent

$$R_\lambda g = \int\limits_0^\infty e^{-\lambda t} T_t^{(1)} g(x)\,dt$$

solves (3.4), then by density of $S(\mathbb{R}^{n+1}_{0+})$ in $L_p(\mathbb{R}^{n+1}_{0+})$ we deduce that $(T_t^{(1)})_{t\ge 0}$ defined on $L_p(\mathbb{R}^{n+1}_{0+})$ is a semigroup generated by $-A_+$.

Rewrite $T_t^{(1)}g(x)$ as

$$T_t^{(1)}g(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{i(x',\xi')-t\psi(\xi')}\{\widehat{g}(\xi', x_{n+1}-t)1_{[0,x_{n+1})}(t) + \widehat{h}(\xi')1_{[x_{n+1},\infty)}(t)\}\,d\xi'.$$

We want to check if the Laplace transform of the semigroup $(T_t^{(1)})_{t\ge 0}$ gives us the solution to (3.4).

Applying the Fubini theorem to $L_{x_{n+1}\to\eta}F_{x'\to\xi'}L_{t\to\lambda}(T_t^{(1)}g)$ we get

$$\begin{aligned}
&L_{t\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T_t^{(1)}g)\\
&= L_{t\to\lambda}L_{x_{n+1}\to\eta}\Big[1_{[t,\infty)}(x_{n+1}) \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{e^{-i(\xi',x')}g(x'-y', x_{n+1}-t)\mathcal{W}_t(y')}{(2\pi)^n}\,dy'\,dx'\Big]\\
&\quad + L_{t\to\lambda}L_{x_{n+1}\to\eta}\Big[1_{[0,t)}(x_{n+1}) \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{e^{-i(\xi',x')}h(x'-y')\mathcal{W}_{x_{n+1}}(y')}{(2\pi)^n}\,dy'\,dx'\Big]\\
&= L_{t\to\lambda}L_{x_{n+1}\to\eta}[\widehat{g}(\xi', x_{n+1}-t)e^{-t\psi(\xi')}1_{[t,\infty)}(x_{n+1}) + 1_{[0,t)}(x_{n+1})\widehat{h}(\xi')e^{-x_{n+1}\psi(\xi')}]\\
&= \frac{\widehat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta} + \frac{\widehat{h}(\xi')}{\lambda(\lambda+\psi(\xi')+\eta)}.
\end{aligned}$$

Thus for $\lambda f(x',0)=h(x')$ we have

$$L_{t\to\lambda}L_{x_{n+1}\to\eta}F_{x'\to\xi'}(T_t^{(1)}g)=\frac{\widehat{g}(\xi',\eta)}{\lambda+\psi(\xi')+\eta}+\frac{\widehat{f}(\xi',0)}{\lambda+\psi(\xi')+\eta},$$

which equals (3.5).

Therefore, the Laplace transform of $T_t^{(1)}g$ indeed gives the solution to (3.4). Since there is one-to-one correspondence between the images and preimages of the Fourier–Laplace transform, we conclude that the operators $(T_t^{(1)})_{t\ge 0}$ form a strongly continuous contraction semigroup with generator $(-A_+,H_p^{\chi,2,1})$, which proves (3.2).

Evidently, (3.3) can be obtained from (3.2) by putting $h(x')=0$. ∎

Remark 3.3. The semigroup generated by $(-A_+,H_p^{\chi,2.1})$ can be different, if we impose different boundary conditions in (3.4). If we take

$$\frac{\partial}{\partial x_{n+1}}f(x',0)=0,$$

then the semigroup generated by $(-A_+,H_p^{\chi,2.1})$ is the following:

$$\begin{aligned}T_t^{(1')}g(x)&=\int\limits_{\mathbb{R}^n}\frac{g(x'-y',x_{n+1}-t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{n/2}}\,1_{[t,\infty)}(x_{n+1})\\&\quad+\int\limits_{\mathbb{R}^n}\frac{g(x'-y',0)\mathcal{W}_t(y')\,dy'}{(2\pi)^{n/2}}\,1_{[0,t)}(x_{n+1}).\end{aligned}\tag{3.6}$$

Analogously, we have

Theorem 3.4. *The operator $(-A_-,H_{p,+}^{\chi,2,1})$ is the generator of the L_p-sub-Markovian semigroup $(T_t^{(3)})_{t\ge0}$ given by*

$$T_t^{(3)}f(x)=\int\limits_{\mathbb{R}^n}\frac{f(x'-y',x_{n+1}+t)\mathcal{W}_t(y')\,dy'}{(2\pi)^{n/2}}.\tag{3.7}$$

Now let us consider the fractional power of $-A_+$, $(-A_+)^\alpha$, $0<\alpha<1$. First consider functions from $D_1=\widetilde{H}_{p,+}^{\chi,2,1}$ and $D_2=H_{p,+}^{\chi,2,1}$. From [J1, Theorem 4.3.7] the domain of the generator A_+ of a strongly continuous contraction semigroup is dense in $D((-A_+)^\alpha)$, and $D(A_+)$ is a core for $(-A_+)^\alpha$, $0<\alpha<1$. Then

$$\overline{D}_1^{\|\cdot\|_{(-A_+)^\alpha}}=\overline{D}_1^{\|\cdot\|_{\Re,2}}=\{f:\|f\|_{\Re,2}<\infty,\ \operatorname{supp}f\subset\mathbb{R}^{n+1}_{0+}\}=\widetilde{H}_{p,+}^{\Re,2},$$

and analogously $\overline{D}_2^{\|\cdot\|_{(-A_+)^\alpha}}=H_{p,+}^{\Re,2}$. For the operator $-(-A_-)^\alpha$ with $\operatorname{symb}(A_-)=\chi_-$, the situation is similar.

To solve the boundary-value problem for the operator $(-A_\pm)^\alpha$, we need the existence of the trace $f(\cdot, x_{n+1})$ if $f \in D((-A_\pm)^\alpha)$. Since

$$(\psi^2(\xi') + \xi_{n+1}^2)^{\alpha/2} \geq \frac{\psi^\alpha(\xi') + |\xi_{n+1}|^\alpha}{2},$$

by the Lizorkin multiplier theorem (see [Liz]) we obtain

$$H_{p,+}^{\Re,2} \hookrightarrow \overline{H_p^{\psi^\alpha,2}(\mathbb{R}^n) \otimes H_p^\alpha(\mathbb{R}_{0+})},$$

where the closure is taken with respect to the graph norm of the operator $\psi(D_{x'})^\alpha + (-\Delta_{x_{n+1}})^{\alpha/2}$. Thus, since the trace in the space $H_p^\alpha(\mathbb{R}_{0+})$ exists for $1/p < \alpha < 1$, for such α the trace will exist in the space $H_{p,+}^{\Re,2}$. Analogously, for $1/p < \alpha < 1$ the trace exists in the space $\widetilde{H}_{p,+}^{\Re,2}$ and is equal to zero.

THEOREM 3.5. *For* $1 < p < \infty$ *and* $1/p < \alpha < 1$ *the operators* $(-(-A_+)^\alpha, \widetilde{H}_{p,+}^{\Re,2})$ *and* $(-(-A_-)^\alpha, H_{p,+}^{\Re,2})$ *with* $\operatorname{symb}(-A_\pm) = \chi_\pm(\xi') = \psi(\xi') \pm i\xi_{n+1}$, *where* ψ *satisfies* (A1), (A2) *and* (3.1), *are the generators of* L_p*-sub-Markovian semigroups* $(T_t^{(4)})_{t\geq 0}$ *and* $(T_t^{(5)})_{t\geq 0}$ *given by*

$$T_t^{(4)} g(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_0^{x_{n+1}} g(x' - y', x_{n+1} - s) \mathcal{W}_s(y') \sigma_\alpha(s,t)\, ds\, dy', \tag{3.8}$$

$$T_t^{(5)} g(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_0^{\infty} g(x' - y', x_{n+1} + s) \mathcal{W}_s(y') \sigma_\alpha(s,t)\, ds\, dy'. \tag{3.9}$$

The statement about the operator $(-(-A_+)^\alpha, \widetilde{H}_{p,+}^{\Re,2})$ was proved in [Kn3] by a straightforward application of the Hille–Yosida theorem (see also [Kn1]); the second statement can be proved in a similar way. We only note that since the operator $(-A_-)^\alpha$, $0 < \alpha < 1$, is an isomorphism between $H_{p,+}^{\Re,2}$ and $L_p(\mathbb{R}_{0,+}^{n+1})$, the boundary condition with which $(-A_-)^\alpha$ generates (3.9) is

$$f(x', 0) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_{\mathbb{R}^n} \frac{g(x' - y', t) \mathcal{W}_t(y')}{(2\pi)^{n/2} t^{1-\alpha}}\, dy'\, dt.$$

In Theorem 3.1 and Remark 3.3 we showed that $(T_t^{(1)})_{t\geq 0}$ and $(T_t^{(1')})_{t\geq 0}$ are strongly continuous contraction semigroups generated by $(-A_+, H_{p,+}^{\chi,2,1})$ with different boundary conditions. By Bochner subordination, the candidates for the semigroups generated by $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$ are the semigroups obtained by subordination with the Bernstein function $f(x) = x^\alpha$, $x > 0$, $0 < \alpha < 1$:

$$\begin{aligned} T_t^{(6)} g(x) = {} & (2\pi)^{-n/2} \int_0^{x_{n+1}} \int_{\mathbb{R}^n} g(x' - y', x_{n+1} - s) \mathcal{W}_s(y') \sigma_\alpha(s,t)\, dy'\, ds \\ & + (2\pi)^{-n/2} \int_{x_{n+1}}^{\infty} \int_{\mathbb{R}^n} h(x' - y') \mathcal{W}_{x_{n+1}}(y') \sigma_\alpha(s,t)\, dy'\, ds, \end{aligned} \tag{3.10}$$

which is obtained by subordination with $f(x)$ from $(T_t^{(1)})_{t\geq 0}$, and

$$T_t^{(6')}g(x) = (2\pi)^{-n/2}\int_0^{x_{n+1}}\int_{\mathbb{R}^n} g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')\sigma_\alpha(s,t)\,dy'\,ds + (2\pi)^{-n/2}\int_{x_{n+1}}^{\infty}\int_{\mathbb{R}^n} g(x'-y',0)\mathcal{W}_s(y')\sigma_\alpha(s,t)\,dy'\,ds, \tag{3.11}$$

which is obtained by subordination with $f(x)$ from $(T_t^{(1')})_{t\geq 0}$. These semigroups are again (by [J1, Theorem 4.3.1]) strongly continuous and contracting. A straightforward application of the Hille–Yosida theorem gives

THEOREM 3.6. *For* $1 < p < \infty$ *and* $1/p < \alpha < 1$, *the operator* $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$ *with* $\operatorname{symb}(-A_+) = \chi_+(\xi) = \psi(\xi') + i\xi_{n+1}$, *where* ψ *satisfies* (A1), (A2) *and* (3.1), *is the generator of the* L_p*-sub-Markovian semigroups* (3.11) *and* (3.12), *depending on the boundary conditions.*

The strongly continuous contraction semigroup $(T_t^{(6)})_{t\geq 0}$ corresponds to the boundary condition $\lambda f(x',0) = h(x')$, which can be shown by applying the Laplace transform to $T_t^{(6)}f(x)$.

Let us show that

$$\frac{\partial f}{\partial x_{n+1}}(x',0) = 0 \quad \text{for } f(x) = L_{t\to\lambda}T_t^{(6')}g(x).$$

We have

$$\begin{aligned} f(x) &= L_{t\to\lambda}T_t^{(6')}g(x) \\ &= \int_0^{x_{n+1}}\int_{\mathbb{R}^n} \frac{g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}}\,dy'\,ds \\ &\quad + \int_{x_{n+1}}^{\infty}\int_{\mathbb{R}^n} \frac{g(x'-y',0)\mathcal{W}_s(y')e'_\alpha(x_{n+1},\lambda)}{-\lambda(2\pi)^{n/2}}\,dy'\,ds. \end{aligned}$$

Differentiating with respect to x_{n+1}, we get

$$\frac{\partial}{\partial x_{n+1}}f(x) = \int_0^{x_{n+1}}\int_{\mathbb{R}^n} \frac{\frac{\partial}{\partial x_{n+1}}g(x'-y', x_{n+1}-s)\mathcal{W}_s(y')e'_\alpha(s,\lambda)}{-\lambda(2\pi)^{n/2}}\,dy'\,ds,$$

which tends to zero a.e. if $x_{n+1}\to 0$:

$$\frac{\partial}{\partial x_{n+1}}f(x',0) = 0 \quad \text{a.e.}$$

Thus, the operator $(-(-A_+)^\alpha, H_{p,+}^{\Re,2})$ with the zero Neumann boundary condition generates the semigroup $(T_t^{(6')})_{t\geq 0}$.

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V. M. Glushkov Institute of Cybernetics
National Academy of Sciences of Ukraine
40, Acad. V. M. Glushkov Ave.
03187 Kiev, Ukraine
E-mail: vic_knopova@yahoo.co.uk

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