COLLOQUIUM MATHEMATICUM

VOL. 101 2004 NO. 2

ON THE TRIVIAL EXTENSIONS OF TUBULAR ALGEBRAS

BY

JERZY BIAŁKOWSKI (Toruń)

Abstract. The aim of this note is to give an affirmative answer to a problem raised in [9] by J. Nehring and A. Skowroński, concerning the number of nonstable $\mathbb{P}_1(K)$-families of quasi-tubes in the Auslander–Reiten quivers of the trivial extensions of tubular algebras over algebraically closed fields K.

1. Introduction. Throughout, by an *algebra* we mean a basic connected, finite-dimensional associative K-algebra with an identity over a (fixed) algebraically closed field K. Any such algebra A can be written as a *bound quiver algebra*, that is, $A \cong KQ/I$, where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q. An algebra A is called *symmetric* if there is a nondegenerate symmetric K-bilinear form $(-,-) : A \times A \to K$ which is associative in the sense that $(ab,c) = (a,bc)$ for all $a,b,c \in A$. An important class of symmetric algebras is formed by the *trivial extensions* $\mathrm{T}(B)$ of algebras B by their minimal injective cogenerators $D(B) = \mathrm{Hom}_K(B,K)$. Recall that $\mathrm{T}(B)$ is the algebra whose K-linear structure is that of the K-vector space $B \oplus D(B)$, and whose multiplication is given by

$$(a,f)(b,g) = (ab, ag + fb)$$

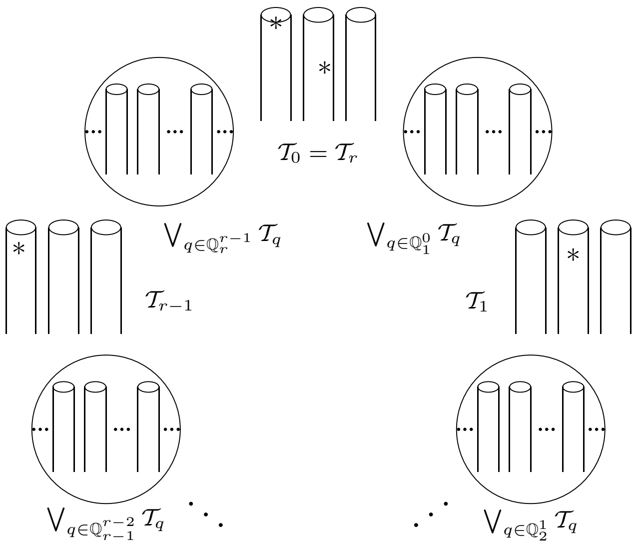
for $a,b \in B$ and $f,g \in {}_BD(B)_B$. We note that the Grothendieck groups $K_0(B)$ and $K_0(\mathrm{T}(B))$ are isomorphic. Recall also that, for a symmetric algebra A, the Auslander–Reiten operator $D\,\mathrm{Tr}$ coincides with the square Ω^2 of Heller's syzygy operator Ω, and hence the $D\,\mathrm{Tr}$-periodicity of modules coincides with their Ω-periodicity. Recently, the class of tame symmetric algebras with all nonprojective indecomposable finite-dimensional modules Ω-periodic has been classified by K. Erdmann and A. Skowroński [4]. In particular, it is shown in [4] that the class of tame symmetric algebras with singular Cartan matrices and all nonprojective indecomposable finite-dimensional modules Ω-periodic coincides with the class of trivial extensions

2000 *Mathematics Subject Classification*: 16D50, 16G20, 16G60, 16G70.

Key words and phrases: tubular algebra, trivial extension, Grothendieck group, reflection.

Supported by the Polish Scientific Grant KBN No. 1 P03A 018 27.

T(B) of tubular algebras B (in the sense of C. M. Ringel [11]). The representation theory of trivial extensions of tubular algebras has been established by J. Nehring and A. Skowroński in [9]. Moreover, it has been proved by Z. Pogorzały and A. Skowroński [10] that the class of trivial extensions of tubular algebras is closed under the stable (respectively, derived) equivalences of module categories. We know from [9] that the Auslander–Reiten quiver $\Gamma_{\mathrm{T}(B)}$ of the trivial extension T(B) of a tubular algebra B is of the form



where, for each $q \in \{0, 1, \ldots, r-1\}$, $\mathcal{T}_q$ is a nonstable $\mathbb{P}_1(K)$-family of quasi-tubes (in the sense of [13]), and for each $q \in \mathbb{Q}_{p+1}^{p} = \mathbb{Q} \cap (p, p+1)$, $0 \le p \le r-1$, $\mathcal{T}_q$ is a $\mathbb{P}_1(K)$-family of stable tubes. Moreover, the number $r = r(\mathrm{T}(B))$ of nonstable $\mathbb{P}_1(K)$-families of quasi-tubes in $\Gamma_{\mathrm{T}(B)}$ is at least 3 and at most the rank $\mathrm{rk}\, K_0(B)$ of the Grothendieck group $K_0(B)$ of B. Recall also that the *tubular algebras* are tubular extensions (equivalently, tubular coextensions) of tame concealed algebras [6] of tubular types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$, and their Grothendieck groups have respectively ranks $6, 8, 9, 10$ [11, Section 5]. Further, it has been proved by D. Happel and C.M. Ringel in [5] that $r(\mathrm{T}(C)) = 3$ for canonical tubular algebras C. In [9, Section 5] J. Nehring and A. Skowroński asked if there are tubular algebras B (necessarily of tubular type $(2, 3, 6)$) with $r(\mathrm{T}(B)) = 10$. The aim of this note is to give an affirmative answer to this problem. In fact, the main result of this note gives a complete answer to the following more general problem: when for a tubular algebra B do we have $r(\mathrm{T}(B)) = \mathrm{rk}\, K_0(B)$?

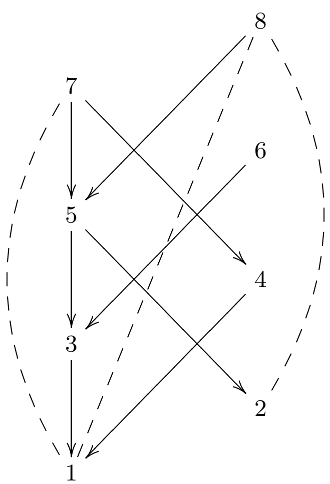
For basic background on the representation theory of algebras considered here we refer to [1], [11], [14], and on selfinjective algebras of tubular type to [2], [8], [9], [12].

2. The main result. Let B be an algebra and $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents of B such that $1 = e_1 + \cdots + e_n$. Denote by Q_B the (Gabriel) quiver of B with the set of vertices $\{1, \ldots, n\}$ corresponding to the set $e_1, \ldots, e_n$. For each vertex $i \in Q_B$, denote by $P_B(i)$ the indecomposable projective B-module e_iB and by $I_B(i)$ the indecomposable injective B-module $\mathrm{D}(Be_i)$. Then, for a sink $i \in Q_B$, the *reflection* S_i^+B *of* B *at* i is the quotient of the one-point extension $B[I_B(i)]$ by the two-sided ideal generated by e_i. The quiver $\sigma_i^+ Q_B$ of S_i^+B is called the *reflection of* Q_B *at* i. Observe that the sink i of Q_B is replaced in $\sigma_i^+ Q_B$ by a source i'. Moreover, we have

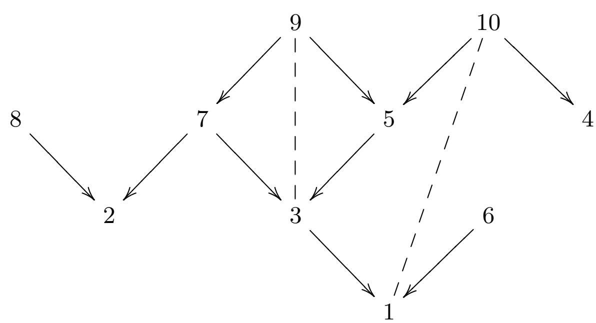
$$\mathrm{T}(B) \cong \mathrm{T}(S_i^+ B).$$

A *reflection sequence of sinks* is a sequence $i_1, \ldots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \cdots \sigma_{i_1}^+ Q_B$ for $1 \le s \le t$ (see [7, (2.8)]).

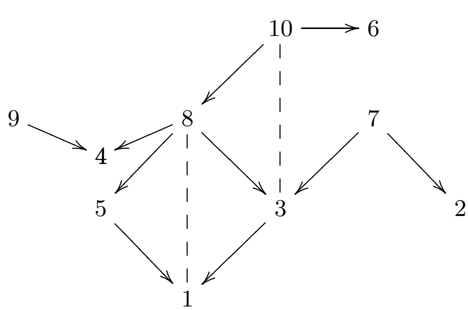
In order to state the main result, consider the following family of bound quiver algebras $B_i = KQ^{(i)}/I^{(i)}$, $1 \le i \le 10$ (where the dashed line means that the sum of the parallel paths indicated by this line is a generator of the ideal $I^{(i)}$):



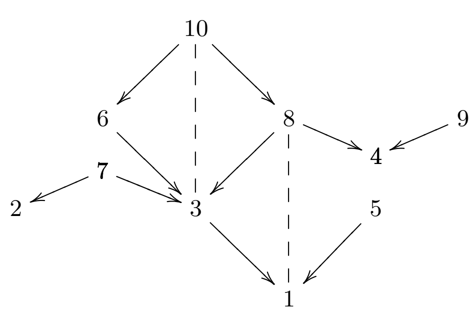
B_1



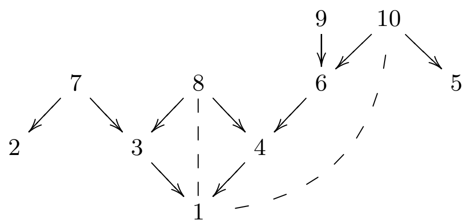
B_2



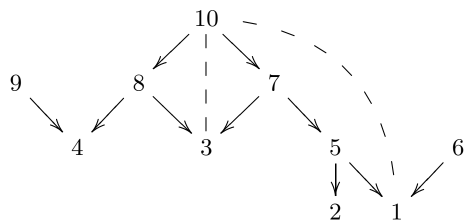
B_3



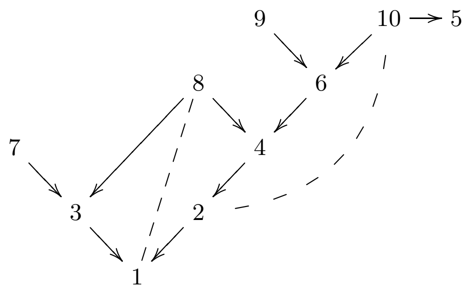
B_4



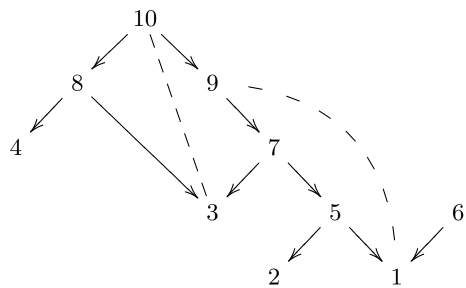
B_5



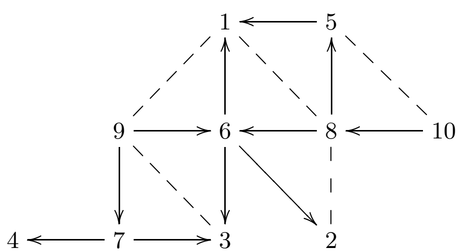
B_6



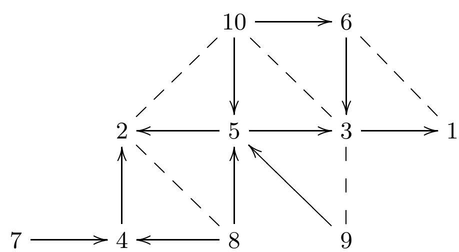
B_7



B_8



B_9



B_{10}

The following theorem is the main result of this note.

THEOREM 2.1. *Let B be a tubular algebra and n be the rank of $K_0(B)$. Then $r(\mathrm{T}(B)) = n$ if and only if B is isomorphic to an algebra of the form $S^+_{i_t} \cdots S^+_{i_1} B_j$ for some j with $1 \le j \le 10$ and a reflection sequence of sinks $i_1, \ldots, i_t$ of $Q_{B_j} = Q^{(j)}$.*

We point out that the iterated reflections $S^+_{i_t} \cdots S^+_{i_1} B_j$, $2 \le j \le 10$, $1 \le t \le 10$, are tubular algebras of type $(2,3,6)$ and give all solutions to the problem raised by J. Nehring and A. Skowroński.

We also obtain the following consequences of Theorem 2.1 and its proof.

COROLLARY 2.2. *The trivial extensions $\mathrm{T}(B_j)$, $1 \le j \le 10$, form a complete family of pairwise nonisomorphic trivial extensions $\mathrm{T}(B)$ of tubular algebras B with $r(\mathrm{T}(B)) = \mathrm{rk}\, K_0(B)$.*

COROLLARY 2.3. *Let B be a tubular algebra such that $r(\mathrm{T}(B)) = \mathrm{rk}\, K_0(B)$. Then B is a tubular extension of a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, or $\widetilde{\mathbb{E}}_8$.*

In the proof of Theorem 2.1 a crucial role is played by the following result, proved in [9, Section 4], describing the relationship between tubular algebras with isomorphic trivial extension algebras.

THEOREM 2.4. *Let B be a tubular algebra and n be the rank of $K_0(B)$. There is a sequence of natural numbers $1 \le t_1 < \cdots < t_r = n$ with $r \ge 3$, uniquely determined by B, and a reflection sequence of sinks $i_1, \ldots, i_{t_1}$, $i_{t_1+1}, \ldots, i_{t_{r-1}}$, $i_{t_{r-1}+1}, \ldots, i_{t_r}$ in Q_B such that:*

(a) $S^+_{i_{t_r}} \cdots S^+_{i_1} B \cong B$.

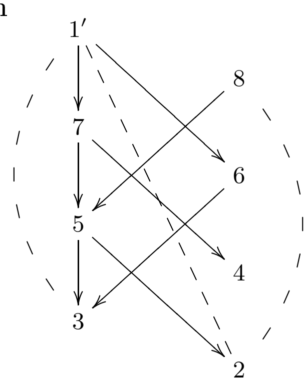
(b) $S^+_{i_{t_j}} \cdots S^+_{i_1} B$, $1 \le j \le r$, *are tubular algebras of the same tubular type as B.*

(c) *Every tubular algebra D with $\mathrm{T}(D) \cong \mathrm{T}(B)$ is isomorphic to $S^+_{i_{t_j}} \cdots S^+_{i_1} B$ for some $1 \le j \le r$.*

Moreover, we have $r = r(\mathrm{T}(B))$.

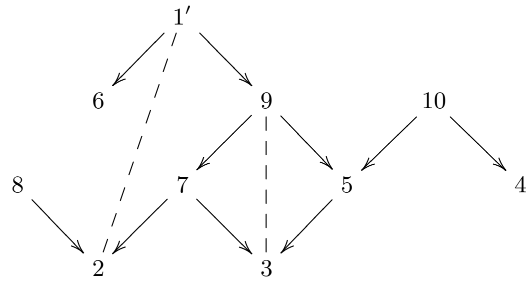
Therefore, in the notation of Theorem 2.4, we have $r(\mathrm{T}(B)) = n = \mathrm{rk}\, K_0(B)$ if and only if there is a reflection sequence of sinks $i_1, \ldots, i_n$ such that $S^+_{i_j} \cdots S^+_{i_1} B$, $1 \le j \le n$, are tubular algebras. We will show that the algebras $B_1, \ldots, B_{10}$ (listed above) are tubular algebras having this property. We will rely heavily on the Bongartz–Happel–Vossieck classification [3], [6] of tame concealed algebras by quivers and relations.

(1) The algebra B_1 is a tubular algebra of type $(3,3,3)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_6$ given by the vertices 1 to 7. Then $S^+_1 B_1$ is of the form

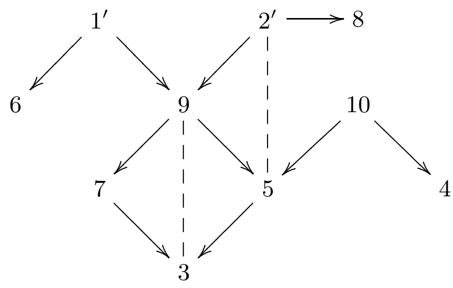


and hence is a tubular algebra of type $(3,3,3)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_6$ given by the vertices 2 to 8. Moreover, we have $S^+_1 B_1 \cong B_1^{\mathrm{op}}$ and $S^+_2 S^+_1 B_1 \cong B_1$ (see [12, Example 3.4]). In particular, the algebras $S_j^+ \cdots S_1^+ B_1$, $1 \le j \le 8$, are tubular algebras of type $(3,3,3)$, and tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_6$.

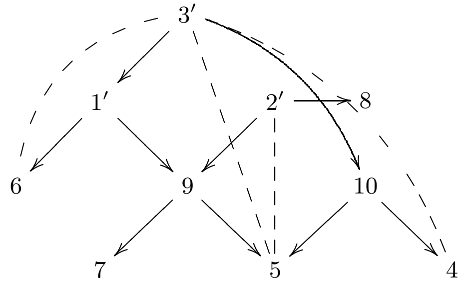
(2) The algebra B_2 is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_7$ given by all the vertices of Q_{B_2} except 4 and 10. Then $S_1^+ B_2$ is of the form



and hence is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_7$ given by all the vertices of $\sigma_1^+ Q_{B_2} = Q_{S_1^+ B_2}$ except $1'$ and 6. Observe also that $S_1^+ B_2 \cong B_2^{\mathrm{op}}$. The algebra $S_2^+ S_1^+ B_2$ is of the form

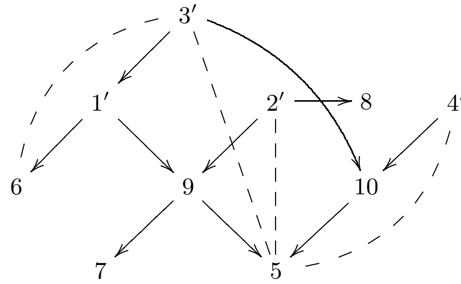


and hence is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_7$ given by all the vertices of $\sigma_2^+ \sigma_1^+ Q_{B_2} = Q_{S_2^+ S_1^+ B_2}$ except $2'$ and 8. The algebra $S_3^+ S_2^+ S_1^+ B_2$ is of the form

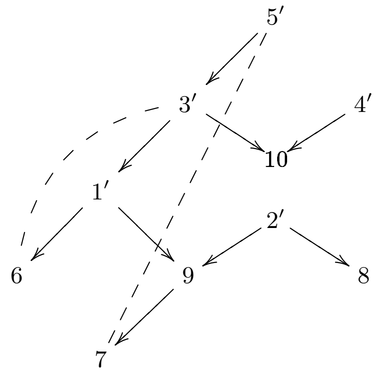


and hence is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_3^+ \sigma_2^+ \sigma_1^+ Q_{B_2} =$

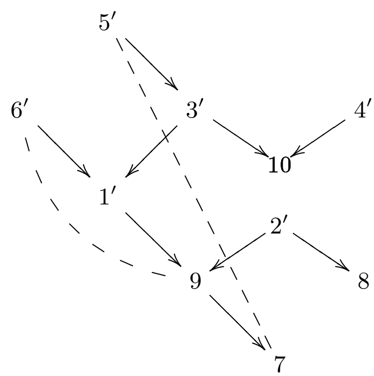
$Q_{S_3^+ S_2^+ S_1^+ B_2}$ except $3'$. The algebra $S_4^+ \cdots S_1^+ B_2$ is of the form



and hence is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_4^+ \cdots \sigma_1^+ Q_{B_2} = Q_{S_4^+ \cdots S_1^+ B_2}$ except $4'$. The algebra $S_5^+ \cdots S_1^+ B_2$ is of the form

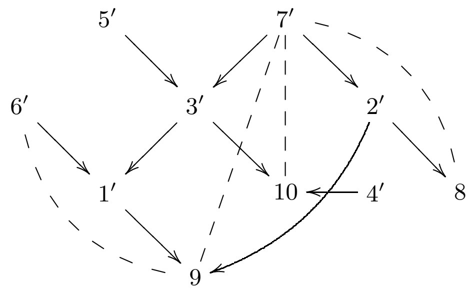


and hence is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_5^+ \cdots \sigma_1^+ Q_{B_2} = Q_{S_5^+ \cdots S_1^+ B_2}$ except $5'$. The algebra $S_6^+ \cdots S_1^+ B_2$ is of the form

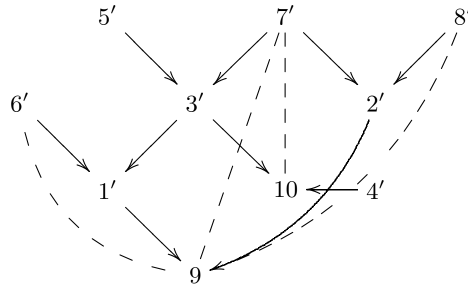


and hence is a tubular algebra of type $(2,3,6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_6^+ \cdots \sigma_1^+ Q_{B_2} =$

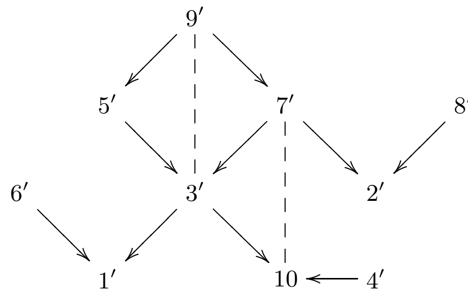
$Q_{S_6^+ \cdots S_1^+ B_2}$ except $6'$. The algebra $S_7^+ \cdots S_1^+ B_2$ is of the form



and hence is a tubular algebra of type $(2, 3, 6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_7^+ \cdots \sigma_1^+ Q_{B_2} = Q_{S_7^+ \cdots S_1^+ B_2}$ except $7'$. The algebra $S_8^+ \cdots S_1^+ B_2$ is of the form

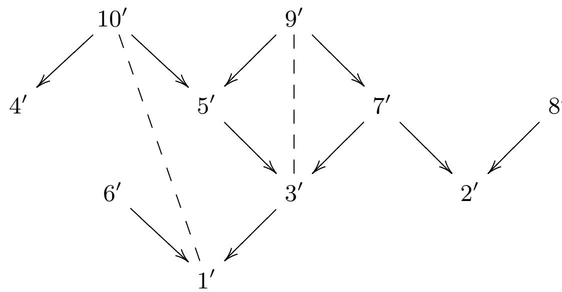


and hence is a tubular algebra of type $(2, 3, 6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_8^+ \cdots \sigma_1^+ Q_{B_2} = Q_{S_8^+ \cdots S_1^+ B_2}$ except $8'$. The algebra $S_9^+ \cdots S_1^+ B_2$ is of the form



and hence is a tubular algebra of type $(2, 3, 6)$ which is a tubular extension of the concealed algebra of type $\widetilde{\mathbb{E}}_8$ given by all the vertices of $\sigma_9^+ \cdots \sigma_1^+ Q_{B_2} =$

$Q_{S_9^+ \cdots S_1^+ B_2}$ except $9'$. Finally, $S_{10}^+ \cdots S_1^+ B_2$ is of the form



and hence is isomorphic to B_2.

Similarly, one proves that:

(3) B_3 and $S_i^+ \cdots S_1^+ B_3$ for $i \in \{4,7\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of tubular type $\widetilde{\mathbb{E}}_7$, while $S_i^+ \cdots S_1^+ B_3$ for $i \in \{1,2,3,5,6,8,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$.

(4) $S_i^+ \cdots S_1^+ B_4$ for $i \in \{1,4,7\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_7$, while B_4 and $S_i^+ \cdots S_1^+ B_4$ for $i \in \{2,3,5,6,8,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$. Moreover $B_4 \cong B_3^{\mathrm{op}}$.

(5) B_5 and $S_7^+ \cdots S_1^+ B_5$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_7$, while $S_i^+ \cdots S_1^+ B_5$ for $i \in \{1,\dots,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$.

(6) $S_i^+ \cdots S_1^+ B_6$, for $i \in \{1,4\}$, are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_7$, while B_6 and $S_i^+ \cdots S_1^+ B_6$ for $i \in \{2,3,5,6,7,8,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$. Moreover $B_6 \cong B_5^{\mathrm{op}}$.

(7) B_7 is a tubular algebra of type $(2,3,6)$ which is a tubular extension of a concealed algebra of type $\widetilde{\mathbb{E}}_7$, while $S_i^+ \cdots S_1^+ B_7$ for $i \in \{1,\dots,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$.

(8) $S_1^+ B_8$ is a tubular algebra of type $(2,3,6)$ which is a tubular extension of a concealed algebra of type $\widetilde{\mathbb{E}}_7$, while B_8 and $S_i^+ \cdots S_1^+ B_8$ for $i \in \{2,\dots,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$. Moreover $B_8 \cong B_7^{\mathrm{op}}$.

(9) B_9 and $S_i^+ \cdots S_1^+ B_9$ for $i \in \{1,\dots,9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$.

(10) B_{10} and $S_i^+ \cdots S_1^+ B_{10}$ for $i \in \{1, \ldots, 9\}$ are tubular algebras of type $(2,3,6)$ which are tubular extensions of concealed algebras of type $\widetilde{\mathbb{E}}_8$. Moreover $B_{10} \cong B_9^{\mathrm{op}}$.

This finishes the proof of the sufficiency part of Theorem 2.1. The necessity for tubular algebras of type $(2,2,2,2)$ follows from [12, (3.3)]. For tubular algebras of types $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$ this is done (see also [2, (5.1), (6.1)]) with the help of computer programs calculating:

- the lists of all tubular algebras of type $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$, using the Bongartz–Happel–Vossieck list [3], [6] of tame concealed algebras,
- the reflection equivalence classes of tubular algebras of types $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$.

For details concerning these calculations we refer to the author's home page (http://www.mat.uni.torun.pl/~jb/en/research/tubular/).

REFERENCES

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[2] J. Białkowski and A. Skowroński, *Selfinjective algebras of tubular type*, Colloq. Math. 94 (2002), 175–194.

[3] K. Bongartz, *Critical simply connected algebras*, Manuscripta Math. 46 (1984), 117–136.

[4] K. Erdmann and A. Skowroński, *Tame symmetric algebras with periodic modules*, in preparation.

[5] D. Happel and C. M. Ringel, *The derived category of a tubular algebra*, in: Representation Theory I, Finite Dimensional Algebras, Lecture Notes in Math. 1177, Springer, 1986, 156–180.

[6] D. Happel and D. Vossieck, *Minimal algebras of infinite representation type with preprojective component*, Manuscripta Math. 42 (1983), 221–243.

[7] D. Hughes and J. Waschbüsch, *Trivial extensions of tilted algebras*, Proc. London Math. Soc. 46 (1983), 347–364.

[8] H. Lenzing and A. Skowroński, *Roots of Nakayama and Auslander–Reiten translations*, Colloq. Math. 86 (2000), 209–230.

[9] J. Nehring and A. Skowroński, *Polynomial growth trivial extensions of simply connected algebras*, Fund. Math. 132 (1989), 117–134.

[10] Z. Pogorzały and A. Skowroński, *Symmetric algebras stably equivalent to the trivial extensions of tubular algebras*, J. Algebra 181 (1996), 95–111.

[11] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[12] A. Skowroński, *Selfinjective algebras of polynomial growth*, Math. Ann. 285 (1989), 177–199.

[13] —, *Algebras of polynomial growth*, in: Topics in Algebra, Banach Center Publ. 26, Part 1, PWN, Warszawa, 1990, 535–568.

[14] K. Yamagata, *Frobenius algebras*, in: Handbook of Algebra, Vol. 1, Elsevier, 1996, 841–887.

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: jb@mat.uni.torun.pl

Received 22 September 2004 (4507)