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HOPF'S RATIO ERGODIC THEOREM BY INDUCING

BY

ROLAND ZWEIMÜLLER (London)

Abstract. We present a very quick and easy proof of the classical Stepanov–Hopf ratio ergodic theorem, deriving it from Birkhoff's ergodic theorem by a simple inducing argument.

During the last few years, there has been some interest in short and easy proofs of (pointwise) ergodic theorems, naturally focussing on the most fundamental one, i.e. on Birkhoff's result for probability preserving transformations (see e.g. [KW], [Ke], [P], and [Sh]). In [KK] a similar proof of an important extension was given, which came shortly after the discovery of the first ergodic theorems ([N] and [B]; see [Z] for historical comments): the Stepanov–Hopf ratio ergodic theorem ([St], [H]), which is the proper version of the pointwise ergodic theorem for infinite measure preserving transformations (there is no way to get a.e. convergence for ergodic sums normalized by a sequence of constants; cf. [A, §2.4]). The aim of the present note is to point out that this result can also be derived as a direct consequence of Birkhoff's theorem, via a (very) simple inducing argument (which does not seem to be available or hinted at in the literature I know).

We are going to prove

THEOREM 1 (Hopf's ratio ergodic theorem). *Let T be a measure preserving transformation on the σ-finite measure space $(X,\mathcal{A},\mu)$. Let $f,g\in L_1(\mu)$ with $g\ge 0$ and $\int_X g\,d\mu>0$. Then there exists a measurable function $Q(f,g):X\to\mathbb{R}$ such that*

$$\frac{\mathbf{S}_n(f)}{\mathbf{S}_n(g)}=\frac{\sum_{k=0}^{n-1} f\circ T^k}{\sum_{k=0}^{n-1} g\circ T^k}\to Q(f,g)\qquad \textit{a.e. on } \{\sup_n \mathbf{S}_n(g)>0\} \textit{ as } n\to\infty.$$

On the conservative part the limit function $Q(f,g)$ is measurable with respect to the σ-algebra $\mathcal{I}\subseteq\mathcal{A}$ of T-invariant sets and satisfies

$$\int_I Q(f,g)\cdot g\,d\mu=\int_I f\,d\mu\qquad \textit{for all } I\in\mathcal{I}.$$

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In particular, if $g > 0$ a.e., then

$$Q(f,g) = \mathbb{E}_{\mu_g}\left[\frac{f}{g}\,\middle\|\,\mathcal{I}\right], \qquad \text{where } d\mu_g := g\,d\mu,$$

and if T is ergodic, then $Q(f,g) = \int_X f\,d\mu / \int_X g\,d\mu$ a.e.

Proof. For the dissipative part of T, where $\sum_{k=0}^{n-1} f \circ T^k < \infty$ a.e. for any $f \in L_1(\mu)$, the assertion is trivial. We can therefore assume without loss of generality that T is conservative. By linearity it is enough to consider nonnegative f.

a) To emphasize the simplicity of the argument, we first consider the special case of an ergodic map T. The main step will be to prove that for any $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$,

$$\text{(1)} \qquad \frac{\mathbf{S}_n(f)}{\mathbf{S}_n(1_Y)} \to \frac{\int_X f\,d\mu}{\mu(Y)} \quad \text{a.e. on } Y.$$

As the set $\{\mathbf{S}_n(f)/\mathbf{S}_n(1_Y) \to \int_X f\,d\mu/\mu(Y)\}$ is T-invariant, we then see that this convergence in fact holds a.e. on X. Applying the same to g yields the assertion of the theorem.

To verify (1), we consider the *first return* (or *induced*) *map* $T_Y : Y \to Y$ given by $T_Y x := T^{\varphi(x)}x$, where $\varphi(x) := \min\{n \geq 1 : T^n x \in Y\}$ is the *first return time* of Y (cf. [Ka]). According to basic classical results, T_Y is a measure preserving transformation on the finite measure space $(Y, \mathcal{A} \cap Y, \mu|_{\mathcal{A}\cap Y})$, ergodic since T is. Moreover, it is well known that μ can be reconstructed from $\mu|_{\mathcal{A}\cap Y}$ via

$$\text{(2)} \qquad \mu(E) = \sum_{j\geq 0} \mu(Y \cap \{\varphi > j\} \cap T^{-j}E) \quad \text{for } E \in \mathcal{A}.$$

In other words, $\int_X 1_E\,d\mu = \int_Y (\sum_{j=0}^{\varphi-1} 1_E \circ T^j)\,d\mu$. An obvious argument using linearity and monotone convergence shows that this extends from indicator functions 1_E to arbitrary measurable $f : X \to [0,\infty)$, i.e. that

$$\text{(3)} \qquad \int_X f\,d\mu = \int_Y f_Y\,d\mu,$$

with $f_Y : Y \to [0,\infty)$ defined by $f_Y := \sum_{j=0}^{\varphi-1} f \circ T^j = \sum_{j\geq 0} 1_{Y\cap\{\varphi>j\}}\,(f \circ T^j)$.

We can therefore apply Birkhoff's ergodic theorem to T_Y and f_Y, thus considering the ergodic sums $\mathbf{S}_m^Y(f_Y) := \sum_{k=0}^{m-1} f_Y \circ T_Y^k$, $m \geq 1$, to see that

$$\text{(4)} \qquad \frac{\mathbf{S}_m^Y(f_Y)}{m} \to \frac{\int_Y f_Y\,d\mu}{\mu(Y)} = \frac{\int_X f\,d\mu}{\mu(Y)} \quad \text{a.e. on } Y.$$

Let $\varphi_m := \mathbf{S}_m^Y(\varphi) = \sum_{k=0}^{m-1} \varphi \circ T_Y^k$, $m \geq 1$, denote the *mth return time* to Y. Then, on Y, $\mathbf{S}_n(1_Y) = m$ for $n \in \{\varphi_{m-1}+1, \ldots, \varphi_m\}$ and $\mathbf{S}_{\varphi_m}(f) = \mathbf{S}_m^Y(f_Y)$, so that

$$\frac{\mathbf{S}_m^Y(f_Y)}{m} = \frac{\mathbf{S}_{\varphi_m}(f)}{\mathbf{S}_{\varphi_m}(1_Y)} \quad \text{for } m \geq 1 \text{ a.e. on } Y,$$

showing that (4) is equivalent to (1) along the subsequence of indices $n = \varphi_m$, $m \geq 1$. To prove convergence of the full sequence, we need only observe that $\mathbf{S}_n(f)$ is nondecreasing in n since $f \geq 0$. Hence,

$$\frac{m-1}{m}\,\frac{\mathbf{S}_{m-1}^Y(f_Y)}{m-1} \leq \frac{\mathbf{S}_n(f)}{\mathbf{S}_n(1_Y)} \leq \frac{\mathbf{S}_m^Y(f_Y)}{m}$$

for $n \in \{\varphi_{m-1}+1, \ldots, \varphi_m\}$, $m \geq 1$, a.e. on Y, and (1) follows from (4).

b) If T is not necessarily ergodic, we first observe that as $\{\sup_n \mathbf{S}_n(g) > 0\}$ is invariant, we may assume without loss of generality that it equals X. Also, the set M on which $\mathbf{S}_n(f)/\mathbf{S}_n(g)$ does not converge to a function Q with the advertised properties belongs to $\mathcal{I}$. Due to σ-finiteness, every set of positive measure has a subset Y with $0 < \mu(Y) < \infty$, and we prove that $\mu(M) > 0$ is impossible by showing that on any such Y the desired convergence holds a.e.

Restricting our attention to the (smallest) invariant set generated by Y, we suppose without loss of generality that $X = \bigcup_{n \geq 0} T^{-n}Y$. By the general form of Birkhoff's theorem, $\mathbf{S}_m^Y(f_Y)/m \to \mathbb{E}_{\mu|\mathcal{A}\cap Y}[f_Y \,\|\, \mathcal{I}_Y]/\mu(Y)$ a.e. on Y, where $\mathcal{I}_Y$ is the σ-algebra of T_Y-invariant sets in $\mathcal{A} \cap Y$. It is a standard fact about first return maps that $\mathcal{I}_Y = \mathcal{I} \cap Y = \{I \cap Y : I \in \mathcal{I}\}$. By exactly the same argument as before we obtain the following parallel to (1):

$$\frac{\mathbf{S}_n(f)}{\mathbf{S}_n(1_Y)} \to \frac{\mathbb{E}_{\mu|\mathcal{A}\cap Y}[f_Y \,\|\, \mathcal{I}_Y]}{\mu(Y)} \quad \text{a.e. on } Y, \tag{5}$$

and analogously for g. Observe now that

$$\{\mathbb{E}_{\mu|\mathcal{A}\cap Y}[g_Y \,\|\, \mathcal{I}_Y] > 0\} = \{\sup_m \mathbf{S}_m^Y(g_Y) > 0\} = Y \cap \{\sup_n \mathbf{S}_n(g) > 0\}.$$

Exploiting T-invariance of lim inf and lim sup of the ratios $\mathbf{S}_n(f)/\mathbf{S}_n(g)$, we therefore conclude that their sequence converges a.e. on X to $Q = Q(f,g)$, the (unique) $\mathcal{I}$-measurable extension of $\mathbb{E}_{\mu|\mathcal{A}\cap Y}[f_Y \,\|\, \mathcal{I}_Y]/\mathbb{E}_{\mu|\mathcal{A}\cap Y}[g_Y \,\|\, \mathcal{I}_Y]$ to X.

It remains to verify the property $\int_I Q \cdot g\, d\mu = \int_I f\, d\mu$ for all $I \in \mathcal{I}$, which uniquely characterizes the T-invariant limit Q. To do so, we notice that by T-invariance, we have $(Q \cdot g)_Y = Q \cdot g_Y$, and hence, for any $I \in \mathcal{I}$,

$$\begin{aligned}\int_I Q \cdot g\, d\mu &= \int_{I\cap Y} (Q\cdot g)_Y\, d\mu = \int_{I\cap Y} Q \cdot g_Y\, d\mu = \int_{I\cap Y} Q \cdot \mathbb{E}_{\mu|\mathcal{A}\cap Y}[g_Y \,\|\, \mathcal{I}_Y]\, d\mu \\ &= \int_{I\cap Y} \mathbb{E}_{\mu|\mathcal{A}\cap Y}[f_Y \,\|\, \mathcal{I}_Y]\, d\mu = \int_{I\cap Y} f_Y\, d\mu = \int_I f\, d\mu,\end{aligned}$$

as required. ∎

REMARK 1. Assuming some familiarity with the *dual operator* $\widehat{T} : L_1(\mu) \to L_1(\mu)$, characterized by $\int_X \widehat{T} g \cdot f \, d\mu = \int_X g \cdot (f \circ T) \, d\mu$ for all $g \in L_1(\mu)$ and $f \in L_\infty(\mu)$, which extends to arbitrary measurable $f, g \geq 0$ in an obvious way, we can avoid the approximation argument used to derive the important relation (3): Observe that (2) becomes $\mu(E) = \int_E \sum_{j \geq 0} \widehat{T}^j 1_{Y \cap \{\varphi > j\}} \, d\mu$ for all $E \in \mathcal{A}$, meaning that $\sum_{j \geq 0} \widehat{T}^j 1_{Y \cap \{\varphi > j\}} = 1$ a.e. on X. This immediately implies (3) via duality:

$$\int_Y f_Y \, d\mu = \int_X \sum_{j \geq 0} 1_{Y \cap \{\varphi > j\}} \cdot (f \circ T^j) \, d\mu = \int_X \sum_{j \geq 0} \widehat{T}^j 1_{Y \cap \{\varphi > j\}} \cdot f \, d\mu = \int_X f \, d\mu.$$

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Mathematics Department
Imperial College London
180 Queen's Gate
London SW7 2AZ, UK
E-mail: r.zweimueller@imperial.ac.uk

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