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ENDOMORPHISM RINGS OF MAXIMAL RIGID OBJECTS IN CLUSTER TUBES

BY

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Abstract. We describe the endomorphism rings of maximal rigid objects in the cluster categories of tubes. Moreover, we show that they are gentle and have Gorenstein dimension 1. We analyse their representation theory and prove that they are of finite type. Finally, we study the relationship between the module category and the cluster tube via the Hom-functor.

Introduction. Cluster categories were defined in [BMRRT] as tools for categorification of Fomin–Zelevinsky cluster algebras [FZ]. They are defined as the orbit categories of the derived category $D^b(\mathcal{H})$ of hereditary abelian categories $\mathcal{H}$ by a certain autoequivalence.

In the situation where $\mathcal{H}$ is the category of finite-dimensional representations of a finite acyclic quiver, the cluster category has been subject to intense investigation. In this case it has been shown that the cluster category and the set of exceptional objects form a good model for the cluster algebra associated with the same quiver.

In this paper we work with a cluster category $\mathcal{C}_n$ defined from a different hereditary abelian category, namely the tube $\mathcal{T}_n$. This category is called the cluster tube and has recently been studied in [BKL1, BKL2] and [BMV]. Although this category is also a Hom-finite triangulated 2-Calabi–Yau category, it does not enjoy all of the nice properties of cluster categories from quivers. In particular, the maximal rigid (also called maximal exceptional) objects do not satisfy the more restrictive definition of cluster-tilting objects.

Moreover, the Gabriel quivers of the endomorphism rings of maximal rigid objects in the cluster tube have loops. Consequently, $\mathcal{C}_n$ with its maximal rigid objects does not carry a cluster structure in the sense of [BIRS]. The axioms for cluster structures can be modified, however, to apply also to cluster tubes (see [BMV]).

The aim of the present paper is to study the endomorphism rings of the maximal rigid objects. We will find a description in terms of quivers

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with relations. Like cluster-tilted algebras, the algebras we consider here are Gorenstein of Gorenstein dimension 1, unless $n = 2$, in which case they are self-injective. However, the proof (from [KR]) for cluster-tilted algebras has no analogy in our setting. Instead, we use the fact that our algebras are gentle, and apply the technique from [GR] to our quivers with relations in order to prove the result. The properties of the algebras we study in this paper are thus reminiscent of those of the algebras recently studied in [ABCP].

Since the endomorphism rings are gentle, they are string algebras. We use the theory of string- and band-modules to show that the endomorphism rings are of finite type. One of the main results about cluster-tilted algebras, which was proved in [BMR1], is the close connection between the module category of the cluster-tilted algebra and the cluster category it arises from. This connection is provided by the Hom-functor. In our situation, the Hom-functor is not full, and therefore there is no analogous theorem. We will nevertheless study the action of the Hom-functor on the objects, and in particular show that it is dense. Indeed, when T is maximal rigid, we find an explicit description of $\mathrm{Hom}_{\mathcal{C}_n}(T, X)$ for every indecomposable X in $\mathcal{C}_n$.

The paper is organised as follows: Section 1 contains the definition of the cluster tube and a description of maximal rigid objects recalled from [BMV]. In Section 2 we give a description of the endomorphism rings, while in Section 3 we study the gentleness and Gorenstein dimension and give some facts about indecomposable representations. Finally, in Section 4 we describe the action of the Hom-functor.

Throughout the paper we will work over some field k, which is assumed to be algebraically closed. Modules over an algebra will always mean left modules, and we will read paths in quivers from right to left.

1. Maximal rigid objects in cluster tubes. We start off by reviewing some properties of cluster tubes. These categories have recently been studied in [BKL1, BKL2] and [BMV], and more details can be found in these papers.

For any integer $n \geq 2$, let $\mathcal{T}_n$ be a *tube of rank* n, that is, the category of nilpotent representations of a cyclically oriented $\tilde{A}_{n-1}$-quiver. It can also be realised as the thick subcategory generated by a tube in the regular part in the AR-quiver of a suitable tame hereditary algebra. All maps in this category are linear combinations of finite compositions of irreducible maps, and are subject to mesh relations in the AR-quiver.

The tube $\mathcal{T}_n$ is a hereditary abelian category, and following the construction introduced in [BMRRT], we form its cluster category, called the *cluster tube of rank* n and denoted $\mathcal{C}_n$. This is by definition the orbit category obtained from the bounded derived category $\mathcal{D}_n = D^b(\mathcal{T}_n)$ by the action of the self-equivalence $\tau^{-1} \circ [1]$. Here, $[1]$ denotes the suspension functor of $\mathcal{D}_n$, while τ is the Auslander–Reiten translation. Unless the actual value of n is important, we will usually suppress the subscript n in notation, and write $\mathcal{T}$, $\mathcal{D}$ and $\mathcal{C}$.

For a finite-dimensional hereditary algebra H, a theorem due to Keller [K] guarantees that the associated cluster category $\mathcal{C}_H$ is triangulated with a canonical triangulated structure inherited from the derived category. Keller's result is not directly applicable in our situation, since $\mathcal{T}$ has no tilting objects. Nevertheless, $\mathcal{C}_n$ also inherits a triangulated structure from $\mathcal{D}_n$ (see [BKL1] for a rigorous treatment of this).

The indecomposable objects of the cluster tube $\mathcal{C}$ are in bijection with the indecomposables in $\mathcal{T}$ itself, and we will sometimes use the same symbol to denote both an object in the tube $\mathcal{T}$ and its image in the cluster tube $\mathcal{C}$. The irreducible maps in $\mathcal{C}$ are the images of the irreducible maps in $\mathcal{D}$, which again are the shifts of the irreducible maps in the tube $\mathcal{T}$. So the AR-quiver of $\mathcal{C}$ is isomorphic to the AR-quiver of $\mathcal{T}$.

For a given rank n, we will use a coordinate system on the indecomposable objects. Choose once and for all a quasisimple object and give it coordinates $(1,1)$. Now give the other quasisimples coordinates $(q,1)$ such that $\tau(q,1) = (q-1,1)$, where q is reduced modulo the rank n. Then give the remaining indecomposables coordinates (a,b) in such a way that there are irreducible morphisms $(a,b) \to (a,b+1)$ for $b \geq 1$ and $(a,b) \to (a+1,b-1)$ for $b \geq 2$. Throughout, the first coordinates will be reduced modulo n. See Figure 1.

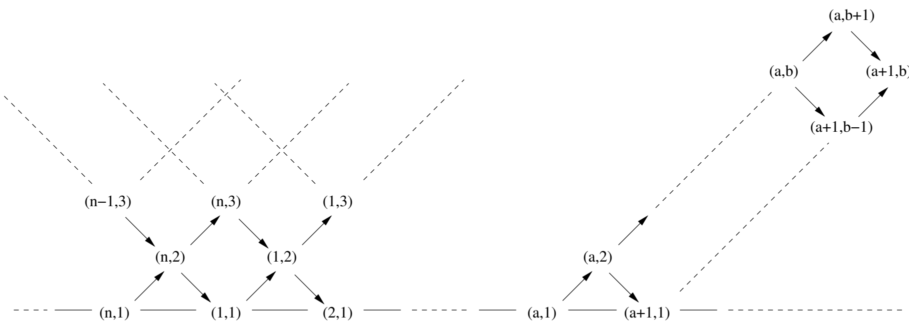


Fig. 1. AR-quiver and coordinate system for $\mathcal{T}_n$ and $\mathcal{C}_n$

The infinite sequence of irreducible maps

$$\mathbf{R}_{(a,i)} = (a,i) \to (a,i+1) \to \cdots \to (a,i+j) \to \cdots$$

is called a *ray*. Similarly, the infinite sequence

$$\mathbf{C}_{(a,i)} = \cdots \to (a-b,i+b) \to \cdots \to (a-1,i+1) \to (a,i)$$

is called a *coray*. Note that the sum of the coordinates is constant, modulo n, for indecomposables located on the same coray.

For an indecomposable object $X = (a, i)$ where $i < n$ we will also need the notion of the *wing* $\mathcal{W}_X$ determined by X. This is by definition the set of indecomposables whose position in the tube is in the triangle with X on top. We will call X the *summit* of $\mathcal{W}_X$. In terms of coordinates, objects in the wing $\mathcal{W}_{(a,i)}$ are (a', i') such that $a' \geq a$ and $a' + i' \leq a + i$. The *height* of $\mathcal{W}_X$ is the quasilength $\operatorname{ql} X$.

Hom-spaces in $\mathcal{C}$ are given by the following lemma, proved in [BMV].

Lemma 1.1. *For X and Y indecomposable in $\mathcal{C}$, we have*

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \simeq \operatorname{Hom}_{\mathcal{T}}(X, Y) \amalg D \operatorname{Hom}_{\mathcal{T}}(Y, \tau^2 X)$$

where D is the usual k-vector space duality $\operatorname{Hom}_k(-, k)$.

When X and Y are indecomposable, the maps in $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ which are images of maps in $\operatorname{Hom}_{\mathcal{D}}(\hat{X}, \tau^{-1}\hat{Y}[1])$ for $\hat{X}, \hat{Y}$ in $\mathcal{T}$ will be called $\mathcal{D}$-*maps*, and those which are images of maps in $\mathcal{T}$ itself will be called $\mathcal{T}$-*maps*. Since $\mathcal{T}$ is hereditary, all maps in $\mathcal{C}$ are linear combinations of maps of these two kinds. The Hom-*hammock* of an indecomposable object (that is, the support of $\operatorname{Hom}_{\mathcal{C}}(X, -)$) is illustrated in Figure 2. Note that the two components in the figure wrap around the tube and intersect. Moreover, if $b \geq n+1$, then each component intersects itself, possibly with several layers, and therefore there exist Hom-spaces of arbitrary finite dimension between indecomposables.

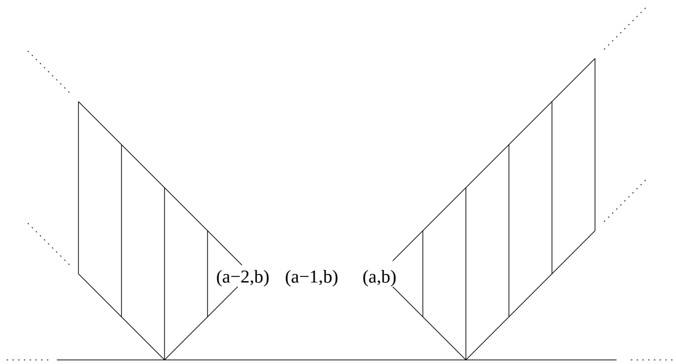


Fig. 2. The Hom-hammock of (a, b). There are $\mathcal{T}$-maps to indecomposables in the right component, and $\mathcal{D}$-maps to indecomposables in the left component.

So for indecomposable X and Y, the existence of a $\mathcal{D}$-map $X \to Y$ is equivalent to the existence of a $\mathcal{T}$-map $Y \to \tau^2 X$. The following lemma is then easily verified:

Lemma 1.2. *Let X be an indecomposable object of $\mathcal{C}_n$. Then there exists a $\mathcal{D}$-endomorphism of X if and only if* $\operatorname{ql} X \geq n - 1$.

We will need the following lemma on the relationship between $\mathcal{T}$-maps and $\mathcal{D}$-maps:

Lemma 1.3. *For X, Y and Z indecomposable objects in $\mathcal{C}_n$, we have the following:*

(i) *Assume that $\operatorname{ql} X \leq n$ and $\operatorname{ql} Y \leq n$. If there are non-zero $\mathcal{D}$-maps $\psi_{XZ} : X \to Z$ and $\psi_{YZ} : Y \to Z$, and an irreducible map $i_{XY} : X \to Y$, then $\psi_{YZ} \circ i_{XY} = \psi_{XZ}$ up to multiplication by a non-zero scalar.*

(ii) *Assume that $\operatorname{ql} X \leq n$. If there are non-zero $\mathcal{D}$-maps $\psi_{XY} : X \to Y$ and $\psi_{XZ} : X \to Z$, and an irreducible map $i_{YZ} : Y \to Z$, then $\psi_{XZ} = i_{YZ} \circ \psi_{XY}$, up to multiplication by a non-zero scalar.*

Remark 1.4. Note that by repeated application, the same applies to compositions of irreducible maps, i.e. to all $\mathcal{T}$-maps, under the assumption that the required Hom-spaces are non-zero for each indecomposable that the composition factors through.

Proof of Lemma 1.3. (i) We lift the maps to the derived category $\mathcal{D}$, and denote by $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ the preimages of the objects in $\mathcal{T}$. Since X and Y have quasilength $\leq n$, the space $\operatorname{Hom}_{\mathcal{D}}(\hat{X}, \tau^{-1}\hat{A}[1])$ of $\mathcal{D}$-maps is at most one-dimensional for any indecomposable $\hat{A} \in \mathcal{T}$, and similarly for Y.

The aim is to show that the map

$$\operatorname{Hom}_{\mathcal{D}}(i_{XY}, \tau^{-1}\hat{Z}[1]) : \operatorname{Hom}_{\mathcal{D}}(\hat{Y}, \tau^{-1}\hat{Z}[1]) \to \operatorname{Hom}_{\mathcal{D}}(\hat{X}, \tau^{-1}\hat{Z}[1])$$

is surjective. We can view this as a map

$$i_{XY}^{*} : \operatorname{Ext}^1_{\mathcal{T}}(\hat{Y}, \tau^{-1}\hat{Z}) \to \operatorname{Ext}^1_{\mathcal{T}}(\hat{X}, \tau^{-1}\hat{Z})$$

or, by duality and the AR-formula,

$$\operatorname{Hom}_{\mathcal{T}}(\tau^{-1}\hat{Z}, \tau i_{XY}) : \operatorname{Hom}_{\mathcal{T}}(\tau^{-1}\hat{Z}, \tau\hat{X}) \to \operatorname{Hom}_{\mathcal{T}}(\tau^{-1}\hat{Z}, \tau\hat{Y}),$$

which we now wish to show is injective. But this is clear from the structure of the tube when the Hom-spaces are non-zero.

(ii) We need to show that the map

$$\operatorname{Hom}_{\mathcal{D}}(\hat{X}, \tau^{-1}i_{YZ}[1]) : \operatorname{Hom}_{\mathcal{D}}(\hat{X}, \tau^{-1}\hat{Y}[1]) \to \operatorname{Hom}_{\mathcal{D}}(\hat{X}, \tau^{-1}\hat{Z}[1])$$

is surjective. As above, by duality this is equivalent to the map

$$\operatorname{Hom}_{\mathcal{T}}(\tau^{-1}i_{YZ}, \tau\hat{X}) : \operatorname{Hom}_{\mathcal{T}}(\tau^{-1}\hat{Z}, \tau\hat{X}) \to \operatorname{Hom}_{\mathcal{T}}(\tau^{-1}\hat{Y}, \tau\hat{X})$$

being injective. But by the combinatorics of the tube, this is clearly an isomorphism, since by assumption both spaces are 1-dimensional. ∎

Let $1 \leq h \leq n-1$, and choose some indecomposable X in $\mathcal{T}_n$ with quasilength $\operatorname{ql} X = h$. Let $\vec{A}_h$ be a linearly oriented quiver with underlying graph the Dynkin diagram A_h. Then the category $\operatorname{mod} k\vec{A}_h$ of finitely generated modules over the path algebra $k\vec{A}_h$ is naturally equivalent to the subcategory $\operatorname{add}_{\mathcal{T}} \mathcal{W}_X$ of $\mathcal{T}_n$. Embedding into $\mathcal{D}_n$ and projecting to $\mathcal{C}_n$ we find that $\operatorname{mod} k\vec{A}_h$ embeds into the subcategory $\operatorname{add}_{\mathcal{C}} \mathcal{W}_X$ of $\mathcal{C}_n$. The image is the subcategory $\operatorname{add}_{\mathcal{C}}^{\mathcal{T}} \mathcal{W}_X$ obtained by deleting the $\mathcal{D}$-maps from $\operatorname{add}_{\mathcal{C}} \mathcal{W}_X$. From now on, we will drop the subscript when we speak of an additive hull as a set of objects, since there is a bijection between the objects of $\mathcal{T}$ and those of $\mathcal{C}$.

The triangulated category $\mathcal{C}$ is a 2-Calabi–Yau category, which in particular means that for any two objects X and Y, we have symmetric Ext-spaces:

$$\operatorname{Ext}^1_{\mathcal{C}}(X,Y) \simeq D\operatorname{Ext}^1_{\mathcal{C}}(Y,X).$$

Two indecomposable objects X and Y will be called *compatible* if $\operatorname{Ext}^1_{\mathcal{C}}(X,Y) = \operatorname{Ext}^1_{\mathcal{C}}(Y,X) = 0$. It is worth noticing that X and Y are compatible if and only if $\operatorname{Ext}^1_{\mathcal{T}}(X,Y) = \operatorname{Ext}^1_{\mathcal{T}}(Y,X) = 0$.

In an abelian or triangulated category $\mathcal{K}$, an object T is called *rigid* if it satisfies $\operatorname{Ext}^1_{\mathcal{K}}(T,T) = 0$. If it is maximal with respect to this property, that is, if $\operatorname{Ext}^1_{\mathcal{K}}(T \amalg X, T \amalg X) = 0$ implies that $X \in \operatorname{add} T$, then it is called *maximal rigid*. The maximal rigid objects in the cluster tube $\mathcal{C}$ do not satisfy the stronger condition of *cluster tilting* (see [BMV]).

The following description of the maximal rigid objects was given in [BMV]:

Proposition 1.5. *There is a natural bijection between the set of maximal rigid objects in $\mathcal{C}_n$ and the set*

$$\{\textit{tilting modules of } k\vec{A}_{n-1}\} \times \{1,\dots,n\},$$

where $\vec{A}_{n-1}$ is a linearly oriented quiver with the Dynkin diagram A_{n-1} as its underlying graph.

The proposition is a consequence of the following considerations, which will be needed for the rest of the paper: All summands of a maximal rigid object in $\mathcal{C}_n$ are concentrated in the wing $\mathcal{W}_{T_1}$ determined by a *top summand* T_1 with $\operatorname{ql} T_1 = n-1$. Now the claim follows from the embedding of $\operatorname{mod} k\vec{A}_{n-1}$ into $\operatorname{add}_{\mathcal{C}} \mathcal{W}_{T_1}$, since it is easily seen that $\operatorname{Ext}^1_{\mathcal{C}_n}(X,Y)$ for two indecomposables X and Y in $\mathcal{W}_{T_1}$ vanishes if and only if both $\operatorname{Ext}^1_{k\vec{A}_{n-1}}(\widetilde{X},\widetilde{Y})$ and $\operatorname{Ext}^1_{k\vec{A}_{n-1}}(\widetilde{Y},\widetilde{X})$ vanish, where $\widetilde{X}$ and $\widetilde{Y}$ are the corresponding $k\vec{A}_{n-1}$-modules. Since there are n choices for the top summand, this provides the bijection.

2. The endomorphism rings. With the description of the maximal rigid objects of $\mathcal{C}$ presented in Section 1, we now proceed to determine their endomorphism rings in terms of quivers and relations.

Let T be a maximal rigid object in the cluster tube $\mathcal{C}_n$, and let M_T denote the tilting $k\vec{A}_{n-1}$-module associated with T according to Proposition 1.5. Since the module category of a hereditary algebra H sits naturally embedded in the cluster category $\mathcal{C}_H$, we can think of the module M_T as a cluster-tilting object in $\mathcal{C}_{A_{n-1}}$. The endomorphism ring, or cluster-tilted algebra, $\widetilde{\Gamma}_T = \mathrm{End}_{\mathcal{C}_{A_{n-1}}}(M_T)^{\mathrm{op}}$ can easily be found from the tilted algebra $\Gamma_T = \mathrm{End}_{k\vec{A}_{n-1}}(M_T)^{\mathrm{op}}$, by the results in [BRe], or more generally in [ABS].

Every minimal relation on the quiver of a tilted algebra of type A is a zero relation of length two. The quiver of the cluster-tilted algebra is then obtained by inserting an arrow α_ρ from the end vertex to the start vertex of each defining relation path ρ. The relations for the cluster-tilted algebra are, as prescribed by [BMR2], the compositions of any two arrows in any of the 3-cycles formed by adding the new arrows.

We can now formulate the main theorem of this section.

Theorem 2.1. *Let T be a maximal rigid object in $\mathcal{C}_n$. Then the endomorphism ring $\Lambda_T = \mathrm{End}_{\mathcal{C}_n}(T)^{\mathrm{op}}$ is isomorphic to the algebra kQ/I where*

(a) *Q is the quiver obtained from the quiver of $\widetilde{\Gamma}_T$ by adjoining a loop ω to the vertex corresponding to the projective-injective $k\vec{A}_{n-1}$-module;*
(b) *I is the ideal generated by the relations in $\widetilde{\Gamma}_T$ and in addition ω^2.*

Before we can present the proof of the theorem, we need some considerations on the combinatorial structure of maximal rigid objects.

We define a *non-degenerate subwing triple* $(X;Y,Z)$ to be a triple X,Y,Z of indecomposables in $\mathcal{C}$ with $3 \le \mathrm{ql}\, X \le n-1$ such that if $X=(a,b)$, then $Y=(a,c)$ and $Z=(a+c+1,b-c-1)$ for some $1 \le c \le b-2$. This means that X is on the ray $\mathbf{R}_Y$ and on the coray $\mathbf{C}_Z$, so in particular $\mathcal{W}_Y$ and $\mathcal{W}_Z$ are contained in $\mathcal{W}_X$. Moreover, there is exactly one quasisimple $(a+c,1)$ which is in $\mathcal{W}_X$ but not in $\mathcal{W}_Y \cup \mathcal{W}_Z$. See Figure 3. A *degenerate subwing triple* $(X;Y,Z)$ is a triple with $2 \le \mathrm{ql}\, X \le n-1$ such that if $X=(a,b)$, then either $Y=(a,b-1)$ and $Z=0$ or $Y=0$ and $Z=(a+1,b-1)$. Note that any subwing triple (degenerate or non-degenerate) is determined by the top indecomposable X and the unique quasisimple which is not in any of the two subwings $\mathcal{W}_Y$ or $\mathcal{W}_Z$.

Lemma 2.2. *Let $(X;Y,Z)$ be a non-degenerate subwing triple. Let $Y' \in \mathcal{W}_Y$ and $Z' \in \mathcal{W}_Z$.*

(i) *There are no $\mathcal{T}$-maps $Z' \to Y'$.*
(ii) *There are no $\mathcal{T}$-maps $Y' \to Z'$.*
(iii) *There is a $\mathcal{D}$-map $Z' \to Y'$ if and only if Z' is on the left edge of $\mathcal{W}_Z$ and Y' is on the right edge of $\mathcal{W}_Z$. In this case, this map factors through the $\mathcal{D}$-map $Z \to Y$.*

(iv) *There is a $\mathcal{D}$-map $Y' \to Z'$ if and only if Z' is on the right edge of $\mathcal{W}_Z$ and Y' is on the left edge of $\mathcal{W}_Y$, and* $\mathrm{ql}\, X = n-1$. *In this case, this map factors through the $\mathcal{D}$-endomorphism of X.*

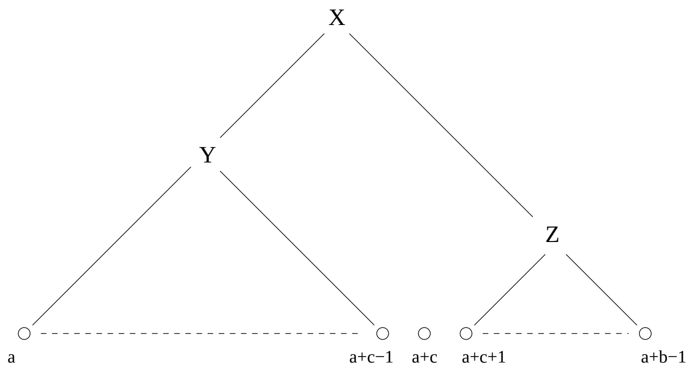


Fig. 3. Non-degenerate subwing triple $(X; Y, Z)$. If $X = (a, b)$ and $Y = (a, c)$ with $1 \leq c \leq b-2$, then $Z = (a+c+1, b-c-1)$.

Proof. Claims (i) and (ii) are easily verified; one must keep in mind that $\mathrm{ql}\, X \leq n-1$ by the definition of subwing triples.

Since the existence of a $\mathcal{D}$-map $Z' \to Y'$ is equivalent to the existence of a $\mathcal{T}$-map $Y' \to \tau^2 Z'$, we see that the only way such a map can arise is when Z' is on the left edge of $\mathcal{W}_Z$, and Y' is on the right edge of $\mathcal{W}_Y$. Now by Lemma 1.3 and Remark 1.4, this $\mathcal{D}$-map factors through the ray $\mathbf{R}_{Z'}$. In particular, it factors as $Z' \xrightarrow{\phi_{Z'Z}} Z \xrightarrow{\psi_{ZY'}} Y'$ where $\phi_{Z'Z}$ is the $\mathcal{T}$-map from Z' to Z, and $\psi_{ZY'}$ is the unique (up to multiplication with scalars) $\mathcal{D}$-map $Z \to Y'$. Applying Lemma 1.3 to $\psi_{ZY'}$, we find that it factors as $Z \xrightarrow{\psi_{ZY}} Y \xrightarrow{\phi_{YY'}} Y'$, where ψ_{ZY} is the $\mathcal{D}$-map from Z to Y and $\phi_{YY'}$ is the $\mathcal{T}$-map from Y to Y'. So claim (iii) holds.

For claim (iv), observe that since $\mathrm{ql}\, X \leq n-1$, a necessary condition for the existence of a $\mathcal{T}$-map $Z' \to \tau^2 Y'$ is that $\mathrm{ql}\, X = n-1$. Moreover, Z' must be on the right edge of $\mathcal{W}_X$ and Y' must be on the left edge of $\mathcal{W}_X$. Now the claim is proved using a similar argument to that for (iii) and the fact from Lemma 1.2 that if $\mathrm{ql}\, X = n-1$ then X has a $\mathcal{D}$-endomorphism. ∎

LEMMA 2.3. *Let $(X; Y, Z)$ be a subwing triple, and let $W \in \mathcal{W}_X$.*

(i) *Y and W are compatible if and only if $W \in \mathcal{W}_Y \cup \mathcal{W}_Z$ or $W \in \mathbf{R}_Y$.*
(ii) *Z and W are compatible if and only if $W \in \mathcal{W}_Y \cup \mathcal{W}_Z$ or $W \in \mathbf{C}_Z$.*

Proof. By the 2-Calabi–Yau property, we have symmetric Ext-groups, so it is enough to check vanishing of $\mathrm{Ext}^1_{\mathcal{C}}(W, Y)$ and $\mathrm{Ext}^1_{\mathcal{C}}(W, Z)$. For this, we use the AR-formula

$$\operatorname{Ext}^1_{\mathcal{C}}(W,Y) \simeq D\operatorname{Hom}_{\mathcal{C}}(Y,\tau W)$$

and similarly for Z. Then consider the intersection of the Hom-hammock of Y with $\tau\mathcal{W}_X$. ∎

Lemma 2.4. *Let $\mathcal{W} = \mathcal{W}_X$ be a wing in $\mathcal{C}_n$ of height $h < n$, and let $\mathcal{X}$ be a set of pairwise compatible indecomposable objects in $\mathcal{W}$.*

(i) *$\mathcal{X}$ has at most h elements.*
(ii) *If $\mathcal{X}$ has less than h elements, there exists a set $\widetilde{\mathcal{X}}$ of h pairwise compatible indecomposable objects, containing $\mathcal{X}$.*
(iii) *If $\mathcal{X}$ has h elements, then X is an element of $\mathcal{X}$.*

Proof. The argument is essentially the same as for Proposition 1.5, and the result follows from the theory of tilting modules applied to $k\vec{A}_h$-modules, upon noting that the projective-injective indecomposable is a summand of every tilting module. ∎

Lemma 2.5. *Let T_k be some indecomposable summand of a maximal rigid object $T = \coprod_{i=1}^{n-1} T_i$ in $\mathcal{C}_n$.*

(i) *There are $\operatorname{ql} T_k$ indecomposable summands of T in $\mathcal{W}_{T_k}$.*
(ii) *Assume $\operatorname{ql} T_k > 1$. Then there is a subwing triple $(T_k; T_{k'}, T_{k''})$ such that $T_{k'}$ and $T_{k''}$ are either summands of T or zero, and all summands of T/T_k which are in $\mathcal{W}_{T_k}$ are in $\mathcal{W}_{T_{k'}} \cup \mathcal{W}_{T_{k''}}$, with $\operatorname{ql} T_{k'}$ summands in $\mathcal{W}_{T_{k'}}$ and $\operatorname{ql} T_{k''}$ summands in $\mathcal{W}_{T_{k''}}$.*

Proof. Given a rank n, we let $T = \coprod_{i=1}^{n-1} T_i$ be a maximal rigid object in $\mathcal{C}_n$. We proceed by reverse induction on the quasilength of the summands of T (that is, from summands of larger quasilength to summands of smaller quasilength). This is OK, since there is a (unique) summand of maximal quasilength.

In accordance with Proposition 1.5, consider the top summand T_1 of T, which has quasilength $n-1$. Without loss of generality assume that this has coordinates $(1, n-1)$. Claim (i) holds for T_1 by the proof of Proposition 1.5.

Assume first that there are no summands of T among the objects $(1,i)$, where $i = 1, \dots, n-2$. Then all $n-2$ summands of T/T_1 are in $\mathcal{W}_{(2,n-2)}$. So by Lemma 2.4, the object $(2, n-2)$ must be a summand of T. Hence in this situation claim (ii) also holds, with the triple $(T_1; 0, (2, n-2))$. A similar argument shows that none of the objects $(i, n-i)$, where $2 \le i \le n-1$, are summands if and only if $(1, n-2)$ is a summand, and thus (ii) holds with $(T_1; (1, n-2), 0)$.

Suppose therefore that $(1, n-2)$ is not a summand, and that there is at least one summand of T with coordinates $(1, i)$, where $1 \le i \le n-3$. Let $T_2 = (1, i_0)$ be the one of these with highest quasilength (that is, maximal i).

Consider the subwing triple $(T_1; T_2, X)$ where $X = (i_0 + 2, n - i_0 - 2)$. By Lemma 2.3 and the maximality of i_0, all $n - 2$ summands of T/T_1 must be in $\mathcal{W}_{T_1} \cup \mathcal{W}_X$. Then it follows from Lemma 2.4 that there must be i_0 summands in $\mathcal{W}_{T_1}$ and $n - i_0 - 2$ summands in $\mathcal{W}_X$ and moreover that X is indeed a summand of T. We conclude that both claims (i) and (ii) hold for the top summand.

Assume now that T_k is some summand of T with $\mathrm{ql}\, T_k > 1$, but not the top summand. Let T_j be a summand of T of smallest quasilength with $\mathrm{ql}\, T_j > \mathrm{ql}\, T_k$ such that $T_k \in \mathcal{W}_{T_j}$. (Such a summand exists, since all summands are in $\mathcal{W}_{T_1}$.) Then by induction, the claims hold for T_j, and by the minimality in the choice of T_j, the subwing triple corresponding to T_j is $(T_j; T_{j'}, T_{j''})$ where either $T_{j'}$ or $T_{j''}$ is T_k, and the other one is also a summand. In any case, since by the induction hypothesis claim (ii) holds for T_j, there are $\mathrm{ql}\, T_k$ summands of T in $\mathcal{W}_{T_k}$, and so (i) holds for T_k.

Now that we know that there are $\mathrm{ql}\, T_k - 1$ summands of T/T_k in $\mathcal{W}_{T_k}$, we can prove that (ii) holds by the same arguments as for T_1 above. ■

A subwing triple $(T_k; T_{k'}, T_{k''})$ such that $T_k, T_{k'}$ and $T_{k''}$ are summands of a maximal rigid object T will be called a T-*subwing triple*.

With Lemma 2.5 we have obtained a combinatorial description of the maximal rigid objects as a system of subwing triples partially ordered by inclusion. Note that for a maximal rigid T with top summand T_1 there is a natural map from the objects in $\mathcal{W}_{T_1}$ to the summands of T given by sending an object X to the summand T_x with smallest quasilength such that $X \in \mathcal{W}_{T_x}$. The restriction of this map to the set of quasisimples is a bijection. See Figure 4a below for an example. Also, we have the following:

LEMMA 2.6. *Let T_i, T_j be summands of a maximal rigid T. Then either $T_i \in \mathcal{W}_{T_j}$, or $T_j \in \mathcal{W}_{T_i}$, or $\mathcal{W}_{T_i}$ and $\mathcal{W}_{T_j}$ have empty intersection.*

Proof. Suppose $T_i \notin \mathcal{W}_{T_j}$ and $T_j \notin \mathcal{W}_{T_i}$. Let T_k be a summand of T such that both $T_i \in \mathcal{W}_{T_k}$ and $T_j \in \mathcal{W}_{T_k}$, and which has minimal quasilength among summands with this property. There is some non-degenerate T-subwing triple $(T_k; T_{k'}, T_{k''})$, and by Lemma 2.5, the summands T_i and T_j must be in $\mathcal{W}_{T_{k'}} \cup \mathcal{W}_{T_{k''}}$. Now by the minimality in the choice of T_k, we know that T_i must be in $\mathcal{W}_{T_{k'}}$ and T_j in $\mathcal{W}_{T_{k''}}$ or vice versa. It follows that $\mathcal{W}_{T_i}$ and $\mathcal{W}_{T_j}$ have empty intersection, since $\mathcal{W}_{T_{k'}}$ and $\mathcal{W}_{T_{k''}}$ have empty intersection. ■

LEMMA 2.7. *Let T_i and T_j be summands of a maximal rigid T. Then the following are equivalent:*

(a) *There exists a non-zero $\mathcal{T}$-map $T_i \to T_j$.*
(b) *Either T_j is on the ray $\mathbf{R}_{T_i}$ or T_i is on the coray $\mathbf{C}_{T_j}$.*

Proof. This follows from the observation that since T_i and T_j are Ext-orthogonal, there is no map $T_i \to \tau T_j$. ∎

LEMMA 2.8. *Let T be a maximal rigid object.*

(i) *If $(T_i; T_j, T_k)$ is a non-degenerate T-subwing triple, there is a $\mathcal{T}$-map $f_{ji} : T_j \to T_i$ and a $\mathcal{T}$-map $f_{ik} : T_i \to T_k$, and these maps are irreducible in* $\operatorname{add}_{\mathcal{C}} T$.
(ii) *If $(T_i; T_j, 0)$ is a degenerate T-subwing triple, there is a $\mathcal{T}$-map $f_{ji} : T_j \to T_i$, and it is irreducible in* $\operatorname{add}_{\mathcal{C}} T$. *A similar statement holds for a degenerate T-subwing triple $(T_i; 0, T_k)$.*
(iii) *There are no irreducible $\mathcal{T}$-maps in* $\operatorname{add}_{\mathcal{C}} T$ *other than those described in* (i) *and* (ii).
(iv) *If $(T_i; T_j, T_k)$ is non-degenerate, the composition $f_{ik} \circ f_{ji}$ is zero.*

Proof. Claims (i) and (ii) are obvious, so consider summands T_x, T_y and assume that there is a $\mathcal{T}$-map $f_{xy} : T_x \to T_y$. By Lemma 2.7, either the summand T_y must be on the ray $\mathbf{R}_{T_x}$ or T_x must be on the coray $\mathbf{C}_{T_y}$. Assume the former case, so T_x is on the left edge of $\mathcal{W}_{T_y}$. We know from Lemma 2.5 that there is some T-subwing triple $(T_z; T_x, T_{x'})$, where T_x must necessarily be on the left edge of $\mathcal{W}_{T_z}$. (If it was on the right, it would violate Lemma 2.6.) Then clearly f_{xy} factors through the $\mathcal{T}$-map $f_{xz} : T_x \to T_z$. A similar argument can be given in the case where T_x is on the coray $\mathbf{C}_{T_y}$ by considering a T-subwing triple $(T_x; T_{x'}, T_{x''})$, so (iii) holds as well.

Claim (iv) also follows from Lemma 2.7. ∎

LEMMA 2.9. *Let T_i, T_j be summands of a maximal rigid T. Then the following are equivalent.*

(a) *There is a non-zero $\mathcal{D}$-map $T_i \to T_j$.*
(b) *One of the following is satisfied:*
    (b′) *There is some non-degenerate T-subwing triple $(T_x; T_y, T_z)$ such that T_i is on the left edge of $\mathcal{W}_{T_z}$ and T_j is on the right edge of $\mathcal{W}_{T_y}$.*
    (b″) *T_i is on the left edge of $\mathcal{W}_{T_1}$ and T_j is on the right edge of $\mathcal{W}_{T_1}$, where T_1 is the top summand.*

Proof. Assume first that $T_i = T_1$, the top summand. Then there is a $\mathcal{D}$-map $T_i \to T_j$ if and only if T_j is on the right edge of $\mathcal{W}_{T_1}$, which is a special case of (b″). Similarly, if $T_j = T_1$, then there is a $\mathcal{D}$-map $T_i \to T_j$ if and only if T_i is on the left edge of $\mathcal{W}_{T_1}$. So under the assumption that at least one of T_i, T_j is the top summand, the claim holds.

Consider therefore the case where neither T_i nor T_j is the top summand. It is easily seen that if $T_j \in \mathcal{W}_{T_i}$ or $T_i \in \mathcal{W}_{T_j}$, there can be no $\mathcal{D}$-map $T_i \to T_j$. So consider the summand T_x of minimal quasilength such that both T_i and T_j are in $\mathcal{W}_{T_x}$. By Lemma 2.5 there is a T-subwing triple $(T_x; T_y, T_z)$, and by the minimality of T_x, either $T_i \in \mathcal{W}_{T_y}$ and $T_j \in \mathcal{W}_{T_z}$ or the other way around. The claim now follows from Lemma 2.2. ∎

We can now prove the main theorem.

Proof of Theorem 2.1. In the argument, we will implicitly use the fact that for indecomposable objects X and Y with $\operatorname{ql} X, \operatorname{ql} Y \leq n-1$, there is up to multiplication by scalars at most one $\mathcal{T}$-map and one $\mathcal{D}$-map from X to Y.

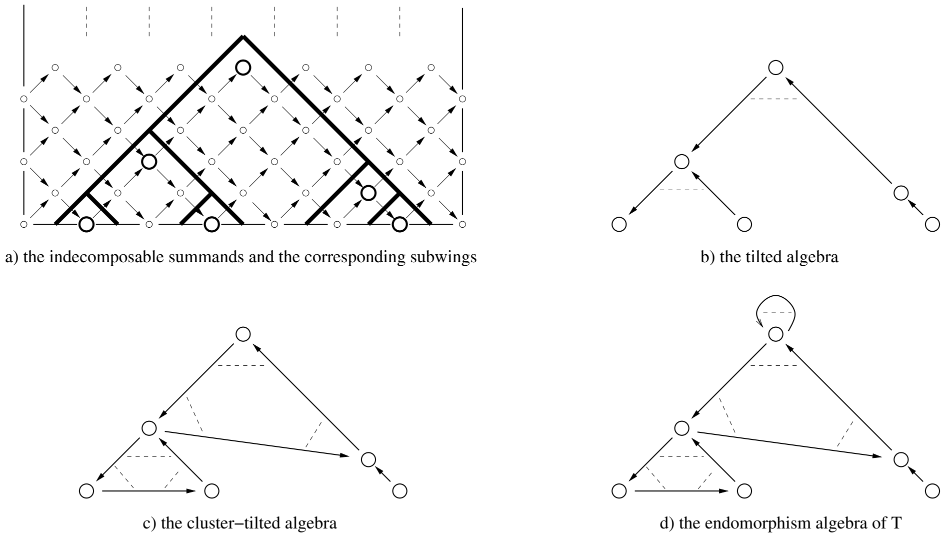
Our task is to determine the quiver Q and defining relations of $\Lambda_T = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. First recall that the functor $\operatorname{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \operatorname{mod} \Lambda_T$ induces an equivalence between $\operatorname{add}_{\mathcal{C}} T$ and the category $\mathcal{P}(\Lambda_T)$ of projective Λ_T-modules. The vertices in Q are therefore in bijection with the indecomposable summands of T, and the arrows correspond to maps which are irreducible in $\operatorname{add}_{\mathcal{C}} T$.

No $\mathcal{T}$-map can factor through a $\mathcal{D}$-map, so let us first consider the arrows in Q coming from $\mathcal{T}$-maps and their relations, that is, the endomorphism ring of T as an object in the subcategory $\operatorname{add}_{\mathcal{C}}^{\mathcal{T}} \mathcal{W}_{T_1}$, where T_1 is the top summand of T. By virtue of the equivalence of this category with the module category $\operatorname{mod} k\vec{A}_{n-1}$, we know that this will be the quiver with relations for the tilted algebra $\Gamma_T = \operatorname{End}_{A_{n-1}}(M_T)^{\mathrm{op}}$.

By Lemma 2.8(i), for each non-degenerate T-subwing triple $(T_i; T_j, T_k)$ there exist $\mathcal{T}$-maps $f_{ji} : T_j \to T_i$ and $f_{ik} : T_i \to T_k$ which are irreducible in $\operatorname{add}_{\mathcal{C}} T$ and hence correspond to arrows $\alpha_{ij} : i \to j$ and $\alpha_{ki} : k \to i$ in Q. Similarly, by Lemma 2.8(ii), for each degenerate T-subwing triple $(T_i; T_j, 0)$ there is an arrow $\alpha_{ij} : i \to j$, and for a T-subwing triple $(T_i; 0, T_k)$ there is an arrow $\alpha_{ki} : k \to i$. Moreover, by Lemma 2.8(iii), these are the only arrows in Q coming from $\mathcal{T}$-maps. Assuming that the triple is non-degenerate, by Lemma 2.8(iv), the composition $f_{ik} \circ f_{ji}$ is zero, and hence $\alpha_{ij}\alpha_{ki}$ is a zero relation for the quiver.

It follows from Lemma 2.7 that there are no other minimal relations on the arrows coming from $\mathcal{T}$-maps, since a path $\alpha_{i_{k-1}i_k} \cdots \alpha_{i_1 i_2} \alpha_{i_0 i_1}$ such that no $\alpha_{i_l i_{l+1}} \alpha_{i_{l-1} i_l}$ comes from a composition $f_{i_l i_{l-1}} \circ f_{i_{l+1} i_l}$ from a non-degenerate T-subwing triple $(T_{i_l}; T_{i_{l+1}}, T_{i_{l-1}})$ corresponds to a map following a ray or a coray. So the above gives a description of the tilted algebra $\Gamma_T = \operatorname{End}_{A_{n-1}}(M_T)^{\mathrm{op}}$. See Figure 4b for an example.

Now consider the $\mathcal{D}$-maps, and postpone for a moment the situation with maps to or from the top summand T_1. By Lemma 2.2(iii), for each non-degenerate T-subwing triple $(T_i; T_j, T_k)$ there is a $\mathcal{D}$-map $g_{kj} : T_k \to T_j$. We claim that this map is irreducible in $\operatorname{add}_{\mathcal{C}} T$. So assume that there exists a summand T_x, not isomorphic to T_k or T_j, and a $\mathcal{D}$-map $g_{xj} : T_x \to T_j$ such that $f_{kj} = g_{xj} \circ h_{kx}$ where $h_{kx} : T_k \to T_x$. Since the composition of two



a) the indecomposable summands and the corresponding subwings
b) the tilted algebra
c) the cluster−tilted algebra
d) the endomorphism algebra of T

Fig. 4. An $n = 7$ example of a maximal rigid object T and the associated algebras Γ_T, $\widetilde{\Gamma}_T$ and Λ_T

$\mathcal{D}$-maps is zero, h_{kx} must be a $\mathcal{T}$-map. So by Lemma 2.7, either T_x is on $\mathbf{R}_{T_k}$, or T_k is on $\mathbf{C}_{T_x}$. The former case contradicts Lemma 2.6, since T_k is on the right side of $\mathcal{W}_{T_i}$ and on the left side of $\mathcal{W}_{T_x}$. The latter case contradicts Lemma 2.2, since T_x is on the right edge of $\mathcal{W}_{T_k}$. A similar argument shows that we cannot have a factorisation $g_{kj} = h_{yj} \circ g_{ky}$ where h_{yj} is a $\mathcal{T}$-map and g_{ky} is a $\mathcal{D}$-map.

Thus g_{kj} is irreducible and corresponds to an arrow $\beta_{jk} : j \to k$ in Q.

For the non-degenerate T-subwing triple $(T_i; T_j, T_k)$, the composition $g_{kj} \circ f_{ik} : T_i \to T_j$ is a $\mathcal{D}$-map. We claim that it must be zero. Assume therefore that there is a $\mathcal{T}$-map from T_j to $\tau^2 T_i$. Then, since T_i sits on $\mathbf{R}_{T_j}$ and there is no $\mathcal{T}$-map $T_j \to \tau T_i$ by the Ext-orthogonality of T_i and T_j, there must be $\mathcal{T}$-maps from T_j to *all* indecomposables of quasilength $\operatorname{ql} T_i$, *except* τT_i. But this would require $\operatorname{ql} T_j = n-1$, and this is impossible, since $\operatorname{ql} T_j < \operatorname{ql} T_1 = n - 1$. We conclude that there is no $\mathcal{D}$-map $T_i \to T_j$, and the composition is zero.

Similarly, the composition $f_{ji} \circ g_{kj} : T_k \to T_i$ is also zero, since there is no $\mathcal{T}$-map from T_i to $\tau^2 T_k$. It follows that the paths $\alpha_{ki}\beta_{jk}$ and $\beta_{jk}\alpha_{ij}$ are zero relations on the quiver Q.

By Lemma 1.2, there is a $\mathcal{D}$-map h_{T_1} which is an endomorphism of the top summand T_1. This map must be irreducible in $\operatorname{add}_{\mathcal{C}} T$, for the only objects in $\mathcal{W}_{T_1}$ to which there are maps from T_1 are the ones on the right edge of $\mathcal{W}_{T_1}$, but there are no maps from any of these to T_1. Thus there is a loop ω at the vertex corresponding to the top summand.

The composition $h_{T_1} \circ h_{T_1}$ is zero, since any composition of two $\mathcal{D}$-maps is the image of a map $\mathcal{T} \to \mathcal{T}[2]$ in $\mathcal{D}$, which must necessarily be zero since $\mathcal{T}$ is hereditary. So ω^2 is a zero relation on the quiver of Λ_T.

By Lemmas 2.2 and 2.9, there are no other irreducible $\mathcal{D}$-maps between summands of T, and also no other minimal relations involving arrows from $\mathcal{D}$-maps: Let β_{jk} be an arrow corresponding to a $\mathcal{D}$-map $T_k \to T_j$ where $(T_i; T_j, T_k)$ is a non-degenerate subwing triangle. Then if $\gamma_1 \cdots \gamma_l \beta_{jk} \gamma_1^* \cdots \gamma_{l'}^*$ is a path in Q such that $\gamma_1 \cdots \gamma_l$ and $\gamma_1^* \cdots \gamma_{l'}^*$ do not traverse any relations, then necessarily $\gamma_1, \ldots, \gamma_l$ are arrows coming from maps on the left edge of $\mathcal{W}_{T_k}$ and $\gamma_1^*, \ldots, \gamma_{l'}^*$ are arrows coming from the right edge of $\mathcal{W}_{T_j}$. Then this path corresponds to a non-zero map by Lemma 2.9.

We now see that the arrows β_{xy} are in bijection with the zero relations for the quiver of Γ_T, and complete the relation paths to oriented cycles, so by [ABS] or [BRe] the arrows α_{zw} and β_{xy} form the quiver of $\widetilde{\Gamma}_T = \mathrm{End}_{\mathcal{C}_{A_{n-1}}}(M_T)^{\mathrm{op}}$. Furthermore, the relations imposed on this quiver coincide with the relations defining $\widetilde{\Gamma}_T$.

So the quiver of Λ_T is obtained from the quiver of $\widetilde{\Gamma}_T$ by adjoining the loop ω at the vertex corresponding to the top summand, which again corresponds to the projective-injective Γ_T-module. Also, the relations for Λ_T are the relations for $\widetilde{\Gamma}_T$ and in addition $\omega^2 = 0$. ■

See Figure 4 for an example of a maximal rigid object T and the tilted algebra Γ_T, the cluster-tilted algebra $\widetilde{\Gamma}_T$ and the endomorphism ring Λ_T.

An explicit description of the quivers for cluster-tilted algebras of type A was given in [S], and also in [BV]. It can be deduced from the type A cluster category model from [CCS]. They are exactly the quivers satisfying the following:

- all non-trivial minimal cycles are oriented and of length 3,
- any vertex has valency at most four,
- if a vertex has valency four, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle,
- if a vertex has valency three, then two of its adjacent arrows belong to a 3-cycle, and the third does not belong to any 3-cycle.

In the first condition, a *cycle* means a cycle in the underlying graph. A *connecting vertex* for such a quiver, as defined in [V], is a vertex which either has valency one, or has valency two and is traversed by a 3-cycle.

Note that for the endomorphism ring Λ_T of a maximal rigid T, the loop vertex is connecting for the quiver of $\widetilde{\Gamma}_T$. There is a sort of converse to Theorem 2.1, so we have a full description of the algebras which can arise:

PROPOSITION 2.10. *Let $\widetilde{\Gamma}$ be a cluster-tilted algebra of type A_{n-1}, and let c be a connecting vertex for $Q_{\widetilde{\Gamma}}$. Then there exists a maximal rigid object T of $\mathcal{C}_n$ such that Q_{Λ_T} is obtained from $Q_{\widetilde{\Gamma}}$ by adjoining a loop at c.*

Proof. We use induction on n. The claim is easily verified for small values.

Given a cluster-tilted algebra $\widetilde{\Gamma}$ of type A_{n-1}, let Q be its quiver and let c be some connecting vertex of Q. Then c has valency 1 or 2 in Q. Consider first the case where c has valency 2. Then c is traversed by a 3-cycle $c \to c_1 \to c_2 \to c$ in Q. The quiver $Q \setminus \{c, c_1 \to c_2\}$ has two disconnected components Q_1 and Q_2, where c_i is connecting for Q_i. Also, Q_i is the quiver of some cluster-tilted algebra $\widetilde{\Gamma}_i$ of type A_{k_i}, where $k_1 + k_2 = n - 2$.

By induction we can assume that for each $i = 1, 2$ there exists a maximal rigid object T_i in $\mathcal{C}_{k_i+1}$ such that the endomorphism ring of T_i is obtained from the quiver of $\widetilde{\Gamma}_i$ by adjoining a loop at c_i. Let M_i be the corresponding tilting $k\vec{A}_{k_i}$-module. We have a natural embedding of the module category $\operatorname{mod} k\vec{A}_{k_1}$ into the wing $\mathcal{W}_{(1,k_1)}$, and of $\operatorname{mod} k\vec{A}_{k_2}$ into $\mathcal{W}_{(k_1+2,k_2)}$. It is now easily seen that the images of the indecomposable summands of M_1 and M_2 under these embeddings are all compatible, and that the direct sum of these can be completed to a maximal rigid object T by adding the object $(1, n-1)$. Then the quiver of T is obtained from Q by adding a loop at c.

The case where c has valency one in Q is easier; one considers $Q \setminus \{c\}$ which is cluster-tilted of type A_{n-2}, and uses induction as above. ∎

REMARK 2.11. There are in fact exactly n maximal rigid objects in $\mathcal{C}_n$ with the prescribed endomorphism algebra, and these form a τ-orbit.

3. Gentleness, Gorenstein dimension and indecomposable modules. In this section we show that the endomorphism rings under discussion are gentle. We use this to determine their Gorenstein dimension.

A finite-dimensional algebra kQ/I where Q is a finite quiver is called *special biserial* [SkW] if

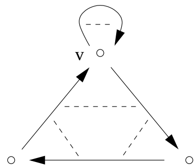
(i) for all vertices v in Q, there are at most two arrows starting at v and at most two arrows ending at v,
(ii) for every arrow β in Q, there is at most one arrow α_1 in Q with $\beta\alpha_1 \notin I$ and at most one arrow γ_1 with $\gamma_1\beta \notin I$.

A special biserial algebra kQ/I is *gentle* [ASk] if moreover

(iii) I is generated by paths of length 2,
(iv) for every arrow β in Q there is at most one arrow α_2 such that $\beta\alpha_2$ is a path and $\beta\alpha_2 \in I$, and at most one arrow γ_2 such that $\gamma_2\beta$ is a path and $\gamma_2\beta \in I$.

THEOREM 3.1. *If T is a maximal rigid object in the cluster tube $\mathcal{C}$, then the endomorphism ring* $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ *is gentle.*

Proof. It follows from the description of cluster-tilted algebras of type A, and Theorem 2.1, that condition (i) is satisfied whenever v is not the loop vertex. If v is the loop vertex, the quiver generally looks locally like this:



where the relations are indicated by dashed lines and it may also happen that one of the other two vertices pictured and consequently the adjacent arrows are not there. We see that also for this vertex, (i) is satisfied.

As proved in [BV], any cluster-tilted algebra of type A is gentle, and therefore if β is not an arrow incident with the loop vertex, (ii) and (iv) are satisfied. If β is incident with the loop vertex, (ii) and (iv) follow from the local description pictured above, with the observation that by the description of the cluster-tilted algebras there are no minimal relations involving both an arrow in the picture and an arrow outside the picture.

Moreover, by Theorem 2.1, the ideal I is generated by paths of length two, so (iii) is satisfied, and the algebra is gentle. ■

It is known from [GR] that all gentle algebras are *Gorenstein*, that is, a gentle algebra G has finite injective dimension both as a left and a right G-module. This dimension is then called the *Gorenstein dimension* of G.

In order to prove the next result, we need to recall the main result of [GR] in more detail. Let $G = kQ/I$ be a gentle algebra. An arrow α in Q is said to be *gentle* if there is no arrow α_0 such that $\alpha\alpha_0$ is a non-zero element of I. A *critical path* in Q is a path $\alpha_t \cdots \alpha_2\alpha_1$ such that $\alpha_{i+1}\alpha_i \in I$ for all $i = 1, \dots, n-1$.

THEOREM 3.2 (Geiß, Reiten [GR]). *Let $G = kQ/I$ be a gentle algebra, and let $n(G)$ be the supremum of the lengths of critical paths starting with a gentle arrow. ($n(G)$ is taken to be zero if there are no gentle arrows.)*

(a) *$n(G)$ is bounded by the number of arrows in Q.*
(b) *If $n(G) > 0$, then G is Gorenstein of Gorenstein dimension $n(G)$.*
(c) *If $n(G) = 0$, then G is Gorenstein of Gorenstein dimension at most one.*

We can use this to find the Gorenstein dimension of our algebras:

PROPOSITION 3.3. *Let T be a maximal rigid object in $\mathcal{C}$. If $n = 2$, the Gorenstein dimension of $\Lambda_T = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is zero, and if $n \geq 3$, the Gorenstein dimension is one.*

Proof. By Theorem 3.1, the algebra Λ_T is gentle, and we can apply Theorem 3.2.

If $n = 2$, then T has only one summand, and the endomorphism algebra Λ_T is isomorphic to the self-injective algebra $k[x]/(x^2)$, so in this case the Gorenstein dimension is zero.

Assume therefore that $n \geq 3$. The gentle arrows are exactly the arrows which are not traversed by any minimal oriented cycle. (In particular, the loop is not gentle.) But if α is any such arrow, then $\beta\alpha$ is a path for at most one arrow β, and $\beta\alpha$ can never be a zero relation, since it is not a part of a 3-cycle. So if gentle arrows exist, the maximal length $n(\Lambda_T)$ of critical paths starting with gentle arrows is 1, and therefore the Gorenstein dimension of Λ_T is also 1. If gentle arrows do not exist, we have $n(\Lambda_T) = 0$, and the Gorenstein dimension is at most 1.

It remains to show that Λ_T cannot be self-injective for $n \geq 3$. For this, consider the indecomposable projective associated with the loop vertex. It is easily seen that it is not injective. ∎

Since $\Lambda_T = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is gentle, it is in particular a string algebra. We will use this to show that Λ_T has finite representation type. For this, we will recall some basic facts about representations of string algebras. More details can be found e.g. in [BRi].

Let kQ/I be a string algebra. We consider words from the alphabet formed by the arrows in Q and their formal inverses. Inverse words are defined in the obvious way. For an inverted arrow α^{-1} we set the end vertex $e(\alpha^{-1})$ equal to the start vertex $s(\alpha)$ of the original arrow and vice versa. A word $w = \alpha_t \cdots \alpha_2\alpha_1$, where no two consecutive α_i are inverses of each other, is called a *string* if $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, t-1$ and moreover no subword of w or its inverse is a zero relation. In addition, there is a trivial string of length zero for each vertex of Q. We say that the start and end vertices of the α_i which appear in the strings are the vertices *traversed* by the string. For technical purposes, we also consider a unique zero (or empty) string of length -1.

To any string $\sigma = \alpha_t \cdots \alpha_2\alpha_1$ of length t in Q there is associated an indecomposable $(t+1)$-dimensional representation $M(\sigma)$ of kQ/I given by one-dimensional vector spaces in each vertex traversed by the string (with multiplicity) and one-dimensional identity maps for each of the arrows (and inverted arrows) appearing in σ. We have $M(\sigma_1) \simeq M(\sigma_2)$ if and only if $\sigma_1 = \sigma_2$ or $\sigma_1 = \sigma_2^{-1}$. The kQ/I-modules given by such representations are called *string modules.* If there exist *closed* strings $\alpha_t \cdots \alpha_2\alpha_1$ (i.e. strings starting and ending with the same vertex) such that powers of the string are strings as well, there will also be infinite families of indecomposables called *band modules.* The string modules and band modules constitute a complete set of representatives of isoclasses of indecomposable modules over kQ/I.

Now we include some remarks on the strings in the quiver of Λ_T, which will be useful in the proof of Theorem 3.8 and in Section 4. In what follows, an arrow α that arises from an irreducible $\mathcal{T}$-map in $\mathrm{add}_{\mathcal{C}}\, T$ will be called a *$\mathcal{T}$-arrow*, and similarly an arrow that arises from a $\mathcal{D}$-map will be called a *$\mathcal{D}$-arrow*.

Since the vertices of the quiver are in a natural bijection with the indecomposable summands of T, we will transfer some terminology about the summands to the vertices. So the quasilength of a vertex is the quasilength of the corresponding summand, and vertices (also arrows, strings) are said to be in a wing if the corresponding summands are in the wing.

LEMMA 3.4. *Let $(T_i; T_j, T_k)$ be a non-degenerate T-subwing triple, and α_{ij}, α_{ki} and β_{jk} the corresponding arrows. Then*

(i) *If $\sigma_2\alpha_{ki}\sigma_1$ is a string, then σ_1 is not of the form $\sigma_1 = \beta_{jk}\sigma_1^*$.*
(ii) *If $\sigma_2\alpha_{ij}\sigma_1$ is a string, then σ_2 is not of the form $\sigma_2 = \sigma_2^*\beta_{jk}$.*

Analogous statements hold for the inverses of the arrows.

Proof. The assertions follow from the fact that $\alpha_{ki}\beta_{jk}$ and $\beta_{jk}\alpha_{ij}$ are both zero relations. ∎

LEMMA 3.5.

(i) *If $\beta : i \to j$ is a $\mathcal{D}$-arrow, and $\sigma_2\beta\sigma_1$ is a string, then σ_2 is in the wing of j and σ_1 is in the wing of i, and similarly for inverses of $\mathcal{D}$-arrows.*
(ii) *A string in the quiver of Λ_T contains at most one $\mathcal{D}$-arrow or inverse of such.*

Proof. (i) Let $\sigma_2\beta\sigma_1$ be a string, where $\beta : i \to j$ is a $\mathcal{D}$-arrow corresponding to a $\mathcal{D}$-map $\psi_{ji} : T_j \to T_i$.

Suppose first that $T_i \neq T_j$. Then there is a subwing triple $(T_k; T_i, T_j)$. Now if the string σ_2 which starts with j is the trivial string for vertex j, then there is nothing to show, since j is definitely in the wing of itself. Assume therefore that σ_2 has length at least one. Then $\sigma_2 = a_k \cdots a_1 a_0$, where a_0 is either an arrow starting at j or the inverse of an arrow ending at j. By Lemma 3.4 we know that a_0 cannot be the arrow $\alpha_{jk} : j \to k$ which connects j to the vertex k associated with T_k. Now the remaining possibilities for a_0 are contained in the wing of j. By Lemma 3.4 again, it follows that none of the a_i can be a $\mathcal{D}$-arrow (or an inverse $\mathcal{D}$-arrow). So σ_2 traverses vertices of successively smaller quasilength and cannot return to j. Consequently, σ_2 is in the wing of j.

If $T_i = T_j$, then T_i is the top summand and β is the loop, in which case the claim follows from the fact that β^2 is a zero relation, and a similar argument to the above.

The statement for σ_1 and i is proved analogously, as also are the statements for inverses of $\mathcal{D}$-arrows.

(ii) This follows from (i). If $i \xrightarrow{\beta_1} j$ and $k \xrightarrow{\beta_2} l$ are two $\mathcal{D}$-arrows, then clearly β_2 cannot be in the wing of i or j when at the same time β_1 is in the wing of k or l. So they cannot both appear in the same string. The same holds if (at least) one of β_1 and β_2 is the inverse of a $\mathcal{D}$-arrow. ∎

REMARK 3.6. It follows from this lemma that if σ is a closed string, then either σ is a trivial string, or σ contains the loop as the only $\mathcal{D}$-arrow.

LEMMA 3.7. *In the following, we consider strings only up to orientation.*

(i) *The strings of length $k-1$ in the quiver of Λ_T which do not contain a $\mathcal{D}$-arrow or inverse of such are in bijection with sequences $T_{i_1}, \dots, T_{i_k}$ such that there are subwing triples $(T_{i_j}; T_{i_{j+1}}, T^*_{i_{j+1}})$ or $(T_{i_j}; T^*_{i_{j+1}}, T_{i_{j+1}})$ for $j = 1, \dots, k-1$.*

(ii) *The strings in the quiver of Λ_T that do not contain a $\mathcal{D}$-arrow or inverse of such are in bijection with pairs T_i, T_j of summands of T such that $T_i \in \mathcal{W}_{T_j}$.*

Proof. (i) Let σ be a string without a $\mathcal{D}$-arrow or an inverse $\mathcal{D}$-arrow. If σ is trivial, the claim is obviously true, so assume σ has length ≥ 1. Choose T_{i_1} to be an indecomposable summand of T which corresponds to a vertex i_1 traversed by σ such that none of the other vertices traversed by σ has higher quasilength. Assume $\operatorname{ql} T_{i_1} > 1$. Then there is some T-subwing triple $(T_{i_1}; T'_{i_1}, T''_{i_1})$. Let $\alpha'' : i''_1 \to i_1$ and $\alpha' : i_1 \to i'_1$ be the corresponding arrows.

Now since there are no $\mathcal{D}$-arrows in σ, and i_1 has maximal quasilength among the vertices traversed by σ, the string σ must be of the form $\sigma = \sigma_2\sigma_1$ where σ_1, if it is non-trivial, is of the form $\sigma_1 = \alpha''\sigma_1^*$ or $\sigma_1 = (\alpha')^{-1}\sigma_1^*$, and similarly $\sigma_2 = \sigma_2^*\alpha'$ or $\sigma_2 = \sigma_2^*(\alpha'')^{-1}$ if it is non-trivial. So for $\sigma_2\sigma_1$ to be a string, one of these has to be the trivial string associated with i_1, since the composition $\alpha'\alpha''$ is zero.

So i_1 is the start or end vertex of σ, and the first arrow (or inverse arrow) connects i_1 to one of the vertices from the T-subwing triple with i_1 on top. By repeating the process with the string σ_1^* or σ_2^*, we get the desired chain of subwing triples.

(ii) Follows directly from (i). ∎

See Figure 5 for an example of strings in the quiver of Λ_T.

THEOREM 3.8. *For a maximal rigid object T in $\mathcal{C}_n$, the endomorphism ring Λ_T is of finite type, and the number of indecomposable representations is*

$$\frac{1}{2}(3n^2 - 5n + 2).$$

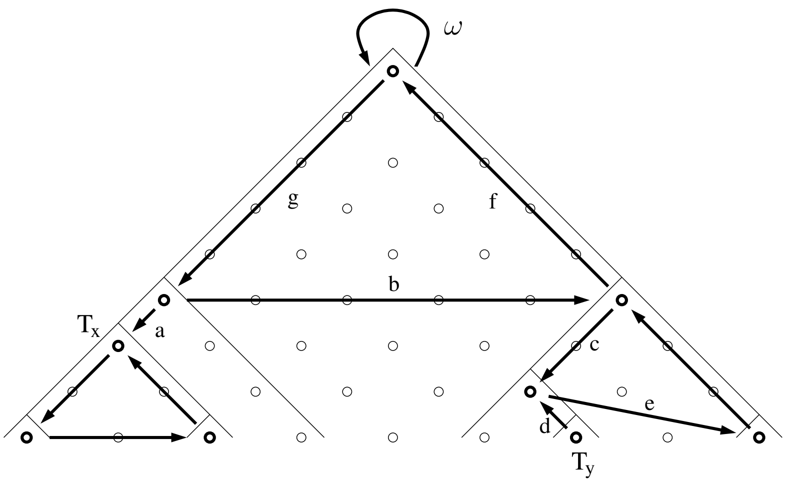


Fig. 5. The string $d^{-1}cba^{-1}$, where b is the only $\mathcal{D}$-arrow, starts at the vertex corresponding to T_x and ends at the vertex corresponding to T_y. Note how a path like ec (or its inverse) cannot be in a string, since it is a zero relation. Also, the path gf is not a string, while both $g\omega f$ and $g\omega^{-1}f$ are.

Proof. Let Q be the quiver of Λ_T and I the relation ideal. Moreover, let l denote the loop vertex, and ω the loop itself. For a string σ we will denote the associated indecomposable representation by $M(\sigma)$.

First consider strings which do not involve the loop. These are in bijection with ordered pairs of vertices: For each pair i, j from Q_0, let T_i, T_j be the associated summands of T. If $T_i \in \mathcal{W}_{T_j}$ or vice versa, there is, as in Lemma 3.7, a string without a $\mathcal{D}$-arrow connecting i and j. So suppose this is not the case. Consider the (unique) summand T_k which is of minimal quasilength such that T_i and T_j are both in $\mathcal{W}_{T_k}$. Then there is some T-subwing triple $(T_k; T'_i, T'_j)$ where $T_i \in \mathcal{W}_{T'_i}$ and $T_j \in \mathcal{W}_{T'_j}$ or the other way around. Now use Lemma 3.7 again to find strings connecting i to the vertex i' associated with T'_i and j to the vertex j' associated with T'_j. Now there is a $\mathcal{D}$-arrow $\beta'_{ij} : i' \to j'$. Connecting the two strings by β'_{ij} yields the desired string, and we can choose the orientation.

In particular, by Remark 3.6, any non-loop string starting and ending at the same vertex is a trivial string.

Denote by $\sigma(i, j)$ the unique non-loop string starting at i and ending at j, so $\sigma(i, j)^{-1} = \sigma(j, i)$. For the corresponding indecomposable representations we have isomorphisms $M(\sigma(i, j)) \simeq M(\sigma(j, i))$. The simple representations are the $M(\sigma(i, i))$. The total number of representations corresponding to non-loop strings is therefore

$$(n-1) + \binom{n-1}{2} = \frac{1}{2}(n^2 - n),$$

where the first term is the number of simple representations and the second is the number of strings of length ≥ 1 up to orientation.

Now for the strings passing through the loop. For each pair i, j of vertices, there are two strings from i to j passing through the loop ω:

$$\sigma_\omega(i,j) = \sigma(l,j)\omega\sigma(i,l), \quad \sigma_\omega^-(i,j) = \sigma(l,j)\omega^{-1}\sigma(i,l),$$

and these are all possible loop strings. We see that $(\sigma_\omega(i,j))^{-1} = \sigma_\omega^-(j,i)$, so $M(\sigma_\omega(i,j)) \simeq M(\sigma_\omega^-(j,i))$ for any choice of i, j, in particular for $i = j$. We deduce that the indecomposable representations associated with loop strings are in bijection with ordered pairs of vertices, and the number of such representations is $(n-1)^2$.

These are all the strings. Since (by Remark 3.6 and Lemma 3.7) the only closed strings are either trivial, ω or of the form $\alpha\sigma\alpha^{-1}$ for some arrow (or inverse arrow) α and some string σ, it follows that there are no band modules. So the string modules we have presented form a complete set of isomorphism classes of Λ_T-modules.

Summarising, we find that the total number of representations is

$$\frac{1}{2}(n^2 - n) + (n-1)^2 = \frac{1}{2}(3n^2 - 5n + 2). \ \blacksquare$$

4. On the behaviour of the Hom-functor. For a cluster-tilted algebra $C_T = \mathrm{End}_{\mathcal{C}_H}(T)^{\mathrm{op}}$ arising from the cluster category $\mathcal{C}_H$ of some hereditary algebra H, there is a close connection between the module category of C_T and the cluster category itself. The main theorem from [BMR1] says that the functor $G = \mathrm{Hom}_{\mathcal{C}_H}(T, -) : \mathcal{C}_H \to \mathrm{mod}\, C_T$ induces an equivalence

$$\bar{G} : \mathcal{C}_H/\mathrm{add}\,\tau T \xrightarrow{\sim} \mathrm{mod}\, C_T.$$

In particular, the cluster-tilted algebra is of finite representation type if and only if $\mathcal{C}_H$ has finitely many objects (which again happens if and only if H is of finite type). By Theorem 3.8, a similar result cannot hold for the cluster tubes. The analogous argument fails because the Hom-functor is not full. In this section, we will study some properties of this functor.

We introduce some notation. For any indecomposable object X in $\mathcal{C}$, let $H(X) = H^{\mathcal{T}}(X) \cup H^{\mathcal{D}}(X)$ be the Hom-hammock of X, where $H^{\mathcal{T}}(X)$ is the set of indecomposables to which X has $\mathcal{T}$-maps, and similarly for $H^{\mathcal{D}}(X)$. Also, consider the *reverse* Hom-*hammock* $R(X) \subset \mathrm{ind}\,\mathcal{C}$, that is, the support of $\mathrm{Hom}_{\mathcal{C}}(-, X)$ among the indecomposables. Like the ordinary Hom-hammock, this has a natural structure as the union of two components, one denoted by $R^{\mathcal{T}}(X)$ containing the indecomposables that have non-zero $\mathcal{T}$-maps to X, and another one denoted by $R^{\mathcal{D}}(X)$ containing those that have $\mathcal{D}$-maps to X. Note that by the description of the Hom-hammocks, $R^{\mathcal{T}}(X) = H^{\mathcal{D}}(\tau^{-2}X)$ and $R^{\mathcal{D}}(X) = H^{\mathcal{T}}(\tau^{-2}X)$. So the shape of $R(X)$ is similar to the shape of $H(X)$ (Figure 2).

For the remainder of this section, $T = \coprod_{i=1}^{n-1} T_i$ will be a maximal rigid object in $\mathcal{C}_n$, and we assume that the top summand of T is $T_1 = (1, n-1)$. Clearly, by redefining the coordinates we can use the results for all maximal rigid objects of $\mathcal{C}$. As in the preceding sections, we will denote by Λ_T the endomorphism ring $\Lambda_T = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.

We define $\mathcal{F}$ to be a certain set of indecomposable objects in $\mathcal{C}$:

$$\mathcal{F} = \{X = (a,b) \mid a + b \leq 2n - 1\}.$$

See Figure 6. The region $\mathcal{F}$ in the tube consists of the rigid part and in addition a triangle of height $n-1$ in the non-rigid part. (We have defined wings only for rigid indecomposables, but we can think of $\mathcal{F}$ as the wing of the object $(1, 2n-2)$.)

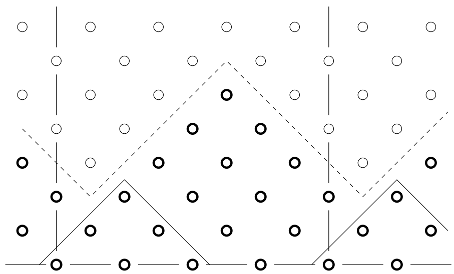


Fig. 6. The set $\mathcal{F}$ in $\mathcal{C}_4$, below the dashed curve. T is concentrated in the indicated wing.

The following claims are easily verified:

Lemma 4.1. *If X is an indecomposable in $\mathcal{W}_{T_1}$, then $H^{\mathcal{T}}(X) \cap \mathcal{F}$ forms one rectangle-shaped subgraph of the tube, and similarly for $H^{\mathcal{D}}(X) \cap \mathcal{F}$.*

Lemma 4.2. *Let $*$ be either $\mathcal{T}$ or $\mathcal{D}$. Then for an indecomposable X, the set $R^*(X)$ contains a unique quasisimple q_X^*, and a necessary condition for an object Y to be in $R^*(X)$ is that $q_X^* \in \mathcal{W}_Y$.*

Lemma 4.3. *Let $X \in \mathcal{F}$. Then $T_1 \notin R(X)$ if and only if $X \in \mathcal{W}_{\tau T_1}$.*

We now want to assign to each indecomposable in $\mathcal{F} \setminus \mathrm{add}\, \tau T$ a uniquely defined string in the quiver of Λ_T. In the main result of this section we will show that the images under the Hom-functor are given by these strings. The first step is to encode information about $\mathcal{T}$-maps and $\mathcal{D}$-maps in separate strings, which will be joined to one string at a later stage.

Lemma 4.4. *Let $*$ be either $\mathcal{T}$ or $\mathcal{D}$. For $X \in \mathcal{F}$ we have the following:*

(i) *$R(X) \cap \mathrm{add}\, T$ is empty if and only if $X \in \mathrm{add}\, \tau T$.*
(ii) *For any T-subwing triple $(T_i; T_j, T_k)$, at most one of T_j and T_k can be in $R^*(X)$.*

(iii) *If* $R^*(X) \cap \operatorname{add} T$ *is non-empty, there is a unique string in the quiver of* Λ_T *traversing each of the vertices corresponding to the indecomposables in* $R^*(X) \cap \operatorname{add} T$ *exactly once (and no other vertex) and ending at the vertex corresponding to the summand in* $R^*(X) \cap \operatorname{add} T$ *of highest quasilength.*

(iv) *A string of the type in part* (iii) *contains no* $\mathcal{D}$*-arrow (or inverse of such).*

Proof. (i) We need to show that $\operatorname{Hom}_{\mathcal{C}}(T, X) = 0$ if and only if $X \in \operatorname{add} \tau T$. If $X \in \operatorname{add} \tau T$, then there are no non-zero maps from T to X since T is rigid.

For the converse, assume that the intersection is empty. Using Lemma 4.3, we get $\operatorname{ql} X \leq n-1$. Moreover, we have

$$\operatorname{Ext}^1_{\mathcal{C}}(T, \tau^{-1} X) = \operatorname{Hom}_{\mathcal{C}}(T, X) = 0.$$

Since $\operatorname{ql} X \leq n-1$, the object X, and consequently $\tau^{-1} X$, is rigid. So $\tau^{-1} X = T_i$ for some i since T is maximal rigid, and we can conclude that $X \in \operatorname{add} \tau T$.

(ii) We know that $\mathcal{W}_{T_j}$ and $\mathcal{W}_{T_k}$ are disjoint. The claim then follows from Lemma 4.2.

(iii) Assume $R^*(X) \cap \operatorname{add} T$ is non-empty, and let T_l and T_h be elements in this set with minimal and maximal quasilength, respectively. By Lemma 4.2, the unique quasisimple q^*_X which is in $R^*(X)$ is now in both $\mathcal{W}_{T_l}$ and $\mathcal{W}_{T_h}$. So in particular $\mathcal{W}_{T_l}$ and $\mathcal{W}_{T_h}$ have non-empty intersection, and therefore by Lemma 2.6 we know that $T_l \in \mathcal{W}_{T_h}$. Also by Lemma 2.6 we see that T_h and T_l are uniquely determined. There is some T-subwing triple $(T_l; T'_l, T''_l)$, and it can easily be seen that if q^*_X were in $\mathcal{W}_{T'_l}$, say, then T'_l would also be in $R^*(X)$, which would violate the minimality condition on T_l. Thus T_l is the summand of smallest quasilength such that q^*_X is in the corresponding wing. See Figure 7.

Also, for summands T_s we see that $T_s \in R^*(X)$ if and only if $\mathcal{W}_{T_l} \subseteq \mathcal{W}_{T_s} \subseteq \mathcal{W}_{T_h}$, again by Lemma 2.6 and the maximality of T_h. Now the desired string is of the type described in Lemma 3.7, oriented in the suitable way.

(iv) By (ii), a $\mathcal{D}$-arrow associated with a non-degenerate T-subwing triple $(T_i; T_j, T_k)$ could not be traversed by a string of the type described in (iii). Moreover, the loop is disallowed as well, since then the loop vertex would be traversed twice, contrary to the condition in (iii). ∎

For an indecomposable $X \in \mathcal{F}$ such that $R^*(X) \cap \operatorname{add} T$ is non-empty, where $*$ is either $\mathcal{T}$ or $\mathcal{D}$, we denote the string in Lemma 4.4(iii) by σ^*_X. If the intersection is empty, we define σ^*_X to be the zero string. The next two lemmas tell us that different objects in $\mathcal{F} \setminus \operatorname{add} \tau T$ can be distinguished by their associated strings.

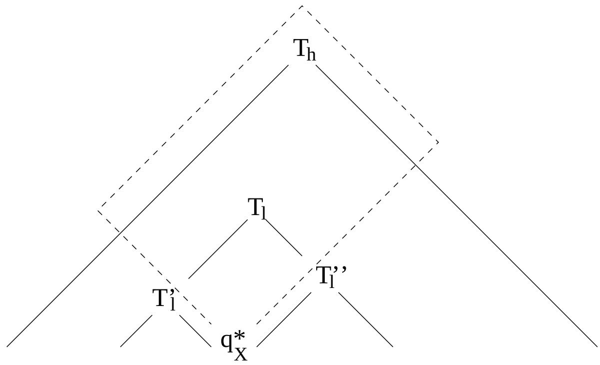


Fig. 7. If T_l and T_h are the summands of T in $R^*(X)$ of lowest and highest quasilength, respectively, then the entire rectangle indicated must be in $R^*(X)$.

LEMMA 4.5. *Let* $X, Y \in \mathcal{F} \setminus \operatorname{add} \tau T$. *If* $\sigma_X^{\mathcal{T}} = \sigma_Y^{\mathcal{T}}$ *and* $\sigma_X^{\mathcal{D}} = \sigma_Y^{\mathcal{D}}$*, then* $X = Y$.

Proof. We note first that for any indecomposable Z, the unique quasisimple $q_Z^{\mathcal{T}}$ in $R^{\mathcal{T}}(Z)$ determines the first coordinate of Z. Similarly, the unique quasisimple $q_Z^{\mathcal{D}}$ in $R^{\mathcal{D}}(Z)$ determines the sum of the coordinates of Z modulo the rank n. So if the first coordinate of Z is known, the quasisimple $q_Z^{\mathcal{D}}$ determines the second coordinate modulo n.

Let now X and Y be in $\mathcal{F} \setminus \operatorname{add} \tau T$ such that $\sigma_X^{\mathcal{T}} = \sigma_Y^{\mathcal{T}}$ and $\sigma_X^{\mathcal{D}} = \sigma_Y^{\mathcal{D}}$. We aim to show that X and Y must be equal.

By Lemma 4.4(i), at least one of $\sigma_X^{\mathcal{T}} = \sigma_Y^{\mathcal{T}}$ and $\sigma_X^{\mathcal{D}} = \sigma_Y^{\mathcal{D}}$ is non-zero. Assume first that both are non-zero. We claim that $q_X^{\mathcal{T}} = q_Y^{\mathcal{T}}$ and $q_X^{\mathcal{D}} = q_Y^{\mathcal{D}}$. As in the proof of Lemma 4.4(iii), we observe that if T_l is the T-summand of smallest quasilength in $R^{\mathcal{T}}(X)$, then $q_X^{\mathcal{T}}$ is the unique quasisimple which is in $\mathcal{W}_{T_l}$ but not in the wing of any T-summand of smaller quasilength. Since T_l is also the summand of smallest quasilength in $R^{\mathcal{T}}(Y)$, we must have $q_X^{\mathcal{T}} = q_Y^{\mathcal{T}}$. Similarly, we deduce that $q_X^{\mathcal{D}} = q_Y^{\mathcal{D}}$.

Thus X and Y have the same first coordinate, and the same second coordinate modulo n. But since X and Y are in $\mathcal{F}$, this means that unless X and Y are equal, one of them is in $\mathcal{W}_{(1,n-2)}$ and the other is in the non-rigid part. If they are not equal, there is then a contradiction to Lemma 4.3: If X is in $\mathcal{W}_{(1,n-2)}$ and Y is in the non-rigid part, then by Lemma 4.3, $T_1 \in R(Y)$ but $T_1 \notin R(X)$, which is impossible since we have assumed $\sigma_X^{\mathcal{T}} = \sigma_Y^{\mathcal{T}}$ and $\sigma_X^{\mathcal{D}} = \sigma_Y^{\mathcal{D}}$. We conclude that if both $\sigma_X^{\mathcal{T}} = \sigma_Y^{\mathcal{T}} \neq 0$ and $\sigma_X^{\mathcal{D}} = \sigma_Y^{\mathcal{D}} \neq 0$, then $X = Y$.

Assume then that $\sigma_X^{\mathcal{T}} = \sigma_Y^{\mathcal{T}} \neq 0$ and $\sigma_X^{\mathcal{D}} = \sigma_Y^{\mathcal{D}} = 0$, and furthermore that $X \neq Y$. Then X and Y have the same first coordinate. Moreover, at least one of X and Y must be in $\mathcal{W}_{(1,n-2)}$, since the only other possible positions for an object $Z \in \mathcal{F}$ such that $\sigma_Z^{\mathcal{T}} \neq 0$ and $\sigma_Z^{\mathcal{D}} = 0$ are on the coray $\mathbf{C}_{(n-1,n)}$.

Suppose (without loss of generality) that X has smaller quasilength than Y. So in particular, X is in $\mathcal{W}_{(1,n-2)}$. Let T_h be the object in $R^{\mathcal{T}}(X) \cap \operatorname{add} T$ with highest quasilength. Since $X \in \mathcal{W}_{(1,n-2)}$, we have $T_h \neq T_1$. Therefore, there is some T-subwing triple $(T_a; T_h, T'_h)$, where T_h is necessarily on the left side since $R^{\mathcal{T}}(X)$ contains whole corays, and so by the maximality of T_h, there can be no more summands of T on $\mathbf{C}_{T_h}$.

Assume first that this triple is non-degenerate. Since $R^{\mathcal{D}}(X)$ does not contain any summands of T, there is in particular no $\mathcal{D}$-map $T'_h \to X$. So X is in $H^{\mathcal{T}}(T_h)$ but not in $H^{\mathcal{D}}(T'_h)$. Moreover, we see that $X \notin H^{\mathcal{T}}(T_a)$, by the maximality of T_h. So X must be on the coray $\mathbf{C}_{\tau T'_h}$. If the triple is degenerate, then X must be on the right edge of $\mathcal{W}_{T_h}$, since $T_a \notin R^{\mathcal{T}}(X)$. In any of these two cases, we get a contradiction: Since Y and X have the same first coordinate, and Y has higher quasilength, T_a must be in $R^{\mathcal{T}}(Y)$. This contradicts the equality of $\sigma^{\mathcal{T}}_X$ and $\sigma^{\mathcal{T}}_Y$, and so our assumption that $X \neq Y$ must be wrong. See Figure 8.

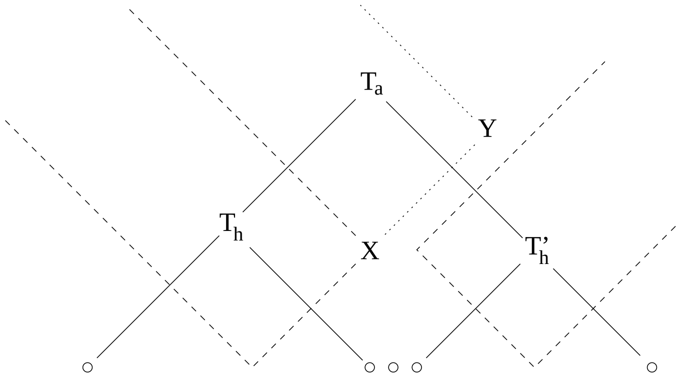


Fig. 8. If $\sigma^{\mathcal{T}}_X = \sigma^{\mathcal{T}}_Y \neq 0$ and $\sigma^{\mathcal{D}}_X = \sigma^{\mathcal{D}}_Y = 0$, then X and Y have the same first coordinate. If X is inside some non-degenerate T-subwing triple, then X must be on the coray $\mathbf{C}_{\tau T'_h}$, since otherwise $T'_h \in R^{\mathcal{D}}(X)$.

The situation where $\sigma^{\mathcal{T}}_X = \sigma^{\mathcal{T}}_Y = 0$ and $\sigma^{\mathcal{D}}_X = \sigma^{\mathcal{D}}_Y \neq 0$ can be proved in a similar manner. ∎

The following lemma is used to show that if two different objects have exactly one associated $\sigma^{\mathcal{T}}$- or $\sigma^{\mathcal{D}}$-string, then the strings are different.

Lemma 4.6. *If $\sigma^{\mathcal{T}}_X = \sigma^{\mathcal{D}}_Y \neq 0$, then $\sigma^{\mathcal{T}}_Y$ is non-zero.*

Proof. Suppose that $\sigma^{\mathcal{T}}_X = \sigma^{\mathcal{D}}_Y \neq 0$, and let T_i be the summand in $R^{\mathcal{T}}(X) \cap \operatorname{add} T = R^{\mathcal{D}}(Y) \cap \operatorname{add} T$ which has highest quasilength. We claim that $T_i = T_1$. To see this, assume that $\operatorname{ql} T_i < n-1$. Then there is some (degenerate or non-degenerate) T-subwing triple $(T_k; T_i, T^*_i)$ or $(T_k; T^*_i, T_i)$. Since $R^{\mathcal{T}}(X)$ contains whole corays, and $R^{\mathcal{D}}(Y)$ contains whole rays, the summand T_k must be in one of these two reverse Hom-hammocks. But this contradicts our choice of T_i. So $T_i = T_1$.

Now observe that if Y is in $\mathcal{F}$, and there is a $\mathcal{D}$-map $T_1 \to Y$, then there is also a $\mathcal{T}$-map $T_1 \to Y$, so in particular $\sigma_Y^{\mathcal{T}} \neq 0$. ∎

We now show that if both strings associated with an indecomposable in $\mathcal{F}$ are non-zero, then there is a larger string containing both of them.

LEMMA 4.7. *Let $X \in \mathcal{F}$. If both $\sigma_X^{\mathcal{T}}$ and $\sigma_X^{\mathcal{D}}$ are non-zero, then there is a $\mathcal{D}$-arrow β_X from the end vertex of $\sigma_X^{\mathcal{T}}$ to the end vertex of $\sigma_X^{\mathcal{D}}$. So in particular, $(\sigma_X^{\mathcal{D}})^{-1}\beta_X\sigma_X^{\mathcal{T}}$ is a well-defined string.*

Proof. We consider four cases, depending on the position of X in $\mathcal{F}$.

The first case is when there is a $\mathcal{D}$-map $T_1 \to X$. One readily verifies that there is then also a $\mathcal{T}$-map $T_1 \to X$, so in this case T_1 is in both $R^{\mathcal{T}}(X)$ and $R^{\mathcal{D}}(X)$, and the claim holds with the loop as β_X.

The second case is when there is a $\mathcal{T}$-map $T_1 \to X$, but no $\mathcal{D}$-map from T_1 to X. This happens exactly when X is on the coray $\mathbf{C}_{(n-1,n)}$, and in this case there are no $\mathcal{D}$-maps from any summands of T to X, so $R^{\mathcal{D}}(X) \cap \operatorname{add} T$ is empty, and there is nothing to prove.

The third case is when X is located on the ray $\mathbf{R}_{(n,1)}$. Then there are no $\mathcal{T}$-maps from T to X, so again there is nothing to prove.

The only remaining situation is when X is in the wing $\mathcal{W}_{(1,n-2)}$. As in the proof of Lemma 4.5, let T_h be the summand in $\operatorname{add} T \cap R^{\mathcal{T}}(X)$ of highest quasilength. Since $T_h \neq T_1$, there is some T-subwing triple $(T_a; T_h, T_h')$ with T_h necessarily on the left, since $R^{\mathcal{T}}(X)$ contains whole corays. Assume first that this triple is non-degenerate. Then, since $T_h \in R^{\mathcal{T}}(X)$ and $T_a \notin R^{\mathcal{T}}(X)$ by the maximality property of T_h, we note the following about the position of X:

- $X \in \mathcal{W}_{T_a}$, but not on the right edge of $\mathcal{W}_{T_a}$, since then there would be a $\mathcal{T}$-map $T_a \to X$;
- $X \notin \mathcal{W}_Y$, where Y is the object on $\mathbf{C}_{T_h'}$ which has an irreducible map to T_h', since then there would be no $\mathcal{T}$-map $T_h \to X$;
- if $X \in \mathcal{W}_{T_h}$, then it is on the right edge, since otherwise there would be no $\mathcal{T}$-map $T_h \to X$.

Our aim is to show that T_h' is in $R^{\mathcal{D}}(X)$, and moreover that it is the summand of T with highest quasilength appearing in $R^{\mathcal{D}}(X)$.

With the above remarks about the position of X, we see that the only allowed positions such that there is no $\mathcal{D}$-map $T_h' \to X$ are positions on the coray $\mathbf{C}_{\tau T_h'}$. But if X were on this coray, then $R^{\mathcal{D}}(X) \cap \operatorname{add} T$ would be empty, contrary to our hypothesis: Namely, assuming this position for X, suppose there were some summand $T_b \in R^{\mathcal{D}}(X)$. An equivalent condition to this (cf. Lemma 1.1) is that there is a non-zero $\mathcal{T}$-map $X \to \tau^2 T_b$, which is again equivalent to the existence of a $\mathcal{T}$-map $\tau^{-1}X \to \tau T_b$. But since $X \in \mathbf{C}_{\tau T'_h} \cap \mathcal{W}_{T_a}$, we see that $\tau^{-1}X$ is on the right edge of $\mathcal{W}_{T_a}$. So there would be a non-zero $\mathcal{T}$-map $T_a \to \tau T_b$, which is impossible since T_a and T_b are Ext-orthogonal.

So $T'_h \in R^{\mathcal{D}}(X)$. Let T_c be the T-summand of highest quasilength which appears in $R^{\mathcal{D}}(X)$. Then, since both $\mathcal{W}_{T'_h}$ and $\mathcal{W}_{T_c}$ contain the quasisimple $q_X^{\mathcal{D}}$ from Lemma 4.2, Lemma 2.6 tells us that $T'_h \in \mathcal{W}_{T_c}$. But T'_h is in $\mathcal{W}_{T_a}$ as well, so by Lemma 2.6 again, either $T_a \in \mathcal{W}_{T_c}$ or $T_c \in \mathcal{W}_{T_a}$. The former case is not possible, since it would imply that $T_a \in R^{\mathcal{D}}(X)$, which is impossible since $X \in \mathcal{W}_{T_a}$. So the remaining possibility is that $T_c = T'_h$, that is, T'_h is the T-summand in $R^{\mathcal{D}}(X)$ of highest quasilength.

By the description of the quiver of Λ_T in Section 2, there is a $\mathcal{D}$-arrow β_X associated with the non-degenerate T-subwing triple, from the vertex corresponding to T_h to the vertex corresponding to T'_h. Since $\sigma_X^{\mathcal{T}}$ ends at the vertex corresponding to T_h, and $(\sigma_X^{\mathcal{D}})^{-1}$ starts at the vertex corresponding to T'_h, the string $(\sigma_X^{\mathcal{D}})^{-1}\beta_X\sigma_X^{\mathcal{T}}$ is well-defined.

It remains to consider the case where the T-subwing triple $(T_a; T_h, T'_h)$ is degenerate, that is, $T'_h = 0$. In this case, since $T_a \notin R^{\mathcal{T}}(X)$, the only option is that X is on the right edge of $\mathcal{W}_{T_h}$. But then $R^{\mathcal{D}}(X) \cap \operatorname{add} T$ is empty: If there were a $\mathcal{D}$-map $T_d \to X$ for some T-summand T_d, then T_a and T_d would have an extension, as can be seen from an argument similar to the above. ∎

By virtue of the preceding considerations, we can now associate to each indecomposable object $X \in \mathcal{F} \setminus \operatorname{add} \tau T$ a unique indecomposable Λ_T-module $M(\sigma_X)$ where

$$\sigma_X = \begin{cases} \sigma_X^{\mathcal{T}} & \text{if } \sigma_X^{\mathcal{D}} \text{ is zero,} \\ \sigma_X^{\mathcal{D}} & \text{if } \sigma_X^{\mathcal{T}} \text{ is zero,} \\ (\sigma_X^{\mathcal{D}})^{-1}\beta_X\sigma_X^{\mathcal{T}} & \text{if both } \sigma_X^{\mathcal{T}} \text{ and } \sigma_X^{\mathcal{D}} \text{ are non-zero.} \end{cases}$$

We can now describe the action of the Hom-functor on objects in $\mathcal{F}$.

THEOREM 4.8. *Let T be a maximal rigid object of $\mathcal{C}$, and $\Lambda_T = \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ the endomorphism ring.*

(1) *Let X be an object in $\mathcal{F} \setminus \operatorname{add} \tau T$. Then the Λ_T-module* $\operatorname{Hom}_{\mathcal{C}}(T, X)$ *is isomorphic to the string module $M(\sigma_X)$.*
(2) *The functor* $\operatorname{Hom}_{\mathcal{C}}(T, -)$ *provides a bijection between $\mathcal{F} \setminus \operatorname{add} \tau T$ and the set of isoclasses of indecomposable Λ_T-modules.*

Proof. (1) By Lemma 4.4(i), the module is non-zero. Let e_i be the idempotent of Λ_T corresponding to the vertex i, which in turn corresponds to the summand T_i of T. Then the vector space $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T,X)$ decomposes

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T,X) = \bigoplus_{i=1}^{n-1} e_i \,\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T,X) = \bigoplus_{i=1}^{n-1} \mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T_i,X) = \bigoplus_{i=1}^{n-1} (\Phi_{iX} \oplus \Psi_{iY}),$$

where each vector space Φ_{iX} and Ψ_{iX} is at most 1-dimensional and is spanned by a $\mathcal{T}$-map $\phi_{iX} : T_i \to X$ and a $\mathcal{D}$-map $\psi_{iX} : T_i \to X$ respectively, these maps being zero if no non-zero such maps exist.

By the definition of $\sigma_X^{\mathcal{T}}$, the vertices for which $\Phi_{iX} \neq 0$ are exactly the vertices that are traversed by $\sigma_X^{\mathcal{T}}$. Similarly, the vertices for which $\Psi_{iX} \neq 0$ are the ones traversed by $\sigma_X^{\mathcal{D}}$. In particular, there is an equality of dimension vectors

$$\underline{\dim}(M(\sigma_X)) = \underline{\dim}(\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T,X)).$$

We need to establish that the action of Λ_T on $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T,X)$ is the same as the action on $M(\sigma_X)$.

Each map which is irreducible in $\mathrm{add}_{\mathcal{C}}\, T$ corresponds to an arrow in the quiver of Λ_T, and the arrow acts by composition with the irreducible map. Unless both the start vertex and the end vertex of this arrow are vertices in the support of $\mathrm{Hom}_{\mathcal{C}_{\mathcal{T}}}(T,X)$, clearly this map (equivalently, this arrow) has a zero action on both the modules $\mathrm{Hom}_{\mathcal{C}}(T,X)$ and $M(\sigma_X)$.

So we must show that each $\mathcal{T}$-arrow $i \to j$ appearing in $\sigma_X^{\mathcal{T}}$ acts by an isomorphism $\Phi_{iX} \to \Phi_{jX}$, and each $\mathcal{T}$-arrow $i \to j$ appearing in $\sigma_X^{\mathcal{D}}$ acts by an isomorphism $\Psi_{iX} \to \Psi_{jX}$, and finally that if $\beta_X : i \to j$ is defined, then the action of this is given by an isomorphism $\Phi_{iX} \to \Psi_{jX}$.

Our first goal is now to show that whenever $i \xrightarrow{\alpha} j$ is a $\mathcal{T}$-arrow such that α itself or α^{-1} appears in σ_X, then the action of α is given by a pair of linear transformations

$$\alpha' : \Phi_{iX} \to \Phi_{jX}, \qquad \alpha'' : \Psi_{iX} \to \Psi_{jX}$$

which are isomorphisms when their domains and codomains are both non-zero. (And necessarily zero otherwise.) Let $\phi_{ji} : T_j \to T_i$ be the irreducible $\mathcal{T}$-map corresponding to α. Then what we need is that if ϕ_{jX} and ϕ_{iX} are both non-zero, then $\phi_{ji} \cdot \phi_{iX} = \phi_{iX} \circ \phi_{ji} = \phi_{jX}$ up to a non-zero scalar, and similarly that if ψ_{iX} and ψ_{jX} are both non-zero, then $\phi_{ji} \cdot \psi_{iX} = \psi_{iX} \circ \phi_{ji} = \psi_{jX}$. The first assertion is clearly true by the structure of the tube. The second assertion holds by an application of Lemma 1.3(i), and Remark 1.4, where we use the fact that ϕ_{ji} must be a composition of maps which are irreducible in $\mathcal{C}_{\mathcal{T}}$, and follows a ray or coray (along the edge of a wing), and thus all the indecomposables that ϕ_{ji} factors through are also in $R^{\mathcal{D}}(X)$.

Next let X be such that the $\mathcal{D}$-arrow $\beta_X : i \to j$ is defined, and thus appears in the string σ_X. Then we know that Φ_{iX} and Ψ_{jX} are non-zero. The action of β_X is given by composition with a $\mathcal{D}$-map $\psi_{ji} : T_j \to T_i$. We wish to show that (up to multiplication by a non-zero scalar) this action is given by a linear transformation

$$\beta'_X : \Phi_{iX} \oplus \Psi_{iX} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} \Phi_{jX} \oplus \Psi_{jX}.$$

In other words, it sends ϕ_{iX} to ψ_{jX} and annihilates ψ_{iX}. The composition $\psi_{iX} \circ \psi_{ji}$ is clearly zero, as all compositions of two $\mathcal{D}$-maps are.

Consider the image $\phi_{iX} \circ \psi_{ji}$ of ϕ_{iX}. We need the observation that the $\mathcal{T}$-map ϕ_{iX} does not factor through any indecomposable to which there is no $\mathcal{D}$-map from T_j. This holds since, by Lemma 4.1, $H^{\mathcal{D}}(T_j) \cap \mathcal{F}$ forms a rectangle-shaped subgraph of the tube, and the map ϕ_{iX} cannot factor through any indecomposable outside this subgraph. We can then conclude from Lemma 1.3(ii) and Remark 1.4 that $\phi_{iX} \circ \psi_{ji} = \psi_{jX}$, which is what we wanted.

It remains to show that if there exists an arrow which connects two vertices in the support of $\mathrm{Hom}_{\mathcal{C}}(T, X)$, but which does not appear in σ_X, then the action of this arrow is zero on $\mathrm{Hom}_{\mathcal{C}}(T, X)$. By Lemmas 3.7, 4.4 and 4.7 the only case to consider is when β_X is the loop vertex, and there is a T-subwing triple $(T_i; T_j, T_k)$ such that $T_j \in R^{\mathcal{T}}(X)$ and $T_k \in R^{\mathcal{D}}(X)$ or vice versa. Since the action of the arrow $\beta : j \to k$ is given by composition with the $\mathcal{D}$-map $\psi_{kj} : T_k \to T_j$, we only need to study the case where $\sigma_X^{\mathcal{T}}$ traverses j and $\sigma_X^{\mathcal{D}}$ traverses k. So we need to show that $\phi_{jX} \circ \psi_{kj} : T_k \to X$ is zero.

But by examining the Hom-hammocks of T_j and T_k, we see that if there is a $\mathcal{T}$-map $T_j \to X$ and a $\mathcal{D}$-map $T_k \to X$, then either X is in $\mathcal{W}_{T_i}$, which contradicts the fact that i must be traversed by σ_X, or ϕ_{iX} factors through objects on the coray $\mathbf{C}_{(n,1)}$. In the latter case, the composition must be zero, since there are no $\mathcal{D}$-maps from T_k to any objects on this coray.

(2) Counting the number of elements of $\mathcal{F}$, we find that it contains $n(n-1)$ objects with quasilength less than n, and $\frac{1}{2}n(n-1)$ with quasilength n or more, that is, a total of $\frac{3}{2}n(n-1)$ elements. Since T has $n-1$ summands, the cardinality of $\mathcal{F} \setminus \mathrm{add}\, \tau T$ is

$$\frac{3}{2}n(n-1) - (n-1) = \frac{1}{2}(3n^2 - 5n + 2),$$

which, by Theorem 3.8, is also the number of indecomposables in $\mathrm{mod}\, \Lambda_T$. By Lemmas 4.5 and 4.6, if X and Y are different objects in $\mathcal{F} \setminus \mathrm{add}\, T$, then $\sigma_X \neq \sigma_Y$. It then follows from part (1) that $\mathrm{Hom}_{\mathcal{C}}(T, X) \not\simeq \mathrm{Hom}_{\mathcal{C}}(T, Y)$. So $\mathrm{Hom}_{\mathcal{C}}(T, -)$ provides a bijection. ∎

We now turn to the indecomposables which are not in $\mathcal{F}$. It is easily seen that Lemma 4.4(ii)–(iv) holds also for indecomposables which are not in $\mathcal{F}$.

So we can define $\sigma_X^{\mathcal{T}}$ and $\sigma_X^{\mathcal{D}}$ in this case as well. The following theorem now completes the description of the action of $\mathrm{Hom}_{\mathcal{C}}(T,-)$ on objects.

THEOREM 4.9. *Let X be an indecomposable object in $\mathcal{C}$, where $X \notin \mathcal{F}$. Then we have the following:*

(1) $\mathrm{Hom}_{\mathcal{C}}(T,X) = 0$ *if and only if* $X = (n, kn-1)$, *where* $k \geq 2$.

(2) *If X is not of the type described in* (1), *then*

$$\mathrm{Hom}_{\mathcal{C}}(T,X) = M(\sigma_X^{\mathcal{T}}) \amalg M(\sigma_X^{\mathcal{D}})$$

where $M(\sigma)$ is the zero module if σ is the zero string.

Proof. (1) When $X \notin \mathcal{F}$ we know that $\mathrm{Hom}_{\mathcal{C}}(T,X) = 0$ if and only if $\mathrm{Hom}_{\mathcal{C}}(T_1,X) = 0$. There are no $\mathcal{T}$-maps $T_1 \to X$ if and only if X is on the ray $\mathbf{R}_{(n,1)}$, that is, $X = (n,t)$ for some $t \geq 1$. Moreover, there are no $\mathcal{D}$-maps $T_1 \to X$ if and only if X is on the coray $\mathbf{C}_{(n,n-1)}$. The indecomposables that are in the intersection of $\mathbf{R}_{(n,1)}$ and $\mathbf{C}_{(n,n-1)}$ and outside $\mathcal{F}$ are exactly the ones with coordinates (n,t) such that $n+t \equiv n+n-1 \bmod n$. The claim follows.

(2) The proof of Theorem 4.8 goes through, with the following exception, which is exactly what is needed. The action of β_X (which in this case is always the loop, as we see from the argument for (1) above) is zero: The $\mathcal{T}$-map $\phi_{1X} : T_1 \to X$ factors through (at least) one object on the coray $\mathbf{C}_{(n,n-1)}$. We know that there are no $\mathcal{D}$-maps from T_1 to any object on this coray. It then follows that the composition $\phi_{1X} \circ \psi_{11}$, where ψ_{11} is the $\mathcal{D}$-endomorphism of T_1, is a zero map. The result follows. ∎

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