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CARDINALITY OF SOME CONVEX SETS AND OF THEIR SETS OF EXTREME POINTS

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Abstract. We show that the cardinality $\mathfrak{n}$ of a compact convex set W in a topological linear space X satisfies the condition that $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$. We also establish some relations between the cardinality of W and that of $\operatorname{extr} W$ provided X is locally convex. Moreover, we deal with the cardinality of the convex set $E(\mu)$ of all quasi-measure extensions of a quasi-measure μ, defined on an algebra of sets, to a larger algebra of sets, and relate it to the cardinality of $\operatorname{extr} E(\mu)$.

1. Introduction. The main body of the paper (*) falls into two parts.

The first part, Sections 2 and 3, is concerned with abstract convex sets in a topological linear space. In Section 2 we show that the cardinality of a compact convex set is an ω-power and so is its algebraic dimension if it is infinite (Theorem 2; a cardinal $\mathfrak{n} \geq 1$ is called an *ω-power* if $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$). This result is deduced from an analogous one concerning closed convex sets in a complete metric linear space (Theorem 1), the dimension part of which is an extension of a result of [16]. In Section 3 we establish some relations between the cardinality of a compact convex set in a locally convex space and that of its set of extreme points (Propositions 1 and 2), and formulate a relevant open problem.

The second part, Sections 4–6, continues the author's study of the convex set $E(\mu)$ of all quasi-measure extensions of a *quasi-measure* μ, i.e., a positive additive function on an algebra $\mathfrak{M}$ of subsets of a set Ω, to a larger algebra $\mathfrak{R}$ of subsets of Ω, and that of the set $\operatorname{extr} E(\mu)$ of its extreme points (see [13]–[15], [17]–[20]). The results of the first part apply here, as $E(\mu)$ is a weak* compact subset of the dual Banach lattice $ba(\mathfrak{R})$. In addition, we use many results from the author's previous papers, especially [15], [18] and [20].

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The relations between the cardinalities of $E(\mu)$ and $\operatorname{extr} E(\mu)$ established in Section 6 (Theorems 5 and 7) involve, in one way or another, a subalgebra $\mathfrak{N}$ of $\mathfrak{R}$, which, jointly with $\mathfrak{M}$, generates $\mathfrak{R}$ as an algebra. In this context, we also relate the cardinality of $E(\mu)$ to that of $\mathfrak{M}(\mu)$, the quotient Boolean algebra of $\mathfrak{M}$ modulo the ideal of μ-null sets (Theorem 6 and Corollary 2).

The notation and terminology used in the second part (as well as in many previous papers by the author, including [13]–[15], [17]–[20]) are explained in Section 4. This section also contains some auxiliary results.

2. Cardinality and dimension of some convex sets. The cardinality of the set I is denoted by $|I|$. Following [4, p. 452], we say that a cardinal $\mathfrak{n} \geq 1$ is an *ω-power* if $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$ for some cardinal $\mathfrak{m}$; equivalently, $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$ (see [3] for relevant information).

Given a subset W of a linear space Y, we denote by $\dim W$ the algebraic dimension of $\operatorname{lin} W$, the linear span of W in Y. The topological linear spaces we consider in this section and the next one are over $\mathbb{R}$ or $\mathbb{C}$ and are assumed to be Hausdorff spaces.

In the proof of our first result, Theorem 1, we shall apply the folowing proposition from general topology: *If X is a complete metric space such that every nonempty open subset of X has weight $\mathfrak{m}$, then $|X| = \mathfrak{m}^{\aleph_0}$.* This proposition goes back to F. K. Schmidt (1932); see [25, Lemma 3.1], [3, p. 184] or [12, p. 52]. In fact, we shall need an equivalent form of it with "weight" replaced by "density character". (We denote by $\mathfrak{d}(X)$ the density character of X.) That those cardinal functions coincide for arbitrary metric spaces is well known (see [6, Theorem 4.1.15]).

THEOREM 1. *Let W be a closed convex set in a complete metric linear space X. Then*

(a) $|W|$ *is an ω-power;*
(b) $\dim W$ *is either finite or an ω-power.*

Proof. (a): We assume that $|W| > 1$. Let U be a nonempty (relatively) open subset of W. We claim that $\mathfrak{d}(U) \geq \mathfrak{d}(W)$. This implies the assertion, by the proposition formulated above. To establish the claim, assume additionally that $0 \in U$. (This can be achieved using a suitable translation of W.) We then have $\mathbb{N} \cdot U \supset W$. Indeed, for each $x \in W$ the mapping $[0, 1] \ni t \mapsto tx \in W$ is continuous and its value at $t = 0$ belongs to U. Hence $(1/n)x$ is in U for n large enough. It follows that U is infinite and

$$\mathfrak{d}(U) = \mathfrak{d}(\mathbb{N} \cdot U) \geq \mathfrak{d}(W).$$

Thus, the claim is established.

(b): Suppose $\dim W \geq \aleph_0$. Then, by [16, Theorem 6], $\dim W \geq 2^{\aleph_0}$. Consequently, $\dim W = |W|$ and an application of (a) completes the proof. ■

The special case of Theorem 1(a) with $W = X$ appears in [12, p. 51, Theorem]. For this special case and X a Banach space see also [1, p. 347, Theorem], [3, p. 184, Theorem] or [11, Lemma 2].

THEOREM 2. *Let W be a compact convex set in a topological linear space X. Then*

(a) $|W|$ *is an ω-power;*
(b) $\dim W$ *is either finite or an ω-power.*

Proof. Without loss of generality, we additionally assume that $|W| > 1$, $\operatorname{lin} W = X$ and W is absolutely convex. As for this last assumption, note that if W is compact and convex, then so are the sets $W - W$ and $[0,1] \cdot W$, and they both have the same cardinality and dimension as W.

Under these additional assumptions, the gauge of W defines a Banach space topology on X stronger than the original one; see [21, Proposition 3.2.2 and Corollary 3.2.5]. (The standing local convexity assumption made in [21] is not used in the proofs of those results.) Thus, (a) and (b) follow from the corresponding parts of Theorem 1. ∎

We note that every ω-power $\mathfrak{n} \geq 1$ is the cardinality and the dimension of some Banach or even Hilbert space (see [1, p. 348], [3, p. 186] or [11, Lemma 2 and Corollary 2.4]). This shows that Theorem 1 is, in some sense, best possible, and so is Theorem 2, in view of the Banach–Alaoglu theorem. In the latter connection see also Remark 3 in Section 3, and [15, Proposition 1(a)] and Remark 4 in Section 4.

3. Relations between the cardinality of a compact convex set and that of its set of extreme points. We shall establish some inequalities and equalities involving the cardinalities in question and present an open problem.

PROPOSITION 1. *Let W be a compact convex set in a locally convex space. Then*

(a) $|\operatorname{extr} W|^{\aleph_0} \leq |W| \leq 2^{2^{\aleph_0 \cdot |\operatorname{extr} W|}}$;
(b) $|W| \leq 2^{\aleph_0 \cdot |\overline{\operatorname{extr} W}|}$;
(c) $|\operatorname{extr} W|^{\aleph_0} = |W|$ *if W is metrizable;*
(d) $|\operatorname{extr} W|^{\aleph_0} = |W|$ *if* $\operatorname{extr} W$ *is a continuous image of a separable metric space* (*in particular, if* $\operatorname{extr} W$ *is countable*).

Proof. (a): The first inequality is seen from Theorem 2(a). By the Krein–Milman theorem, W contains a dense subset of cardinality $\aleph_0 \cdot |\operatorname{extr} W|$. Thus, the second inequality of (a) follows from a cardinal inequality for general Hausdorff spaces (see [6, Theorem 1.5.3]).

(b): By [22, Proposition 1.2],

$$|W| \leq |S(\overline{\operatorname{extr} W})|,$$

where $S(\overline{\operatorname{extr} W})$ stands for the set of Radon probability measures on $\overline{\operatorname{extr} W}$, and so the result follows from Proposition 3(a) below.

(c) follows from (a) and the Krein–Milman theorem.

(d) follows from (c), since the additional assumption on $\operatorname{extr} W$ implies that W is metrizable due to a result of Corson and Haydon (see [9, Theorem 4.6]). In the countable case, one can alternatively apply the result that W then coincides with the σ-convex hull of $\operatorname{extr} W$ (see [8, Theorem 2]). ∎

PROPOSITION 2. *If W is a weakly compact convex set in a Banach space, then* $|\operatorname{extr} W|^{\aleph_0} = |W|$.

Proof. In view of Theorem 2(a), we only need to show the inequality "$\geq$". Combining the Krein–Milman theorem and a theorem due essentially to Mazur (see [5], Theorem V.3.13), we get that W coincides with the norm closure of $\operatorname{conv} \operatorname{extr} W$. It follows that

$$|W| \leq |\operatorname{conv} \operatorname{extr} W|^{\aleph_0} \leq (2^{\aleph_0} \cdot |\operatorname{extr} W|)^{\aleph_0} = |\operatorname{extr} W|^{\aleph_0}$$

provided $|W| > 1$. ∎

EXAMPLE 1. For every cardinal $\mathfrak{m} \geq 1$ there exists a weakly compact convex set W in a Hilbert space such that $|\operatorname{extr} W| = \mathfrak{m}$. This is plain if $\mathfrak{m}$ is finite. In the opposite case, let Ω be a set of cardinality $\mathfrak{m}$, and denote by e_ω, $\omega \in \Omega$, the elements of the standard orthonormal basis of $l_2(\Omega)$. Set

$$W = \Big\{ \sum_{\omega \in \Omega} t_\omega e_\omega : t_\omega \geq 0 \text{ and } \sum_{\omega \in \Omega} t_\omega \leq 1 \Big\}.$$

Clearly, W is a convex subset of the unit ball of $l_2(\Omega)$. Moreover, it is seen to be weakly closed, and so weakly compact, by the Banach–Alaoglu theorem. Finally, we have

$$\operatorname{extr} W = \{e_\omega : \omega \in \Omega\} \cup \{0\}.$$

Recall that a topological space is said to be *scattered* if it contains no dense-in-itself nonempty subset (see [24, p. 147]). Given a compact (Hausdorff) space Z, we denote by $S(Z)$ the set of all probability Radon measures on Z. This notation follows essentially [24, Section 18.2.3].

PROPOSITION 3. *Let Z be a compact space. Then*

(a) $|Z|^{\aleph_0} \leq |S(Z)| \leq 2^{\aleph_0 \cdot |Z|}$*;*

(b) $|Z|^{\aleph_0} = |S(Z)|$ *if Z is scattered.*

For part (a) see [7, p. 172]. Part (b) follows from the fact that a compact space Z is scattered if and only if every Radon measure on Z is atomic (see [24, Theorem 19.7.6]).

Fremlin and Plebanek [7, Theorem 3A] construct, under Martin's axiom, a (zero-dimensional) compact space Z such that $|Z| = 2^{\aleph_0}$ and $|S(Z)| = 2^{2^{\aleph_0}}$. This shows that the equality of Proposition 3(b) does not hold in general.

Remark 1. By the Riesz representation theorem (see [24, Section 18]), $S(Z)$ can be identified with a compact convex subset of the dual Banach space $C(Z)^*$ equipped with its weak* topology. Moreover, $\operatorname{extr} S(Z)$ can be identified with Z (cf. [5, Lemma V.8.6]). Thus, the first inequality of Proposition 3(a) is, in fact, a special case of the corresponding inequality of Proposition 1(a) while Proposition 1(b) is an extension of the second inequality of Proposition 3(a).

Theorem 2(a) and Propositions 1–3 suggest the following problem. For which pairs $\mathfrak{m}$, $\mathfrak{n}$ of cardinals does there exist a compact convex set W in a locally convex space such that

$$|W| = \mathfrak{n} \quad \text{and} \quad |\operatorname{extr} W| = \mathfrak{m}?$$

The following restrictions on $\mathfrak{m}$ and $\mathfrak{n}$ are either obvious or are direct consequences of Proposition 1(a) and Theorem 2(a):

$$1 \le \mathfrak{m} \le \mathfrak{n} \le 2^{2^{\aleph_0 \cdot \mathfrak{m}}} \quad \text{and} \quad \mathfrak{n}^{\aleph_0} = \mathfrak{n}.$$

By Remark 4 in Section 4 and [15, Proposition 1(a)], each pair $\mathfrak{m}$, $\mathfrak{m}^{\aleph_0}$, where $\mathfrak{m} \ge 1$, has the property in question. So does, under Martin's axiom, the pair $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, according to the Fremlin–Plebanek example mentioned above (cf. Remark 1). Thus, $\mathfrak{m} = 2^{\aleph_0}$ does not determine the desired $\mathfrak{n}$ uniquely. However, if $1 \le \mathfrak{m} \le \aleph_0$, then $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$ (for $\mathfrak{m} = \aleph_0$ see Proposition 1(d)).

The problem formulated above seems to be open even in various specialized forms. For example, one can restrict attention to unit balls of dual Banach spaces or to sets $E(\mu)$ considered in the rest of the paper (see the next section for definition). However, in two important special cases a complete answer can be given without difficulty.

Remark 2. Let W be a metrizable compact convex set in a locally convex space and let $\mathfrak{m}$ and $\mathfrak{n}$ be as above. Then either $\mathfrak{n} = 1$ or $\mathfrak{n} = 2^{\aleph_0}$. In the former case $\mathfrak{m} = 1$ while in the latter case $\mathfrak{m}$ is either finite and > 1 or $\aleph_0$ or $2^{\aleph_0}$. (This is because $\operatorname{extr} W$ is then a G_δ-set, see [22, Proposition 1.3].) Moreover, all those possibilities can occur, even for subsets of $\mathbb{R}^2$.

Remark 3. Let W be a weakly compact convex set in a Banach space and let $\mathfrak{n}$ and $\mathfrak{m}$ be as in the paragraph following Remark 1. Then $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$ and each $\mathfrak{m} \ge 1$ can occur (see Proposition 2 and Example 1).

4. Notation, terminology and auxiliary results on quasi-measures. Throughout the rest of the paper, Ω stands for a nonempty set. The algebra of subsets of Ω generated by $\mathfrak{C} \subset 2^{\Omega}$ is denoted by $\mathfrak{C}_b$.

Let $\mathfrak{M}$ be an algebra of subsets of Ω, and denote by $ba(\mathfrak{M})$ the Banach lattice of all real-valued bounded additive functions on $\mathfrak{M}$ (see [2, Section 2.2]). As usual, $|\varphi|$ stands for the modulus of $\varphi \in ba(\mathfrak{M})$ and $\wedge$ for the minimum operation in $ba(\mathfrak{M})$.

If $\mathfrak{M}$ is a σ-algebra, we set

$$ca(\mathfrak{M}) = \{\varphi \in ba(\mathfrak{M}) : \varphi \text{ is countably additive}\}.$$

Let $\mu \in ba_+(\mathfrak{M})$. We denote by $\mathfrak{M}(\mu)$ the quotient Boolean algebra of $\mathfrak{M}$ modulo the ideal of μ-null sets. We set

$$[0, \mu] = \{\nu \in ba_+(\mathfrak{M}) : \nu \le \mu\}.$$

We say that μ is (*purely*) *atomic* if there exist a (at most) two-valued $\mu_i \in ba_+(\mathfrak{M})$ such that $\mu = \sum_{i=1}^{\infty} \mu_i$ (see [2, p. 213]). We say that μ is *nonatomic* provided for every $\varepsilon > 0$ there exists an $\mathfrak{M}$-partition $\{M_1, \ldots, M_n\}$ of Ω with $\mu(M_i) < \varepsilon$ for all i (see [2, Definition 5.1.4], where the term *strongly continuous* is used).

We set

$$pa(\mathfrak{M}) = \{\mu \in ba_+(\mathfrak{M}) : \mu(\Omega) = 1\},$$
$$ult(\mathfrak{M}) = \{\mu \in pa(\mathfrak{M}) : \mu \text{ is two-valued}\}.$$

For $\mu, \nu \in ba_+(\mathfrak{M})$ we write $\nu \ll \mu$ if ν is absolutely continuous with respect to μ, i.e., the familiar ε–δ condition holds (see [2, Definition 6.1.1]).

In the proof of Theorem 6 in Section 6 we shall need the following lemma.

LEMMA 1. *For every $\mu \in ba_+(\mathfrak{M})$ we have*

$$|[0, \mu]| = |\mathfrak{M}(\mu)|^{\aleph_0}.$$

Proof. For every $N \in \mathfrak{M}$ define μ_N in $[0, \mu]$ by

$$\mu_N(M) = \mu(M \cap N), \quad M \in \mathfrak{M}.$$

Given $N_1, N_2 \in \mathfrak{M}$, we then have

$$\mu_{N_1} - \mu_{N_2} = \mu_{N_1 \setminus N_2} - \mu_{N_2 \setminus N_1},$$

and so

$$\|\mu_{N_1} - \mu_{N_2}\| = \mu(N_1 \triangle N_2).$$

This shows that

$$|\{\mu_N : N \in \mathfrak{M}\}| = |\mathfrak{M}(\mu)|.$$

Given $\nu \in [0, \mu]$ and $\varepsilon > 0$, there exist $N_1, \ldots, N_n \in \mathfrak{M}$ and rational numbers $t_1, \ldots, t_n$ such that

$$\Big\|\nu - \sum_{i=1}^{n} t_i \mu_{N_i}\Big\| < \varepsilon$$

(see [2, Theorem 6.3.4, (i)⇒(iii)]). This implies that

$$|\{\mu_N : N \in \mathfrak{M}\}|^{\aleph_0} \geq |[0,\mu]|.$$

In view of Theorem 1(a), the converse inequality also holds, and so the assertion follows. ∎

Let $\mathfrak{R}$ be an algebra of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let $\mu \in ba_+(\mathfrak{M})$. We set

$$E(\mu) = \{\varrho \in ba_+(\mathfrak{R}) : \varrho|\mathfrak{M} = \mu\}.$$

Occasionally, we shall use the more comprehensive notation $E(\mu, \mathfrak{R})$ instead of $E(\mu)$.

Recall that $E(\mu)$ is always nonempty. In fact, we even have

(C) $\operatorname{extr} E(\mu) \neq \emptyset$

(see [15, pp. 351–352] for references). In the proof of Lemma 2 below we shall apply the following extremality criterion due to Plachky [23, Theorem 1]; see [15, p. 352] for more references.

(D) *Let* $\varrho \in E(\mu)$. *Then* $\varrho \in \operatorname{extr} E(\mu)$ *if and only if for every* $R \in \mathfrak{R}$ *and every* $\varepsilon > 0$ *there exists* $M \in \mathfrak{M}$ *with* $\varrho(R \triangle M) < \varepsilon$.

A direct consequence of (D) is

(D)$'$ *If* μ *is two-valued, then* $\operatorname{extr} E(\mu) = \{\varrho \in E(\mu) : \varrho(\mathfrak{R}) = \mu(\mathfrak{M})\}$.

Note that for $\mathfrak{M} = \{\emptyset, \Omega\}$ and $\mu(\Omega) = 1$ we have $E(\mu) = pa(\mathfrak{R})$, and so (D)$'$ then takes the following form:

(D)$''$ $\operatorname{extr} pa(\mathfrak{R}) = ult(\mathfrak{R})$

(see [23, Remark 1 on Theorem 1]). This equality will be used in the proofs of Proposition 3$'$ and Theorem 3 in the next section.

Throughout the rest of the paper, $\mathfrak{M}$ *and* $\mathfrak{R}$ *stand for algebras of subsets of* Ω *with* $\mathfrak{M} \subset \mathfrak{R}$.

Part (a) of the next result will be used in the proof of Theorem 5 in Section 6. Some conditions sufficient for the inequality of (a) to turn into equality are given in the next two sections (see Theorem 3(b), Remark 7 and Theorem 5(a)).

COROLLARY 1. *For every* $\mu \in ba_+(\mathfrak{M})$ *we have*

(a) $|E(\mu)| \geq |\operatorname{extr} E(\mu)|^{\aleph_0}$;
(b) $|E(\mu)| = |\operatorname{extr} E(\mu)|$ *is an* ω*-power if* $E(\mu)$ *is weakly compact and* $\operatorname{extr} E(\mu)$ *is infinite.*

Proof. Part (a) is a direct consequence of Theorem 1(a), since $E(\mu)$ is closed in the Banach space $ba(\mathfrak{R})$. Part (b) follows from Proposition 2 and [19, Corollary 1(b)]. ∎

Remark 4 (cf. Proposition 3′(b) below). For every cardinal $\mathfrak{m} \geq 1$ there exist Ω, $\mathfrak{M}$, $\mathfrak{R}$ and μ as above such that

$$|E(\mu)| = \mathfrak{m}^{\aleph_0} \quad \text{and} \quad |\operatorname{extr} E(\mu)| = \mathfrak{m}.$$

Indeed, let $\mathfrak{m} > 1$ and let Ω be a set with $|\Omega| = \mathfrak{m}$. Take for $\mathfrak{R}$ the algebra of finite subsets of Ω and their complements, and note that $|pa(\mathfrak{R})| = \mathfrak{m}^{\aleph_0}$ and $|ult(\mathfrak{R})| = \mathfrak{m}$. Let $\mathfrak{M} = \{\emptyset, \Omega\}$ and $\mu(\Omega) = 1$, and apply (D)″.

Remark 5 (cf. Remarks 2 and 3). Let $E(\mu)$, where $\mu \in ba_+(\mathfrak{M})$, be weakly compact, and set

$$\mathfrak{n} = |E(\mu)| \quad \text{and} \quad \mathfrak{m} = |\operatorname{extr} E(\mu)|.$$

Then either $1 < \mathfrak{m} < \aleph_0$ and $\mathfrak{n} = 2^{\aleph_0}$ (by Proposition 2) or $\mathfrak{m}$ is an ω-power and $\mathfrak{n} = \mathfrak{m}$ (by Corollary 1(b)). Moreover, all those possibilities can occur (see Remark 4, and [18, Example 3] and [15, Theorem 1(a)]). In the special case where $E(\mu)$ is strongly compact we have either $\mathfrak{m} = \mathfrak{n} = 1$ or $1 < \mathfrak{m} < \aleph_0$ and $\mathfrak{n} = 2^{\aleph_0}$ or $\mathfrak{m} = \mathfrak{n} = 2^{\aleph_0}$, and all those possibilities can occur (see [19, Remark 1]).

The following two lemmas seem to be of some interest in themselves. Lemma 2 will be applied in the proof of Lemma 3. The latter will be applied, in turn, in the proof of Theorem 4(b) in Section 6.

Lemma 2. *Let $\mu, \nu \in ba_+(\mathfrak{M})$ satisfy $\nu \geq \mu$. If $\pi \in \operatorname{extr} E(\mu)$, then there exists $\sigma \in \operatorname{extr} E(\mu)$ with $\sigma \geq \pi$.*

Proof. We first establish the assertion under the additional assumption that $\nu \ll \mu$. Setting

$$d_\pi(R_1, R_2) = \pi(R_1 \triangle R_2), \qquad R_1, R_2 \in \mathfrak{R},$$

we define a pseudometric on $\mathfrak{R}$. By (D), $\mathfrak{M}$ is dense in $\mathfrak{R}$ with respect to d_π. Moreover, ν is uniformly continuous on $(\mathfrak{R}, d_\pi)$, since $\nu \ll \mu$ is assumed. Let σ be a (unique) continuous extension of ν to $(\mathfrak{R}, d_\pi)$. Clearly, σ is in $ba_+(\mathfrak{R})$, $\sigma \ll \pi$ and $\sigma \geq \pi$. In view of (D), σ is in $\operatorname{extr} E(\nu)$.

In the general case, according to the Lebesgue decomposition theorem (see [2, Theorem 6.2.5]), there exist $\nu_1, \nu_2 \in ba_+(\mathfrak{M})$ with

$$\nu = \nu_1 + \nu_2, \qquad \nu_1 \ll \mu \quad \text{and} \quad \nu_2 \wedge \mu = 0.$$

We then have $\nu_1 \wedge \nu_2 = 0$ and $\nu_1 \geq \mu$, by Remark 6.1.20(ii) and Theorem 2.2.1(b) of [2], respectively. Applying what we have proved so far, we get σ_1 in $\operatorname{extr} E(\nu_1)$ with $\sigma_1 \geq \pi$. Choose σ_2 in $\operatorname{extr} E(\nu_2)$ arbitrarily (see (C)), and set $\sigma = \sigma_1 + \sigma_2$. In view of [15, Lemma 2(b)], σ is in $\operatorname{extr} E(\nu)$. ∎

LEMMA 3. *Let* $\mu_1, \mu_2 \in ba_+(\mathfrak{M})$ *and let* $\pi_1 \in \operatorname{extr} E(\mu_1)$. *Then there exists* $\pi_2 \in \operatorname{extr} E(\mu_2)$ *such that*

$$\|\pi_2 - \pi_1\| = \|\mu_2 - \mu_1\|.$$

Proof. Set $\mu = \mu_1 \wedge \mu_2$ and $\mu_i' = \mu_i - \mu$, $i = 1, 2$. By [15, Lemma 2(a)], there exist $\pi_1' \in \operatorname{extr} E(\mu_1')$ and $\pi \in \operatorname{extr} E(\mu)$ with $\pi_1 = \pi_1' + \pi$. Choose π_2 in $\operatorname{extr} E(\mu_2)$ with $\pi_2 \geq \pi$ (see Lemma 2), and set $\pi_2' = \pi_2 - \pi$. Clearly, π_2' is in $E(\mu_2')$. In view of [2, Theorem 2.2.1(7)], $\pi_1' \wedge \pi_2' = 0$, since $\mu_1' \wedge \mu_2' = 0$. It follows that

$$\|\pi_2' - \pi_1'\| = \|\pi_2'\| + \|\pi_1'\| = \|\mu_2'\| + \|\mu_1'\| = \|\mu_2' - \mu_1'\|$$

(see [2, Theorems 1.5.4(21) and 2.2.2(7)]). This implies the assertion. ∎

Recall that an algebra $\mathfrak{N}$ of sets is said to be *superatomic* if every subalgebra of $\mathfrak{N}$ is atomic (see [10, Proposition 17.5] or [2, Definition 5.3.4]). This notion appears below in Proposition 3′(b), Remark 7 and Theorem 5, and in the proof of Theorem 3.

5. $E(\mu)$ for atomic μ.

Recall that, as before, $\mathfrak{M}$ and $\mathfrak{R}$ denote algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$. We shall need the following version of Proposition 3.

PROPOSITION 3′. *We have*

(a) $|ult(\mathfrak{R})|^{\aleph_0} \leq |pa(\mathfrak{R})| \leq 2^{\aleph_0 \cdot |ult(\mathfrak{R})|}$;

(b) $|ult(\mathfrak{R})|^{\aleph_0} = |pa(\mathfrak{R})|$ *if* $\mathfrak{R}$ *is superatomic.*

Proof. We shall establish both assertions in two different ways.

1. Denote by Z the Stone space of $\mathfrak{R}$. By [10, Remark 17.2], $\mathfrak{R}$ is superatomic if and only if Z is scattered. In view of [20, Proposition 1(a)], $pa(\mathfrak{R})$ and $S(Z)$ are, in particular, equipotent. Thus, (a) and (b) follow from the corresponding parts of Proposition 3.

2. The first inequality of (a) follows from the injectivity of the mapping

$$(\pi_n) \mapsto \sum_{n=1}^{\infty} \frac{2}{3^n} \pi_n$$

of $(ult(\mathfrak{R}))^{\mathbb{N}}$ into $pa(\mathfrak{R})$. As for the second one, we have $|\mathfrak{R}| \leq \aleph_0 \cdot |ult(\mathfrak{R})|$, by a standard result (see [10, Theorem 5.31]), and so

$$|pa(\mathfrak{R})| \leq |\mathbb{R}^{\mathfrak{R}}| \leq 2^{\aleph_0 \cdot |ult(\mathfrak{R})|}.$$

The assumption of (b) implies that every element of $pa(\mathfrak{R})$ is atomic (see [2, Theorem 5.3.6]). Thus, (b) follows from (a). ∎

The next result extends a part of Proposition 3′.

THEOREM 3. *Let* $\mu \in ba_+(\mathfrak{M})$.

(a) *If* μ *is atomic, then*

$$|E(\mu)| \le 2^{\aleph_0 \cdot |\operatorname{extr} E(\mu)|}.$$

(b) *If each element of* $E(\mu)$ *is atomic, then*

$$|E(\mu)| = |\operatorname{extr} E(\mu)|^{\aleph_0}.$$

Proof. Under the assumption of (a), $E(\mu)$ is affinely isomorphic to the countable Cartesian product $\prod_j pa(\mathfrak{R}_j)$, where $\mathfrak{R}_j$ are algebras of sets, by [20, Theorem 1(a)]. Under the assumption of (b), those algebras can be chosen, in addition, superatomic, by [20, Theorem 3, (ii)⇒(iv)]. In view of (D)$''$, (a) and (b) now follow from the corresponding parts of Proposition 3$'$. ∎

Alternatively, in view of [15, Proposition 1(a) and Theorem 2(a)], Theorem 3(a) is a consequence of Proposition 1(b).

REMARK 6. The inequality of Theorem 3(a) and the corresponding inequality of Propositon 3$'$(a) cannot be improved, at least under Martin's axiom, due to the Fremlin–Plebanek example mentioned after the proof of Proposition 3. As for the latter, this is seen from its first proof. Now, the former is, in fact, an extension of the latter. For the same reason, Theorems 6 and 7 in Section 6 cannot be improved either.

REMARK 7. The equality of Theorem 3(b) also holds if $E(\mu)$ is weakly compact or $\operatorname{extr} E(\mu)$ is countable. The former assertion is a special case of Proposition 2 while the latter follows from Proposition 1(d) combined with [15, Proposition 1(a)]. Alternatively, we can deduce the latter from [17, Theorem 5] and Proposition 3$'$(b), since every algebra $\mathfrak{R}$ of sets with $ult(\mathfrak{R})$ countable is superatomic (see [24, Proposition 8.5.7] and [10, Remark 17.2]). For two other conditions sufficient for the equality of Theorem 3(b) to hold see Theorem 5 in Section 6.

6. $E(\mu)$ for arbitrary μ.

Throughout this section, $\mathfrak{N}$ *denotes an algebra of subsets of* Ω *with* $(\mathfrak{M} \cup \mathfrak{N})_b = \mathfrak{R}$.

Part (a) of our next result is an extension of the author's version of a theorem due to R. Bierlein (see [13, Theorem 3] and [14, Theorem 3(d)]). It is applied in the proof of part (b) thereof, which is, in turn, applied in the proof of Theorem 5(a).

THEOREM 4. *Let* $\mu \in ba_+(\mathfrak{M})$ *and* $\varrho \in E(\mu)$.

(a) *If* $\varrho(\mathfrak{N})$ *is finite, then there exist* $t_i > 0$ *and* $\pi_i \in \operatorname{extr} E(\mu)$ *such that* $\sum_{i=1}^{\infty} t_i = 1$ *and* $\varrho = \sum_{i=1}^{\infty} t_i \pi_i$.

(b) *If* $\varrho|\mathfrak{N}$ *is atomic, then* ϱ *is in the strong closure of* $\operatorname{conv}\operatorname{extr} E(\mu)$.

Proof. (a): Set

$$\mathfrak{I} = \{N \in \mathfrak{N} : \varrho(N) = 0\}, \qquad \mathfrak{M}_{\mathfrak{I}} = (\mathfrak{M} \cup \mathfrak{I})_b \quad \text{and} \quad \varrho_{\mathfrak{I}} = \varrho|\mathfrak{M}_{\mathfrak{I}}.$$

Clearly, $\mathfrak{I}$ is an ideal in $\mathfrak{N}$ and

$$\mathfrak{M}_{\mathfrak{I}} = \{M \triangle N : M \in \mathfrak{M} \text{ and } N \in \mathfrak{I}\}.$$

By (D), $\varrho_{\mathfrak{I}}$ is in $\operatorname{extr} E(\mu, \mathfrak{M}_{\mathfrak{I}})$. The assumption on ϱ implies that the quotient Boolean algebra $\mathfrak{N}/\mathfrak{I}$ is finite. Therefore, there exists a finite partition $\mathfrak{E}$ of Ω such that

$$\mathfrak{N} = (\mathfrak{E} \cup \mathfrak{I})_b,$$

and so $\mathfrak{R} = (\mathfrak{M}_{\mathfrak{I}} \cup \mathfrak{E})_b$. By [13, Theorem 3] or [14, Theorem 3(d)], there exist $t_i > 0$ and $\pi_i \in \operatorname{extr} E(\varrho_{\mathfrak{I}}, \mathfrak{R})$ with

$$\sum_{i=1}^{\infty} t_i = 1 \quad \text{and} \quad \varrho = \sum_{i=1}^{\infty} t_i \pi_i.$$

Clearly, π_i is in $\operatorname{extr} E(\mu)$.

(b): By assumption, there exist at most two-valued $\nu_j \in ba_+(\mathfrak{N})$ such that

$$\varrho|\mathfrak{N} = \sum_{j=1}^{\infty} \nu_j.$$

In view of [15, Lemma 2(a)], we can find $\varrho_j \in E(\nu_j, \mathfrak{R})$ with $\varrho = \sum_{j=1}^{\infty} \varrho_j$. Fix $\varepsilon > 0$, and choose n so that

$$\Big\| \varrho - \sum_{j=1}^{n} \varrho_j \Big\| < \varepsilon.$$

Set

$$\varrho' = \sum_{j=1}^{n} \varrho_j, \qquad \mu_j = \varrho_j|\mathfrak{M} \quad \text{and} \quad \mu' = \sum_{j=1}^{n} \mu_j.$$

We have $\varrho' \in E(\mu')$ and $\varrho'(\mathfrak{N})$ is finite. It now follows from (a) that there exist $t_i > 0$ and $\pi_i' \in \operatorname{extr} E(\mu')$ such that

$$\sum_{i=1}^{\infty} t_i = 1 \quad \text{and} \quad \varrho' = \sum_{i=1}^{\infty} t_i \pi_i'.$$

According to Lemma 3, we can find $\pi_i \in \operatorname{extr} E(\mu)$ with

$$\|\pi_i - \pi_i'\| = \|\mu - \mu'\| < \varepsilon.$$

It follows that

$$\Big\| \varrho - \sum_{i=1}^{\infty} t_i \pi_i \Big\| \le \|\varrho - \varrho'\| + \Big\| \sum_{i=1}^{\infty} t_i (\pi_i - \pi_i') \Big\| < 2\varepsilon.$$

This implies the assertion. ∎

THEOREM 5. *Let $\mu \in ba_+(\mathfrak{M})$ and let $\mathfrak{N}$ be superatomic or countable. Then*

(a) $|E(\mu)| = |\mathrm{extr}\, E(\mu)|^{\aleph_0}$*;*
(b) $|E(\mu)| = |\mathrm{extr}\, E(\mu)|$ *if μ is nonatomic.*

Proof. (a): In view of Corollary 1(a), we only need to show the inequality "$\le$". For $\mathfrak{N}$ superatomic this inequality is a consequence of Theorem 4(b) and [2, Theorem 5.3.6]. Suppose then $\mathfrak{N}$ is countable, and denote its elements by $N_1, N_2, \ldots$. Set

$$\mathfrak{R}_i = (\mathfrak{M} \cup \{N_1, \ldots, N_i\})_b.$$

Clearly, the mapping

$$\varrho \mapsto (\varrho|\mathfrak{R}_i)_{i=1}^{\infty}$$

of $E(\mu)$ into $\prod_{i=1}^{\infty} E(\mu, \mathfrak{R}_i)$ is injective. Moreover, we have

$$\Big|\prod_{i=1}^{\infty} E(\mu, \mathfrak{R}_i)\Big| \le \prod_{i=1}^{\infty} |\mathrm{extr}\, E(\mu, \mathfrak{R}_i)|^{\aleph_0} \le |\mathrm{extr}\, E(\mu)|^{\aleph_0},$$

with the first inequality following from [13, Theorem 3], an extension of which is Theorem 4(a) above, or [15, Theorem 1(a)] and Proposition 2, and the second one being a consequence of (C). This completes the proof of (a).

(b): This follows from (a) and [18, Theorem 1]. ∎

Recall that the equality of Theorem 5(b) also holds if $E(\mu)$ is weakly compact and $\mathrm{extr}\, E(\mu)$ is infinite (see Corollary 1(b)). Here is one more example where this equality holds.

EXAMPLE 2. Let $\Omega = [0,1]$ and $\mathfrak{R} = 2^{\Omega}$. Moreover, let $\mathfrak{M}$ be the σ-algebra of Lebesgue measurable subsets of Ω and let μ be the Lebesgue measure on $\mathfrak{M}$. Both $E(\mu)$ and $\mathrm{extr}\, E(\mu)$ then have cardinality $2^{2^{\mathfrak{c}}}$ (see [18, Example 2] for details).

THEOREM 6. *For every $\mu \in ba_+(\mathfrak{M})$ we have*

$$|E(\mu)| \le |\mathfrak{M}(\mu)|^{\aleph_0 \cdot |\mathfrak{N}|}.$$

Proof. We consider two cases. First, let $\mathfrak{N}$ be finite. We can then find a partition $E_1, \ldots, E_n$ of Ω with

$$\mathfrak{N} = \{E_1, \ldots, E_n\}_b.$$

It follows that

$$\mathfrak{R} = \Big\{\bigcup_{i=1}^{n} M_i \cap E_i : M_i \in \mathfrak{M},\ i = 1, \ldots, n\Big\}.$$

For $\varrho \in E(\mu)$ we define ϱ_i, $i = 1, \ldots, n$, by

$$\varrho_i(M) = \varrho(M \cap E_i), \qquad M \in \mathfrak{M}$$

(cf. [14, Theorem 3(b)]). We have

$$\varrho_i(M) \le \varrho(M) = \mu(M), \qquad M \in \mathfrak{M},$$

and so ϱ_i is in $[0,\mu]$. The mapping

$$\varrho \mapsto (\varrho_1, \ldots, \varrho_n)$$

of $E(\mu)$ into $[0,\mu]^n$ is injective. Indeed, fix $\varrho, \varrho' \in E(\mu)$ with $\varrho \ne \varrho'$. Then

$$\varrho\Big(\bigcup_{i=1}^{n} M_i \cap E_i\Big) \ne \varrho'\Big(\bigcup_{i=1}^{n} M_i \cap E_i\Big)$$

for some $M_1, \ldots, M_n \in \mathfrak{M}$. Consequently, $\varrho(M_i \cap E_i) \ne \varrho'(M_i \cap E_i)$ for at least one i, that is, $\varrho_i \ne \varrho_i'$. An application of Lemma 1 now yields

$$|E(\mu)| \le |\mathfrak{M}(\mu)|^{\aleph_0},$$

which shows that the assertion holds for finite $\mathfrak{N}$.

Let now $\mathfrak{N}$ be infinite, and set

$$\mathcal{P} = \{(\mathfrak{M} \cup \mathfrak{E})_b : \mathfrak{E} \text{ is a finite subfamily of } \mathfrak{N}\}.$$

We have $|\mathcal{P}| \le |\mathfrak{N}|$. Moreover, the mapping

$$\varrho \mapsto (\varrho|\mathfrak{P})_{\mathfrak{P}\in\mathcal{P}}$$

of $E(\mu)$ into $\prod_{\mathfrak{P}\in\mathcal{P}} E(\mu,\mathfrak{P})$ is injective. Using the finite case of the assertion established above, we get

$$\Big|\prod_{\mathfrak{P}\in\mathcal{P}} E(\mu,\mathfrak{P})\Big| \le |\mathfrak{M}(\mu)|^{\aleph_0\cdot|\mathcal{P}|} = |\mathfrak{M}(\mu)|^{|\mathfrak{N}|}$$

and the infinite case follows. ■

Corollary 2. *Let $\mathfrak{M}$ be a σ-algebra, let $\mathfrak{N}$ be countable, and let $\mu \in ca_+(\mathfrak{M})$ have infinite range. Then*

$$|E(\mu)| \le |\mathfrak{M}(\mu)|.$$

Proof. Under our assumptions, $\mathfrak{M}(\mu)$ is a complete Boolean algebra. Moreover, it is infinite. Thus, $|\mathfrak{M}(\mu)|$ is an ω-power, according to a theorem of R. S. Pierce (see [4, Corollary 11.6]). An application of Theorem 6 completes the proof. ■

Proposition 3 of [18] shows that the inequality of Corollary 2 can turn into equality even if $\mathfrak{N}$ consists of only four sets.

Theorem 7. *Let $\mu \in ba_+(\mathfrak{M})$ and let $M \cap N \ne \emptyset$ for all $M \in \mathfrak{M}$ with $\mu(M) > 0$ and nonempty $N \in \mathfrak{N}$. Then*

$$|E(\mu)| \le 2^{\aleph_0\cdot|\mathrm{extr}\, E(\mu)|}.$$

Proof. We may assume that $\mu(\Omega) = 1$ and there is an N_0 in $\mathfrak{N}$ with $N_0 \neq \emptyset, \Omega$. In view of [18, Proposition 3], we then have

$$|\mathrm{extr}\, E(\mu, (\mathfrak{M} \cup \{N_0\})_b)| \geq |\mathfrak{M}(\mu)|.$$

This implies $|\mathrm{extr}\, E(\mu)| \geq |\mathfrak{M}(\mu)|$, by (C). On the other hand, $|\mathrm{extr}\, E(\mu)| \geq |ult(\mathfrak{N})|$, by [18, Proposition 2], and so

$$\aleph_0 \cdot |\mathrm{extr}\, E(\mu)| \geq |\mathfrak{N}|,$$

by [10, Theorem 5.31]. Using Theorem 6, we conclude that

$$|E(\mu)| \leq 2^{|\mathfrak{M}(\mu)| \cdot \aleph_0 \cdot |\mathfrak{N}|} \leq 2^{\aleph_0 \cdot |\mathrm{extr}\, E(\mu)|}. \blacksquare$$

REFERENCES

[1] M. Bhaskara Rao and K. P. S. Bhaskara Rao, *Cardinalities of Banach spaces*, J. Indian Math. Soc. 37 (1973), 347–349.

[2] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges. A Study of Finitely Additive Measures*, Academic Press, London, 1983.

[3] W. W. Comfort, *A survey of cardinal invariants*, Gen. Topology Appl. 1 (1971), 163–199.

[4] E. K. van Douven, *Cardinal functions on Boolean algebras*, Chapter 11 in: Handbook of Boolean Algebras, Vol. 2, J. D. Monk (ed.), North-Holland, Amsterdam, 1989.

[5] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience Publ., New York–London, 1958.

[6] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., Warszawa, 1977.

[7] D. H. Fremlin and G. Plebanek, *Large families of mutually singular Radon measures*, Bull. Polish Acad. Sci. Math. 51 (2003), 169–174.

[8] B. Fuchssteiner, *Lattices and Choquet's theorem*, J. Funct. Anal. 17 (1974), 377–387.

[9] R. Haydon, *An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford (2) 27 (1976), 379–385.

[10] S. Koppelberg, *Handbook of Boolean Algebras, Vol. 1*, J. D. Monk (ed.), North-Holland, Amsterdam, 1989.

[11] A. H. Kruse, *Badly incomplete normed linear spaces*, Math. Z. 83 (1964), 314–320.

[12] S. V. Kurochkin, *The cardinality of an F-space*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1982, no. 5, 51–52 (in Russian); English transl.: Moscow Univ. Math. Bull. 37 (1982), no. 5, 63–65.

[13] Z. Lipecki, *On extreme extensions of quasi-measures*, Arch. Math. (Basel) 58 (1992), 288–293.

[14] —, *Components in vector lattices and extreme extensions of quasi-measures and measures*, Glasgow Math. J. 35 (1993), 153–162.

[15] —, *On compactness and extreme points of some sets of quasi-measures and measures*, Manuscripta Math. 86 (1995), 349–365.

[16] —, *Remarks on independent sequences and dimension in topological linear spaces*, Collect. Math. 49 (1998), 53–65.

[17] —, *Quasi-measures with finitely or countably many extreme extensions*, Manuscripta Math. 97 (1998), 469–481.

[18] —, *Cardinality of the set of extreme extensions of a quasi-measure*, ibid. 104 (2001), 333–341.

[19] Z. Lipecki, *On compactness and extreme points of some sets of quasi-measures and measures. III*, ibid. 117 (2005), 463–473.
[20] —, *On compactness and extreme points of some sets of quasi-measures and measures. IV*, ibid. 123 (2007), 133–146.
[21] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland, Amsterdam, 1987.
[22] R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand, Princeton, 1966.
[23] D. Plachky, *Extremal and monogenic additive set functions*, Proc. Amer. Math. Soc. 54 (1976), 193–196.
[24] Z. Semadeni, *Banach Spaces of Continuous Functions, Vol. 1*, PWN–Polish Sci. Publ., Warszawa, 1971.
[25] A. H. Stone, *Cardinals of closed sets*, Mathematika 6 (1959), 99–107.

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