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CHARACTER PSEUDO-AMENABILITY OF BANACH ALGEBRAS

BY

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Abstract. Let $\mathcal{A}$ be a Banach algebra and let ϕ be a nonzero character on $\mathcal{A}$. We introduce and study a new notion of amenability for $\mathcal{A}$ based on existence of a ϕ-approximate diagonal by modifying the concepts of ϕ-amenability and pseudo-amenability. We then apply these results to characterize ϕ-pseudo-amenability of various Banach algebras related to locally compact groups such as group algebras, measure algebras, certain dual algebras and Lebesgue–Fourier algebras.

1. Introduction. The class of amenable Banach algebras was first introduced and studied by Johnson [14] in 1972; a Banach algebra $\mathcal{A}$ is amenable precisely when it has a *bounded approximate diagonal*, that is, a bounded net $(\mathbf{m}_\alpha)$ in the projective tensor product $\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{A}$ such that

$$\|\pi(\mathbf{m}_\alpha)a-a\|\to 0 \quad\text{and}\quad \|a\cdot\mathbf{m}_\alpha-\mathbf{m}_\alpha\cdot a\|\to 0$$

for all $a\in\mathcal{A}$, where π denotes the product morphism from $\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{A}$ into $\mathcal{A}$ given by $\pi(a\otimes b)=ab$ for all $a,b\in\mathcal{A}$.

The concept of amenability has played an important role in the theory of Banach algebras, and several authors have introduced modifications of this notion. Motivated by the fact that amenability is restrictive, Ghahramani and Zhang [11] have recently introduced and studied the notion of *pseudo-amenability*, the existence of an approximate diagonal in $\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{A}$.

For a nonzero character ϕ on $\mathcal{A}$, Kaniuth, Lau and Pym [16], [17] have recently introduced and studied the interesting notion of *ϕ-amenability*, the existence of a bounded linear functional F on the dual space $\mathcal{A}^*$ satisfying

$$F(\phi)=1 \quad\text{and}\quad a\odot F=\phi(a)F$$

for all $a\in\mathcal{A}$, where $\odot$ is the first Arens multiplication on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ defined by the equations

$$(F\odot H)(f)=F(Hf),\quad (Hf)(a)=H(fa),\quad (fa)(b)=f(ab)$$

for all $F,H\in\mathcal{A}^{**}$, $f\in\mathcal{A}^*$, and $a,b\in\mathcal{A}$. Any such F is called a *ϕ-mean*.

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The notion of ϕ-amenability is a considerable generalization of left amenability for a *Lau algebra*, that is, a Banach algebra $\mathcal{L}$ which is the predual of a W^*-algebra $\mathcal{M}$ such that the identity u of $\mathcal{M}$ is a character on $\mathcal{L}$; the large class of Lau algebras was introduced and studied by Lau [18] who called them F-algebras. Later on, in his useful monograph, Pier [26] introduced the name "Lau algebra". Several authors have investigated the concept of left amenability of Lau algebras; see for example [1], [8], [19] and [21].

Moreover, the notion of character amenability was introduced and studied by Monfared [24]; he called $\mathcal{A}$ *character amenable* if it has a bounded right approximate identity and it is ϕ-amenable for all nonzero characters ϕ on $\mathcal{A}$. It is shown by Hu, Monfared and Traynor [13] that $\mathcal{A}$ is ϕ-amenable if and only if it has a bounded (right) ϕ-approximate diagonal; see also [3], [7], [9], [23] and [25].

Our goal in this paper is to introduce and study ϕ-pseudo-amenability of $\mathcal{A}$ as a new notion of amenability based on existence of a (right) ϕ-approximate diagonal (not necessarily bounded).

In Section 2, we characterize ϕ-pseudo-amenability of $\mathcal{A}$ in terms of certain nets in $\mathcal{A}$ and show that both ϕ-amenability and pseudo-amenability of $\mathcal{A}$ are sufficient conditions for ϕ-pseudo-amenability of $\mathcal{A}$, but they are not necessary conditions. In particular, we complete the following schematic diagram that illustrates the implications between character pseudo-amenability and some related notions of amenability:

$$\begin{tikzcd}
 & \text{amenability} \arrow[dl, shift right, "\text{def}"'] \arrow[dr, shift left, "\text{def}"] & \\
\text{character amenability} \arrow[ur, shift right, "/" marking, "{[24]}"'] \arrow[dr, shift right, "2.6"'] & & \text{pseudo-amenability} \arrow[ul, shift left, "/" marking, "{[11]}"] \arrow[dl, shift left, "2.6"] \\
 & \text{character pseudo-amenability} \arrow[ul, shift right, "/" marking, "2.7"'] \arrow[ur, shift left, "/" marking, "2.7"] &
\end{tikzcd}$$

In Section 3, we study character pseudo-amenability of certain Banach algebras related to a locally compact group G. First, we show that for each nonzero character ϕ on the group algebra $L^1(G)$, ϕ-pseudo-amenability of $L^1(G)$ is equivalent to amenability of G. We then describe character pseudo-amenability of the dual of certain left introverted closed subspaces of $L^\infty(G)$. Finally, we study character amenability and character pseudo-amenability of the Lebesgue–Fourier algebra $\mathcal{L}A(G)$.

2. Character pseudo-amenability. Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$, the spectrum of all nonzero characters on $\mathcal{A}$. Recall that a (*right*) *ϕ-approximate diagonal for* $\mathcal{A}$ is a net $(\mathbf{m}_\alpha)$ in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that

$$\phi(\pi(\mathbf{m}_\alpha)) \to 1 \quad \text{and} \quad \|a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. The notion of ϕ-approximate diagonal was introduced and studied by Hu, Monfared and Traynor [13].

We begin this section by introducing a new concept of amenability for Banach algebras which is the main objective of our paper.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. We say that $\mathcal{A}$ is *ϕ-pseudo-amenable* if there is a ϕ-approximate diagonal for $\mathcal{A}$. We also say that $\mathcal{A}$ is *character pseudo-amenable* if $\mathcal{A}$ has a right approximate identity and it is ϕ-pseudo-amenable for all $\phi \in \sigma(\mathcal{A})$.

We continue with the following property of ϕ-pseudo-amenability of Banach algebras. First, let us remark that if $\mathcal{E}$ and $\mathcal{F}$ are Banach right $\mathcal{A}$-modules, then the Banach space $B(\mathcal{E}, \mathcal{F})$ of all bounded operators from $\mathcal{E}$ into $\mathcal{F}$ is an $\mathcal{A}$-bimodule with the module operations

$$(a \cdot T)(\xi) = T(\xi \cdot a), \quad (T \cdot a) = T(\xi) \cdot a$$

for all $a \in \mathcal{A}$, $\xi \in \mathcal{E}$ and $T \in B(\mathcal{E}, \mathcal{F})$. We say that a short exact sequence

$$0 \to \mathcal{F} \xrightarrow{\lambda} \mathcal{X} \xrightarrow{\rho} \mathcal{E} \to 0$$

of Banach right $\mathcal{A}$-modules *approximately splits* if there exist nets (λ_α) in $B(\mathcal{X}, \mathcal{F})$ of left inverse maps to λ and (ρ_α) in $B(\mathcal{E}, \mathcal{X})$ of right inverse maps to ρ such that

$$a \cdot \lambda_\alpha - \lambda_\alpha \cdot a \to 0 \quad \text{and} \quad a \cdot \rho_\alpha - \rho_\alpha \cdot a \to 0$$

in the strong operator topology.

Proposition 2.2. *Let $\mathcal{A}$ be a ϕ-pseudo-amenable Banach algebra with $\phi \in \Delta(\mathcal{A})$. Then each admissible short exact sequence*

$$0 \to \mathcal{F} \xrightarrow{\lambda} \mathcal{X} \xrightarrow{\rho} \mathcal{E} \to 0$$

of Banach right $\mathcal{A}$-modules with $\xi \cdot a = \phi(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{F}$ approximately splits.

Proof. By assumption, there exists a right inverse map $\varrho \in B(\mathcal{E}, \mathcal{X})$ for ρ. Consider the bounded linear map $D : \mathcal{A} \to B(\mathcal{E}, \mathcal{X})$ defined by

$$D(a) = a \cdot \varrho - \varrho \cdot a$$

for all $a \in \mathcal{A}$. It is easy to see that

$$D(\mathcal{A}) \subseteq B(\mathcal{E}, \ker(\rho)) = B(\mathcal{F}, \mathrm{Im}(\lambda)).$$

It follows that $\lambda^{-1} \circ D : \mathcal{A} \to B(\mathcal{E}, \mathcal{F})$ is a bounded linear map such that for each $a, b \in \mathcal{A}$,

$$\lambda^{-1}(D(ab)) = a \cdot \lambda^{-1}(D(b)) + \lambda^{-1}(D(a)) \cdot b.$$

Now, let $(\mathbf{m}_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ be a ϕ-approximate diagonal for $\mathcal{A}$ and let $\varPhi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to B(\mathcal{E}, \mathcal{F})$ be the bounded linear mapping specified by

$$\varPhi(a \otimes b) = \phi(b)\, \lambda^{-1}(D(a))$$

for all $a, b \in \mathcal{A}$. Then

$$\|\varPhi\| \leq \|\lambda^{-1} \circ D\|$$

and

$$\varPhi(a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha) = a \cdot \varPhi(\mathbf{m}_\alpha) + \phi(\pi(\mathbf{m}_\alpha))\lambda^{-1}(D(a)) - \phi(a)\varPhi(\mathbf{m}_\alpha)$$

for all $a \in \mathcal{A}$. Next, set

$$Q_\alpha := -\varPhi(\mathbf{m}_\alpha),$$

and note that for each $a \in \mathcal{A}$ we have

$$\phi(\pi(\mathbf{m}_\alpha))\lambda^{-1}(D(a)) = a \cdot Q_\alpha - \phi(a)Q_\alpha + \varPhi(a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha).$$

On the other hand,

$$\|\varPhi(a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha)\| \leq \|\lambda^{-1} \circ D\|\, \|a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. It follows that

$$\lambda^{-1}(D(a)) = \lim_\alpha (a \cdot Q_\alpha - Q_\alpha \cdot a)$$

for all $a \in \mathcal{A}$. So, if we put

$$\rho_\alpha := \varrho - \lambda \circ Q_\alpha$$

for all α, the net (ρ_α) is as needed. A similar argument gives the required (λ_α). ∎

Our next result gives a characterization of ϕ-pseudo-amenability in terms of approximate ϕ-means; following Kaniuth, Lau and Pym [17], we say that a net $(a_\alpha) \subseteq \mathcal{A}$ is an *approximate ϕ-mean* in $\mathcal{A}$ if $\phi(a_\alpha) \to 1$ and $\|aa_\alpha - \phi(a)a_\alpha\| \to 0$ for all $a \in \mathcal{A}$. Passing to a subnet if necessary, we can assume that $\phi(a_\alpha) = 1$ for all α and $\|aa_\alpha - \phi(a)a_\alpha\| \to 0$. A *weak approximate ϕ-mean* is defined similarly, namely the convergence in the definition of approximate ϕ-mean is only assumed to hold in the weak topology instead of the norm topology. We shall use this result without explicit reference.

Proposition 2.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is ϕ-pseudo-amenable.*
(ii) *$\mathcal{A}$ has a weak approximate ϕ-mean.*
(iii) *$\mathcal{A}$ has an approximate ϕ-mean.*

Proof. (i)⇒(ii). Let $(\mathbf{m}_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ be a ϕ-approximate diagonal for $\mathcal{A}$, and define

$$a_\alpha := \pi(\mathbf{m}_\alpha).$$

Then (a_α) is a weak approximate ϕ-mean; indeed, for each $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$ we have $\phi(a_\alpha) \to 1$ and

$$|\langle f, aa_\alpha - \phi(a)a_\alpha\rangle| \leq \|f \circ \pi\| \, \|a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha\| \to 0.$$

(ii)⇒(iii). Let (a_α) be a weak approximate ϕ-mean in $\mathcal{A}$. So, $\phi(a_\alpha) = 1$ for all α and the net $(aa_\alpha - \phi(a)a_\alpha)$ in $\mathcal{A}$ converges to 0 weakly for all $a \in \mathcal{A}$. Now, for each finite subset $\mathfrak{F} = \{a_1, \ldots, a_k\}$ of $\mathcal{A}$, let

$$\Sigma_{\mathfrak{F}} := \{(a_1 a - \phi(a_1)a, \ldots, a_k a - \phi(a_k)a) : a \in \mathcal{A},\ \phi(a) = 1\}.$$

Next, consider the k-fold ℓ^1-direct sum $\ell^1\text{-}\bigoplus_{i=1}^k \mathcal{A}$ of the Banach space $\mathcal{A}$, and note that 0 is in the weak closure of $\Sigma_{\mathfrak{F}}$ in $\ell^1\text{-}\bigoplus_{i=1}^k \mathcal{A}$; in fact, the net

$$(a_1 a_\alpha - \phi(a_1)a_\alpha, ..., a_k a_\alpha - \phi(a_k)a_\alpha) \subseteq \Sigma_{\mathfrak{F}}$$

converges to $0 \in \ell^1\text{-}\bigoplus_{i=1}^k \mathcal{A}$ weakly. Since $\Sigma_{\mathfrak{F}}$ is convex, it follows that 0 is in the norm closure of $\Sigma_{\mathfrak{F}}$ in $\ell^1\text{-}\bigoplus_{i=1}^k \mathcal{A}$. Thus, for each $\varepsilon > 0$, there is $a_{\varepsilon,\mathfrak{F}} \in \mathcal{A}$ such that $\phi(a_{\varepsilon,\mathfrak{F}}) = 1$ and, for each $a \in \mathfrak{F}$,

$$\|aa_{\varepsilon,\mathfrak{F}} - \phi(a)a_{\varepsilon,\mathfrak{F}}\| < \varepsilon.$$

Now, let $\beth$ be the set of all $\beta := (\varepsilon, \mathfrak{F})$ where $\varepsilon > 0$ and $\mathfrak{F} \subseteq \mathcal{A}$ is a finite set. Then $\beth$ is a directed set by setting $(\varepsilon', \mathfrak{F}') \succcurlyeq (\varepsilon, \mathfrak{F})$ if and only if $\varepsilon' \leq \varepsilon$ and $\mathfrak{F}' \supseteq \mathfrak{F}$. So, the net $(a_\beta)_{\beta\in\beth}$ is an approximate ϕ-mean.

(iii)⇒(i). Choose an element $b \in \mathcal{A}$ such that $\phi(b) = 1$ and define

$$\mathbf{m}_\alpha := a_\alpha \otimes b$$

for all α, where (a_α) is an approximate ϕ-mean in $\mathcal{A}$. It is easy to show that the net $(\mathbf{m}_\alpha)$ is a ϕ-approximate diagonal for $\mathcal{A}$. ∎

THEOREM 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\psi \in \sigma(\mathcal{B})$. If there is a continuous epimorphism $\Theta : \mathcal{A} \to \mathcal{B}$ and $\mathcal{A}$ is $\psi \circ \Theta$-pseudo-amenable, then $\mathcal{B}$ is ψ-pseudo-amenable.*

Proof. By assumption, there is a net $(a_\alpha) \subseteq \mathcal{A}$ such that

$$\psi(\Theta(a_\alpha)) \to 1 \quad \text{and} \quad \|aa_\alpha - \psi(\Theta(a))a_\alpha\| \to 0 \quad \text{for all } a \in \mathcal{A}.$$

Define $b_\alpha := \Theta(a_\alpha)$ and note that $\psi(b_\alpha) \to 1$. Also, for each $b \in \mathcal{B}$, there exists $a \in \mathcal{A}$ such that $\Theta(a) = b$, and hence

$$\begin{aligned}\|bb_\alpha - \psi(b)b_\alpha\| &= \|\Theta(a)\Theta(a_\alpha) - \psi(\Theta(a))\Theta(a_\alpha)\| \\ &\leq \|\Theta\| \, \|aa_\alpha - \psi(\Theta(a))a_\alpha\|.\end{aligned}$$

So, (b_α) is an approximate ψ-mean in $\mathcal{B}$, whence $\mathcal{B}$ is ψ-pseudo-amenable. ∎

In the following result, we give another characterization of ϕ-pseudo-amenability.

PROPOSITION 2.5. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is ϕ-pseudo-amenable.*

(ii) *There is a net $(F_\alpha) \subseteq \mathcal{A}^{**}$ such that for each $a \in \mathcal{A}$,*

$$F_\alpha(\phi) \to 1 \quad \textit{and} \quad \|a \odot F_\alpha - \phi(a)F_\alpha\| \to 0.$$

(iii) *There is a net $(F_\alpha) \subseteq \mathcal{A}^{**}$ such that for each $a \in \mathcal{A}$,*

$$F_\alpha(\phi) \to 1 \quad \textit{and} \quad a \odot F_\alpha - \phi(a)F_\alpha \to 0$$

in the weak* topology of $\mathcal{A}^{**}$.

Proof. The implication (i)⇒(ii) follows from Proposition 2.3, and the implication (ii)⇒(iii) is trivial. Now, suppose that (iii) holds. Let Γ be the set of all $\gamma := (\varepsilon, \mathfrak{F}, \mathfrak{H})$ for which $\varepsilon > 0$ and $\mathfrak{F} \subseteq \mathcal{A}$, $\mathfrak{H} \subseteq \mathcal{A}^*$ are finite sets. Then Γ is a directed set by setting $(\varepsilon', \mathfrak{F}', \mathfrak{H}') \succcurlyeq (\varepsilon, \mathfrak{F}, \mathfrak{H})$ if and only if $\varepsilon' \leq \varepsilon$, $\mathfrak{F}' \supseteq \mathfrak{F}$ and $\mathfrak{H}' \supseteq \mathfrak{H}$; moreover, for each $\gamma \in \Gamma$, there is $F_{\alpha_\gamma} \in \mathcal{A}^{**}$ such that

$$|F_{\alpha_\gamma}(\phi) - 1| < \varepsilon$$

and

$$|\langle fa - \phi(a)f, F_{\alpha_\gamma}\rangle| = |\langle f, a \odot F_{\alpha_\gamma} - \phi(a)F_{\alpha_\gamma}\rangle| < \varepsilon$$

for all $a \in \mathfrak{F}$ and $f \in \mathfrak{H}$. By Goldstine's theorem, there is an element $a_\gamma \in \mathcal{A}$ such that

$$|\phi(a_\gamma) - 1| < \varepsilon$$

and

$$\langle fa - \phi(a)f, a_\gamma\rangle| = |\langle f, aa_\gamma - \phi(a)a_\gamma\rangle| < \varepsilon$$

for all $a \in \mathfrak{F}$ and $f \in \mathfrak{H}$. Thus $(a_\gamma)_{\gamma\in\Gamma}$ is a weak approximate ϕ-mean in $\mathcal{A}$. That is, (i) holds. ∎

As a consequence, we have the following result which shows that both ϕ-amenability and pseudo-amenability are sufficient conditions for ϕ-pseudo-amenability.

THEOREM 2.6. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. If $\mathcal{A}$ is ϕ-amenable or pseudo-amenable, then $\mathcal{A}$ is ϕ-pseudo-amenable.*

Proof. It is shown in [13] that ϕ-amenability is equivalent to the existence of a bounded ϕ-approximate diagonal. This together with Proposition 2.3 shows that ϕ-amenability implies ϕ-pseudo-amenability. To prove the second implication, suppose that $\mathcal{A}$ is pseudo-amenable, and define the linear map $\Lambda : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ by

$$\Lambda(a \otimes b) = \phi(b)a$$

for all $a, b \in \mathcal{A}$. Then Λ is bounded and $\|\Lambda\| \le \|\phi\| \le 1$. Choose an approximate diagonal $(\mathbf{m}_\alpha) \subseteq \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$ for $\mathcal{A}$. For each α, write

$$\mathbf{m}_\alpha := \sum_{i=1}^{\infty} a_i^{(\alpha)} \otimes b_i^{(\alpha)},$$

where $a_i^{(\alpha)}, b_i^{(\alpha)} \in \mathcal{A}$ for all $i \ge 1$. So, if we put $a_\alpha := \Lambda(\mathbf{m}_\alpha)$, then for each $a \in \mathcal{A}$,

$$\begin{aligned}\|a a_\alpha - \phi(a) a_\alpha\| &= \Big\| \sum_{i=1}^{\infty} \phi({b_i}^{(\alpha)})\, a\, {a_i}^{(\alpha)} - \sum_{i=1}^{\infty} \phi({b_i}^{(\alpha)} a)\, {a_i}^{(\alpha)} \Big\| \\ &= \|\Lambda(a \cdot \mathbf{m}_\alpha) - \Lambda(\mathbf{m}_\alpha \cdot a)\| \\ &\le \|a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a\| \to 0\end{aligned}$$

and

$$\phi(a_\alpha) = \phi(\pi(\mathbf{m}_\alpha)) \to 1.$$

That is, (a_α) is an approximate ϕ-mean in $\mathcal{A}$ and thus $\mathcal{A}$ is ϕ-pseudo-amenable. ∎

In the following example we show that the converse of Theorem 2.6 is not valid; in fact, we give an example of a ϕ-pseudo-amenable algebra which is neither ϕ-amenable nor pseudo-amenable.

EXAMPLE 2.7. Consider the Banach algebra $\ell^2(\mathbb{N})$ of all sequences $a := (a(n))$ of complex numbers with

$$\|a\| := \sum_{n=1}^{\infty} |a(n)|^2 < \infty$$

endowed with the pointwise product. It is clear that

$$\sigma(\ell^2(\mathbb{N})) = \{\phi_n : n \in \mathbb{N}\},$$

where $\phi_n(a) = a(n)$ for all $a \in \ell^2(\mathbb{N})$. So,

$$\sigma(\ell^2(\mathbb{N})^\sharp) = \{\phi_n : n \in \mathbb{N}\} \cup \{\phi^\infty\},$$

where $\ell^2(\mathbb{N})^\sharp$ is the unitization of $\ell^2(\mathbb{N})$ and

$$\phi^\infty(a, \lambda) = \lambda$$

for all $(a, \lambda) \in \ell^2(\mathbb{N})^\sharp$. Now, let (e_α) be an approximate identity for $\ell^2(\mathbb{N})$. It is not hard to see that $(-e_\alpha, 1)$ is an approximate ϕ^∞-mean in $\ell^2(\mathbb{N})^\sharp$. Since $\mathbb{N}$ is not finite, it follows that $\ell^2(\mathbb{N})$ has no bounded approximate identity and hence $\ell^2(\mathbb{N})^\sharp$ is not ϕ^∞-amenable by [24, Proposition 2.8(ii)]. Moreover, $\ell^2(\mathbb{N})^\sharp$ is not pseudo-amenable (see [11, Theorem 3.1], and [6, Theorem 4.1]).

We end this section with another example. To prepare the setting, we give the following result; first, recall that a *point derivation at a character*

ϕ of an algebra $\mathcal{A}$ is a linear functional δ on $\mathcal{A}$ such that for each $a, b \in \mathcal{A}$,

$$\delta(ab) = \delta(a)\phi(b) + \phi(a)\delta(b).$$

PROPOSITION 2.8. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. If $\mathcal{A}$ is ϕ-pseudo-amenable, then any bounded point derivation at ϕ on A is trivial.*

Proof. Let δ be a bounded point derivation at ϕ and choose an approximate ϕ-mean (a_α) in $\mathcal{A}$. On the one hand, for each $a \in \mathcal{A}$,

$$\delta(aa_\alpha - \phi(a)a_\alpha) = \delta(aa_\alpha) - \phi(a)\delta(a_\alpha) = \phi(a_\alpha)\delta(a).$$

Thus $\delta(aa_\alpha - \phi(a)a_\alpha) \to \delta(a)$ for all $a \in \mathcal{A}$. On the other hand,

$$|\delta(aa_\alpha - \phi(a)a_\alpha)| \leq \|\delta\| \, \|aa_\alpha - \phi(a)a_\alpha\| \to 0.$$

Hence $\delta(a) = 0$ for all $a \in \mathcal{A}$. ∎

EXAMPLE 2.9. (a) Consider the Banach algebra ℓ^1 of all sequences $a := (a(n))$ of complex numbers with

$$\|a\| := \sum_{n=1}^{\infty} |a(n)| < \infty$$

endowed with the product $\diamond$ defined by

$$(a \diamond b)(n) = \begin{cases} a(n)b(n), & n = 1, \\ a(1)b(n) + b(1)a(n) + a(n)b(n), & n > 1, \end{cases}$$

for all $a, b \in \ell^1$. It is easy to check that

$$\sigma(\ell^1) = \{\phi_1\} \cup \{\phi_1 + \phi_n : n \geq 2\},$$

where $\phi_n(a) = a(n)$ for all $a \in \ell^1$. On the one hand, there is no bounded approximate ϕ_1-mean in ℓ^1; indeed, if (a_α) is an approximate ϕ_1-mean in ℓ^1, then $a_\alpha(1) = \phi_1(a_\alpha) \to 1$ and

$$\|a \diamond a_\alpha - \phi_1(a)a_\alpha\| \to 0$$

for all $a \in \ell^1$. In particular

$$|a_\alpha(1) + a_\alpha(n)| = \|\delta_n \diamond a_\alpha - \phi_1(\delta_n)a_\alpha\| \to 0,$$

where δ_n is the characteristic function of $\{n\}$ for all $n \geq 2$. Since $a_\alpha(1) \to 1$, it follows that $a_\alpha(n) \to -1$ as α increases for all $n \geq 2$. Thus $\sup_\alpha \|a_\alpha\| = \infty$. This shows that (a_α) is not bounded in ℓ^1. Consequently, ℓ^1 is not ϕ_1-amenable. On the other hand, it is easy to check that the sequence $(c_k) \subseteq \ell^1$ defined by

$$c_k(n) = \begin{cases} 1, & n = 1, \\ -1, & 1 < n \leq k, \\ 0, & n > k, \end{cases}$$

is an approximate ϕ_1-mean in ℓ^1. Thus ℓ^1 is ϕ_1-pseudo-amenable. Moreover, for each $n \geq 2$, the element $\delta_n \in \ell^1$ is a $\phi_1+\phi_n$-mean in ℓ^1, and consequently ℓ^1 is $\phi_1 + \phi_n$-amenable.

(b) Consider the discrete convolution algebra ℓ^1 of all $a := (a(n))$ with

$$\|a\| := \sum_{n=1}^{\infty} |a(n)| < \infty.$$

Then

$$\sigma(\ell^1) = \{\psi_z : z \in \mathbb{C},\, |z| \leq 1\}, \quad \text{where} \quad \psi_z(a) = \sum_{n=1}^{\infty} a(n) z^n$$

for all $a \in \ell^1$. If $z \in \mathbb{C}$ and $|z| < 1$, then the map $f \mapsto f'(z)$ is a nontrivial bounded point derivation at ψ_z, and thus ℓ^1 is not ψ_z-pseudo-amenable by Proposition 2.8. But it is shown in [16] that ℓ^1 is ψ_z-amenable for all $z \in \mathbb{C}$ with $|z| = 1$.

(c) Consider the Banach algebra $\mathcal{A} = C^1([0,1])$. Then for each $t \in [0,1]$, the linear functional $\delta : \mathcal{A} \to \mathbb{C}$ defined by

$$\delta(f) = f'(t)$$

is a nontrivial bounded point derivation at ϕ_t, where $\phi_t \in \sigma(\mathcal{A})$ is given by $\phi_t(f) = f(t)$ for all $f \in \mathcal{A}$. This together with Proposition 2.8 implies that $\mathcal{A}$ is not ϕ_t-pseudo-amenable.

(d) Consider the Banach algebra $\mathcal{A}$ of all upper-triangular 3×3 matrices over $\mathbb{C}$. We have

$$\sigma(\mathcal{A}) = \{\phi_1, \phi_2, \phi_3\},$$

where for each $[\lambda_{ij}] \in \mathcal{A}$,

$$\phi_k([\lambda_{ij}]) = \lambda_{kk} \quad (k = 1, 2, 3).$$

Moreover, $\mathcal{A}$ is ϕ_1-pseudo-amenable; indeed, if we put

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $ME_{11} = \phi_1(M)E_{11}$ and $\phi_1(E_{11}) = 1$ for all $M \in \mathcal{A}$. Thus $\mathcal{A}$ is even ϕ_1-amenable. However, $\mathcal{A}$ is not ϕ_2-pseudo-amenable; in fact, if $\mathcal{A}$ is ϕ_2-pseudo-amenable, then there is a net $(M_\alpha) \subseteq \mathcal{A}$ such that for each $M \in \mathcal{A}$ we have

$$\|MM_\alpha - \phi_2(M)M_\alpha\| \to 0,$$

and $\phi_2(M_\alpha) \to 1$. Now, we set

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

thus by assumption we have

$$\left\| \begin{bmatrix} 0 & \lambda_{22}^{(\alpha)} & \lambda_{23}^{(\alpha)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| = \|E_{12}M_\alpha - \phi_2(E_{12})M_\alpha\| \to 0,$$

where $M_\alpha = [\lambda_{ij}^{(\alpha)}]$ for all α. So,

$$\phi_2(M_\alpha) = \lambda_{22}^{(\alpha)} \to 0,$$

which is impossible. Similarly, $\mathcal{A}$ is not ϕ_3-pseudo-amenable.

3. Applications to group algebras. Let G be a locally compact group with left Haar measure λ_G and let $L^1(G)$ be the group algebra of G as defined in [12] endowed with the norm $\|\cdot\|_1$ and the convolution product $*$. Let $L^\infty(G)$ be the usual Lebesgue space with the essential supremum norm $\|\cdot\|_\infty$, and let $M(G)$ be the measure algebra of G as defined in [12]. Let also $\widehat{G}$ denote the dual group of G consisting of all continuous homomorphisms from G into the circle group $\mathbb{T}$. For $\rho \in \widehat{G}$, define $\psi_\rho \in \sigma(L^1(G))$ to be the character induced by ρ on $L^1(G)$; that is,

$$\psi_\rho(h) = \int_G \overline{\rho(x)}\, h(x)\, d\lambda_G(x) \qquad (h \in L^1(G)).$$

It is clear that ψ_ρ has an extension $\phi_\rho \in \sigma(M(G))$, where

$$\phi_\rho(\mu) = \int_G \overline{\rho(x)}\, d\mu(x) \qquad (\mu \in M(G)).$$

Let us recall that amenability of G is equivalent to ψ_1-amenability of $L^1(G)$. It is well-known from [27] that all locally compact abelian groups and compact groups are amenable, but the free group $\mathbb{F}_2$ on two generators is not amenable.

THEOREM 3.1. *Let G be a locally compact group and $\rho \in \widehat{G}$. Then the following statements are equivalent:*

(a) *$M(G)$ is ϕ_ρ-pseudo-amenable.*
(b) *$L^1(G)$ is ψ_ρ-pseudo-amenable.*
(c) *G is amenable.*

Proof. (a)⇒(b). Let (μ_α) be an approximate ϕ_ρ-mean for $M(G)$. Choose $h_0 \in L^1(G)$ such that $\phi_\rho(h_0) = 1$ and define

$$h_\alpha := \mu_\alpha * h_0 \in L^1(G).$$

Then (h_α) is an approximate ψ_ρ-mean in $L^1(G)$; in fact,

$$\psi_\rho(h_\alpha) = \phi_\rho(\mu_\alpha * h_0) = \phi_\rho(\mu_\alpha) \to 1$$

and for each $h \in L^1(G)$ we have

$$\begin{aligned}\|h * h_\alpha - \psi_\rho(h)h_\alpha\|_1 &= \|h * (\mu_\alpha * h_0) - \psi_\rho(h)\mu_\alpha * h_0\|_1 \\ &\le \|h * \mu_\alpha - \phi_\rho(h)\mu_\alpha\| \, \|h_0\|_1 \to 0.\end{aligned}$$

(b)⇒(c). It is easy to see that $L^1(G)$ is ψ_1-pseudo-amenable if and only if it is ψ_ρ-pseudo-amenable. This is an immediate consequence of the equations

$$\|\mu * h - \psi_\rho(\mu)h\|_1 = \|(\overline{\rho}\mu) * (\overline{\rho}h) - \psi_1(\overline{\rho}\mu)(\overline{\rho}h)\|_1$$

for all $h \in L^1(G)$ and $\mu \in M(G)$. Now, suppose that $(h_\alpha) \subseteq L^1(G)$ is an approximate ψ_1-mean. We may assume that $\|h_\alpha\|_1 \ge 1/2$ for all α. So, if we put

$$k_\alpha := |h_\alpha|/\|h_\alpha\|_1$$

for all α, then we have $k_\alpha \ge 0$, $\|k_\alpha\|_1 = 1$ and

$$\|\delta_x * k_\alpha - k_\alpha\|_1 \to 0$$

for all $x \in G$, where δ_x denotes the Dirac measure at $x \in G$. Indeed,

$$\big\|\delta_x * |h_\alpha| - |h_\alpha|\big\|_1 = \big\||\delta_x * h_\alpha| - |h_\alpha|\big\|_1 \le \|\delta_x * h_\alpha - h_\alpha\|_1 \to 0.$$

So, (k_α) is a bounded approximate ψ_1-mean in $L^1(G)$, and thus (c) holds.

(c)⇒(a). Suppose that G is amenable. It follows from Corollary 4.2 of [18] that $M(G)$ is ϕ_1-amenable. It is easy to check that $M(G)$ is ϕ_1-amenable if and only if it is ϕ_ρ-amenable. ∎

COROLLARY 3.2. *Let G be a locally compact group. Then $L^1(G)$ is character pseudo-amenable if and only if G is amenable.*

Proof. It is well-known that $\sigma(L^1(G)) = \{\psi_\rho : \rho \in \widehat{G}\}$; see for example Theorem 2.7.2 of [15] or Theorem 23.7 of [12]. So, the result follows immediately from Theorem 3.1 together with the fact that $L^1(G)$ always has at least one character, namely the augmentation character ψ_1. ∎

COROLLARY 3.3. *Let G be a locally compact group. Then $M(G)$ is character pseudo-amenable if and only if G is discrete and amenable.*

Proof. Suppose that $M(G)$ is character pseudo-amenable. It follows from Proposition 2.8 that there is no nonzero point derivation on $M(G)$ at the discrete augmentation character

$$\mu \mapsto \sum_{x \in G} \mu(\{x\}) \qquad (\mu \in M(G)).$$

This shows that G is discrete; see Dales, Ghahramani, and Helemskii [4]. Now, appeal to Theorem 3.1 to conclude that G is also amenable. The converse follows from Corollary 3.2. ∎

Let X be a *left introverted subspace* of $L^\infty(G)$, that is, $Ff \in X$ for all $F \in X^*$ and $f \in X$, where

$$(Ff)(h) = F(fh) \quad \text{and} \quad (fh)(k) = f(h * k)$$

for all $h, k \in L^1(G)$ and $F \in X^*$. In this case, the first Arens multiplication $\odot$ defined by

$$(F \odot H)(f) = F(Hf)$$

for all $F, H \in X^*$ is well-defined on X^*, and X^* is a Banach algebra with respect to this multiplication. Examples of closed left introverted subspaces of $L^\infty(G)$ include the space $\mathrm{AP}(G)$ of almost periodic functions on G, the space $\mathrm{WAP}(G)$ of all weakly almost periodic functions on G, the space $\mathrm{LUC}(G)$ of all left uniformly continuous functions on G.

Now we prove a character pseudo-amenability version of Theorem 3.9 in [13] for the class of maximally almost periodic groups, which contains all abelian groups and compact groups.

THEOREM 3.4. *Let G be a maximally almost periodic locally compact group and let X be a left introverted subspace of $L^\infty(G)$ containing $\mathrm{AP}(G)$. Then X^* is character pseudo-amenable if and only if G is finite.*

Proof. The "if" part is trivial. To prove the converse, suppose that X^* is character pseudo-amenable. Then $\mathrm{AP}(G)^*$ is also character pseudo-amenable; this follows from 2.4 together with the fact that the restriction map

$$X^* \to \mathrm{AP}(G)^*, \qquad f \to f|_{\mathrm{AP}(G)},$$

is a continuous surjective homomorphism. Let bG be the Bohr compactification of G and note that

$$M(bG) \cong \mathrm{AP}(G)^*$$

is character pseudo-amenable. By Corollary 3.3, bG must be discrete, and hence is finite. Since for a maximally almost periodic group G, the canonical homomorphism from G into bG is injective, it follows that G is finite. ∎

Let us remark that Theorem 3.4 does not remain true for all left introverted subspaces of $L^\infty(G)$ by Corollary 3.3; for example, $M(\mathbb{Z})$ is character pseudo-amenable whereas $\mathbb{Z}$ is not finite. In the following results we investigate implications of character pseudo-amenability of the algebra $L^1(G)^{**}$ and the space $\mathrm{LUC}(G)^*$ on the structure of locally compact group G.

PROPOSITION 3.5. *Let G be a locally compact group. Then the following statements are equivalent:*

(a) $\mathrm{LUC}(G)^*$ *is character pseudo-amenable.*
(b) $L^1(G)^{**}$ *is character pseudo-amenable.*
(c) G *is finite.*

Proof. (a)⇒(b). Suppose that $\mathrm{LUC}(G)^*$ is character pseudo-amenable and note that the restriction map

$$\Theta : \mathrm{LUC}(G)^* \to M(G)$$

is a continuous epimorphism. This together with Theorem 2.4 implies that $M(G)$ is character pseudo-amenable, and so G is discrete by Corollary 3.3. Thus $L^\infty(G) = \mathrm{LUC}(G)$.

(b)⇒(c). Suppose that $L^1(G)^{**}$ is character pseudo-amenable. Then $L^1(G)^{**}$ does not have any nonzero continuous point derivation corresponding to any character $\phi \in \sigma(L^1(G)^{**})$. It follows from [5, Theorem 11.17] that G is finite.

(c)⇒(a). This is trivial. ∎

Proposition 3.5 leads us to the conjecture that Theorem 3.4 is true for all locally compact groups. Here, we consider another left introverted subspace of $L^\infty(G)$, namely the space $L_0^\infty(G)$ of all $f \in L^\infty(G)$ which vanish at infinity; in fact,

$$L_0^\infty(G) = \{f \in L^\infty(G) : \text{for } K \text{ compact}, \|f\chi_{G\setminus K}\|_\infty \to 0 \text{ as } K \uparrow G\}.$$

This space was introduced and studied extensively by Lau and Pym [20]; see also [22].

Proposition 3.6. *Let G be a locally compact group. Then the following statements are equivalent:*

(a) $L_0^\infty(G)^*$ *is character amenable.*
(b) $L_0^\infty(G)^*$ *is character pseudo-amenable.*
(c) G *is discrete and amenable.*

Proof. That (a) implies (b) is trivial. Suppose that (b) holds. By [20, Theorem 2.11], for each right identity E of $L^\infty(G)^*$ with norm one, the space $E \odot L_0^\infty(G)^*$ is isometrically isomorphic to $M(G)$. Also, the map $F \mapsto E \odot F$ for each $F \in L_0^\infty(G)^*$ is an epimorphism from $L_0^\infty(G)^*$ onto $M(G)$, and so $M(G)$ is character pseudo-amenable by Theorem 2.4. Hence G is discrete and amenable by Corollary 3.3.

Now, suppose that G is amenable and discrete. Then

$$L_0^\infty(G)^* = M(G) = \ell^1(G),$$

and so $L_0^\infty(G)^*$ is amenable. In particular, $L_0^\infty(G)^*$ is character amenable. ∎

Let $A(G)$ denote the Fourier algebra of G and set

$$\mathcal{L}A(G) := L^1(G) \cap A(G).$$

For each $f \in \mathcal{L}A(G)$ define

$$|||f||| := \|f\|_1 + \|f\|_{A(G)}.$$

Then $\mathcal{L}A(G)$ with the norm $|||\cdot|||$ is a Banach space; this space was introduced and studied extensively by Ghahramani and Lau [10]; see also [2].

We recall that $\mathcal{L}A(G)$ with convolution product is a Banach algebra and it is a dense left ideal of $L^1(G)$ such that

$$|||h * f||| \leq \|h\|_1 |||f|||$$

for all $f \in \mathcal{L}A(G)$ and $h \in L^1(G)$. Ghahramani and Lau called $\mathcal{L}A(G)$ endowed with convolution product the *Lebesgue–Fourier algebra* of G. In fact, $\mathcal{L}A(G)$ is an abstract Segal subalgebra of $L^1(G)$, whence it follows from [3, Lemma 2.2] that

$$\sigma(\mathcal{L}A(G)) = \{\psi|_{\mathcal{L}A(G)} : \psi \in \sigma(L^1(G))\}.$$

PROPOSITION 3.7. *Let G be a locally compact group and let $\psi \in \sigma(L^1(G))$. Then the Lebesgue–Fourier algebra $\mathcal{L}A(G)$ is $\psi|_{\mathcal{L}A(G)}$-pseudo-amenable if and only if G is amenable.*

Proof. Suppose that G is amenable. Then $L^1(G)$ is ψ-pseudo-amenable by Theorem 3.1. Thus there is an approximate ψ-mean in $L^1(G)$, say (h_α). Fix $f_0 \in \mathcal{L}A(G)$ such that $\psi(f_0) = 1$ and set

$$f_\alpha := h_\alpha * f_0 \in \mathcal{L}A(G)$$

for all α; consequently,

$$\psi(f_\alpha) = \psi(h_\alpha) \to 1$$

and for each $f \in \mathcal{L}A(G)$ we have

$$|||f * f_\alpha - \psi(f) f_\alpha||| \leq \|f * h_\alpha - \psi(f) h_\alpha\|_1 |||f_0||| \to 0.$$

Thus $\mathcal{L}A(H)$ is ψ-pseudo-amenable.

Conversely, suppose that $\mathcal{L}A(G)$ is ψ-pseudo-amenable. Then there is an approximate ψ-mean (f_α) in $\mathcal{L}A(H)$. Fix $f_0 \in \mathcal{L}A(G)$ such that $\psi(f_0) = 1$ and set

$$h_\alpha := f_0 * f_\alpha$$

for all α. Since $\mathcal{L}A(G)$ is a left ideal in $L^1(G)$, it follows that

$$\psi(h_\alpha) = \psi(f_\alpha) \to 1$$

and

$$\begin{aligned}
&\|h * h_\alpha - \psi(h) h_\alpha\|_1 \\
&\quad = \|h * f_0 * f_\alpha - \phi(h) f_0 * f_\alpha\|_1 \\
&\quad \leq \|h * f_0 * f_\alpha - \psi(h)\psi(f_0) f_\alpha\|_1 + \|\psi(h)\psi(f_0) f_\alpha - \psi(h) f_0 * f_\alpha\|_1 \\
&\quad \leq |||h * f_0 * f_\alpha - \psi(h * f_0) f_\alpha||| + |\psi(h)|\, |||\psi(f_0) f_\alpha - f_0 * f_\alpha||| \to 0
\end{aligned}$$

for all $h \in L^1(G)$. Thus (h_α) is an approximate ψ-mean in $L^1(G)$, so $L^1(G)$ is ψ-pseudo-amenable. Therefore G is amenable by Theorem 3.1. ∎

Several notions of amenability for $\mathcal{L}A(G)$ were investigated in [2] and [10]. Here we describe character amenability and its relation to character pseudo-amenability.

PROPOSITION 3.8. *Let G be a locally compact group. Then the following statements are equivalent:*

(a) *The Lebesgue–Fourier algebra $\mathcal{L}A(G)$ is character amenable.*
(b) *The Lebesgue–Fourier algebra $\mathcal{L}A(G)$ is character pseudo-amenable and has a bounded right approximate identity.*
(c) *G is discrete and amenable.*

Proof. That (a) implies (b) is trivial. Suppose that (b) holds. By assumption, there is a bounded right approximate identity $(u_\alpha) \subseteq \mathcal{L}A(G)$ with $|||u_\alpha||| \le K$ for all α. Since

$$|||f * u_\alpha||| \le \|f\|_1 |||u_\alpha|||$$

for all $f \in \mathcal{L}A(G)$ and α, it follows that

$$|||f||| \le K\|f\|_1.$$

On the other hand,

$$\|f\|_1 \le |||f|||$$

by definition of $|||\cdot|||$. Thus the norms $\|\cdot\|_1$ and $|||\cdot|||$ are equivalent on $\mathcal{L}A(G)$, and so $\mathcal{L}A(G) = L^1(G)$; this is because $\mathcal{L}A(G)$ is dense in $L^1(G)$ under $\|\cdot\|_1$. Thus G is discrete by [10, Proposition 2.3]; moreover, G is amenable by Proposition 3.7 together with the fact that $L^1(G)$ always has at least one character.

Now, suppose that G is discrete and amenable. Then $l^1(G)$ is character amenable by [24, Corollary 2.4]. Since $\ell^1(G) = \mathcal{L}A(G)$, the norms $|||\cdot|||$ and $\|\cdot\|_1$ are equivalent on $\mathcal{L}A(G)$ by the open mapping theorem. Thus $\mathcal{L}A(G)$ is character amenable. ∎

As a consequence, we have the following result.

COROLLARY 3.9. *Let G be a locally compact group. Then the Lebesgue–Fourier algebra $\mathcal{L}A(G)$ is character pseudo-amenable if and only if G is amenable and $\mathcal{L}A(G)$ has a right approximate identity.*

Let us recall from [10] that $\mathcal{L}A(G)$ with pointwise product is a Banach algebra which is a dense ideal of $A(G)$ and

$$|||k \cdot f||| \le \|k\|_{A(G)} |||f|||$$

for all $f \in \mathcal{L}A(G)$ and $k \in A(G)$.

Proposition 3.10. *Let G be a locally compact group. Then $\mathcal{L}A(G)$ endowed with pointwise product is character amenable if and only if G is compact.*

Proof. Suppose that $\mathcal{L}A(G)$ is character amenable. Then $\mathcal{L}A(G)$ has a bounded approximate identity. Hence G is compact by Proposition 2.6 of [10].

Conversely, if G is compact, then $A(G)$ is character amenable by [24, Corollary 2.4]. But $\mathcal{L}A(G) = A(G)$. Thus the norms $\||\cdot\||$ and $\|\cdot\|_{A(G)}$ are equivalent on $\mathcal{L}A(G)$ by the open mapping theorem, and so $\mathcal{L}A(G)$ is character amenable. ∎

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