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GLOBAL WELL-POSEDNESS FOR THE 2-D BOUSSINESQ SYSTEM WITH TEMPERATURE-DEPENDENT THERMAL DIFFUSIVITY

BY

QIONGLEI CHEN (Beijing) and LIYA JIANG (Hangzhou)

Abstract. We prove the global well-posedness of the 2-D Boussinesq system with temperature dependent thermal diffusivity and zero viscosity coefficient.

1. Introduction. The following 2-D Boussinesq system is one of the most popular models in fluid and geophysical fluid dynamics:

$$\begin{cases} \partial_t u - \nabla\cdot(\nu\nabla u) + u\cdot\nabla u + \nabla p = \theta e_2, \quad e_2 = (0,1), \\ \partial_t\theta - \nabla\cdot(\kappa\nabla\theta) + u\cdot\nabla\theta = 0, \\ \nabla\cdot u = 0, \\ u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x). \end{cases} \tag{1.1}$$

Here u and θ denote the velocity and temperature of the fluid, respectively. The viscosity ν and the thermal diffusivity κ depend on the temperature.

Owing to the similarity with the incompressible Navier–Stokes equation, system (1.1) has been studied extensively by many researchers. In the case when ν and κ are positive constants, global well-posedness results were proved by numerous authors in various function spaces (see [3, 16] and the references therein). For the case that one of ν and κ is zero and the other is a positive constant, results on global well-posedness in various function spaces can be found in [1, 5, 6, 7, 9, 10, 11]. There is also extensive literature on the global well-posedness of the anisotropic Boussinesq system (see [4, 8, 13, 14]). Recently, using methods based on the De Giorgi technique, Wang and Zhang [19] proved global well-posedness results for system (1.1) with $\nu = \nu(\theta)$ and $\kappa = \kappa(\theta)$, where $\nu(\cdot)$ and $\kappa(\cdot)$ are smooth functions satisfing

$$C_0^{-1} \le \nu(\theta) \le C_0, \quad C_0^{-1} \le \kappa(\theta) \le C_0, \quad \theta\in\mathbb{R}, \tag{1.2}$$

for some positive constant C_0.

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In this paper, we consider the case $\nu = 0$ and $\kappa = \kappa(\theta)$, i.e.,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \theta e_2, \quad e_2 = (0,1), \\ \partial_t\theta - \nabla\cdot(\kappa\nabla\theta) + u\cdot\nabla\theta = 0, \\ \nabla\cdot u = 0, \\ u(0,x) = u_0(x), \quad \theta(0,x) = \theta_0(x). \end{cases} \tag{1.3}$$

Our main result reads as follows.

THEOREM 1.1. *Let $s > 2$ and $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$. Assume that $\kappa(\theta)$ satisfies* (1.2). *Then the Boussinesq system* (1.3) *has a unique global in time solution (u, θ) such that*

$$u \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)), \qquad \theta \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{s+1}(\mathbb{R}^2)).$$

2. Preliminaries. We first recall the nonhomogeneous Littlewood–Paley decomposition and some classical spaces. Choose a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ supported in the unit ball and satisfying $\varphi(\xi) = 1$ for $|\xi| \le 1/2$. Let $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$, so $\psi \in C_0^\infty(\mathbb{R}^d)$ is supported in $\{1/2 \le |\xi| \le 2\}$ and satisfies the identity

$$\varphi(\xi) + \sum_{j\ge 0} \psi(2^{-j}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^n.$$

We denote by Δ_j and S_j the convolution operators with symbols respectively $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, and set $\Delta_{-1} f = S_0 f$, $\Delta_k f = 0$ for $k \le -2$. We can easily verify that

$$\Delta_j \Delta_k \equiv 0 \quad \text{if } |j-k| \ge 3, \qquad \Delta_j(S_{k-1} f \Delta_k g) \equiv 0 \quad \text{if } |j-k| \ge 4. \tag{2.1}$$

The Sobolev space $H^{s,p}(\mathbb{R}^d)$ $(1 < p < \infty)$ is defined by

$$H^{s,p}(\mathbb{R}^d) = \Big\{ f \in \mathcal{D}'(\mathbb{R}^d) : \|f\|_{H^{s,p}} \sim \Big\| \Big(\sum_{j\ge -1} 2^{2sj} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p < \infty \Big\}.$$

If $p = 2$, it is just the classical Sobolev space $H^s(\mathbb{R}^d)$ whose norm is defined by $\|\Lambda^s f\|_2$, where Λ^s is the Fourier multiplier operator with symbol $(1+|\xi|^2)^{s/2}$. Moreover, we introduce the following space-time Sobolev spaces:

$$L^\infty(0,T; H^s) = \big\{ f \in \mathcal{D}'((0,T)\times\mathbb{R}^d) : \\ \|f\|_{L^\infty(0,T;H^s)} \sim \big\| \|f\|_{H^s} \big\|_{L^\infty(0,T)} < \infty \big\},$$

$$\tilde{L}^\infty_T(H^s) = \Big\{ f \in \mathcal{D}'((0,T)\times\mathbb{R}^d) : \\ \|f\|_{\tilde{L}^\infty_T(H^s)} \sim \Big(\sum_{j\ge -1} 2^{2sj} \|\Delta_j f\|^2_{L^\infty(0,T;L^2)} \Big)^{1/2} < \infty \Big\}.$$

It is obvious that $\tilde{L}^\infty_T(H^s) \subset L^\infty(0,T;H^s)$.

Next we recall some lemmas which will be used throughout this paper.

LEMMA 2.1 (see [12]). *Let $1 < p < \infty$ and $s > 0$. Assume that $f, g \in H^{s,p}(\mathbb{R}^d)$. Then there exists a constant C independent of f, g such that*

$$\|[\Lambda^s, g]f\|_p \le C(\|\nabla g\|_{p_1}\|f\|_{H^{s-1,p_2}} + \|g\|_{H^{s,p_3}}\|f\|_{p_4})$$

with $p_2, p_3 \in (1,\infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

where $[\ ,\ ]$ is the commutator.

LEMMA 2.2 (see [18]). *Let $s > 0$ and $f \in H^s(\mathbb{R}^d)$. Assume that $F(\cdot)$ is a smooth function on $\mathbb{R}$ with $F(0) = 0$. Then*

$$\|F(f)\|_{H^s} \le C(1 + \|f\|_\infty)^{[s]+1}\|f\|_{H^s},$$

where the constant C depends on $\sup_{k\le[s]+2,\,|t|\le\|f\|_\infty}\|F^{(k)}(t)\|_\infty$.

LEMMA 2.3 (see [19]). *Let $s > d/2$ and $f \in H^s(\mathbb{R}^d)$. Then*

$$\|f\|_\infty \le C(1 + \|f\|_{H^{d/2}})\log^{1/2}(e + \|f\|_{H^s}).$$

LEMMA 2.4 (see [15]). *Let $s > 1 + d/2$ and $f \in H^s(\mathbb{R}^d)$. Then*

$$\|\nabla f\|_\infty \le C(1 + \|\operatorname{curl} f\|_\infty)\log(e + \|f\|_{H^s}).$$

LEMMA 2.5. *Let $s > 0$ and $f, g \in H^s(\mathbb{R}^d) \cap W^{1,\infty}$. Then*

$$\Big(\sum_{j\ge -1} 2^{2sj}\|[\Delta_j, f]\cdot\nabla g\|_2^2\Big)^{1/2} \le C(\|\nabla f\|_\infty\|g\|_{H^s} + \|\nabla g\|_\infty\|f\|_{H^s}).$$

Proof. The proof is standard; we give a sketch for the sake of completeness. Recall Bony's decomposition (see [2])

$$fg = T_f g + T_g f + R(f,g),$$

where

$$T_f g = \sum_{j\ge -1} S_{j-3} f \Delta_j g, \quad R(f,g) = \sum_{j\ge -1} \Delta_j f \tilde{\Delta}_j g, \quad \tilde{\Delta}_j := \sum_{\nu=-2}^{2} \Delta_{j+\nu}.$$

Then we decompose

$$\begin{aligned}
[\Delta_j, f]\cdot\nabla g &= [\Delta_j, f_i]\partial_i g \\
&= [\Delta_j, T_{f_i}]\partial_i g - T_{\Delta_j\partial_i g} f_i - R(\Delta_j\partial_i g, f_i) \\
&\quad + \Delta_j(T_{\partial_i g}) f_i + \Delta_j(R(f_i, \partial_i g)) \\
&= I - II - III + IV + V,
\end{aligned}$$

where the Einstein convention on the summation over repeated indices $i = 1, 2$ is used. Thanks to the condition (2.1), and denoting $h = \mathcal{F}^{-1}\psi$, we have

$$\begin{aligned} I &= \sum_{j'\sim j} [\Delta_j, S_{j'-3}f_i]\partial_i \Delta_{j'} g \\ &= \sum_{j'\sim j} \int_{\mathbb{R}^2} 2^{2j} h(2^j(x-y))(S_{j'-3}f_i(y) - S_{j'-3}f_i(x))\partial_i \Delta_{j'} g(y)\, dy \\ &= -\sum_{j'\sim j} \int_{\mathbb{R}^2} 2^{3j} (\partial_i h)(2^j(x-y))(S_{j'-3}f_i(y) - S_{j'-3}f_i(x))\partial_i \Delta_{j'} g(y)\, dy \\ &\quad -\sum_{j'\sim j} \int_{\mathbb{R}^2} 2^{2j} h(2^j(x-y))\partial_i(S_{j'-3}f_i)(y)\partial_i \Delta_{j'} g(y)\, dy. \end{aligned}$$

Applying Taylor's formula and the usual convolution inequalities yields

$$\|I\|_2 \le C\|\nabla f\|_\infty \sum_{j'\sim j} \|\Delta_{j'} g\|_2.$$

Thus we get the desired estimate

$$\Big(\sum_{j\ge -1} 2^{2sj}\|I\|_2^2\Big)^{1/2} \le C\|\nabla f\|_\infty \|g\|_{H^s}.$$

For the term II, we can write

$$|II| = \Big| \sum_{j'\ge j-3} S_{j'-3}\Delta_j \partial_i g \Delta_{j'} f_i \Big| \le C\|\nabla g\|_\infty \sum_{j'\ge j-3} |\Delta_{j'} f_i|.$$

Then thanks to the convolution inequality for series we get, for $s > 0$,

$$\begin{aligned} \Big(\sum_{j\ge -1} 2^{2sj}\|II\|_2^2\Big)^{1/2} &\le C\|\nabla g\|_\infty \Big\| \sum_{j'\ge j+2} 2^{(j-j')s} 2^{j's}\|\Delta_{j'} f_i\|_2 \Big\|_{\ell^2} \\ &\le C\|\nabla g\|_\infty \|f\|_{H^s}. \end{aligned}$$

For the term III, it is easy to see that

$$|III| = \Big| \sum_{j'\sim j} \Delta_{j'}(\Delta_j \partial_i g)\tilde{\Delta}_{j'} f_i \Big| \le C\|\nabla g\|_\infty \sum_{j'\sim j} \tilde{\Delta}_{j'} f_i;$$

hence

$$\Big(\sum_{j\ge -1} 2^{2sj}\|III\|_2^2\Big)^{1/2} \le C\|\nabla g\|_\infty \|f\|_{H^s}.$$

By the same argument, we obtain

$$\Big(\sum_{j\ge -1} 2^{2sj}\|IV\|_2^2\Big)^{1/2} \le C\|\nabla g\|_\infty \|f\|_{H^s}.$$

The last term can be written as

$$V = \sum_{j' \geq j-5} \Delta_j(\Delta_{j'} \partial_i g \tilde{\Delta}_{j'} f_i).$$

Hence

$$\|V\|_2 \leq C\|\nabla g\|_\infty \sum_{j' \geq j-5} \|\tilde{\Delta}_{j'} f_i\|_2,$$

and again using the convolution inequality for series as for II, we get, for $s > 0$,

$$\Big(\sum_{j \geq -1} 2^{2sj}\|V\|_2^2\Big)^{1/2} \leq C\|\nabla g\|_\infty \|f\|_{H^s}.$$

Thus the lemma is completely proved.

3. The proof of the main theorem. We divide the proof into three parts. In the following, the same generic constant C will be used to denote various constants that depend on C_0, T and $\|u_0\|_{H^2}, \|\theta_0\|_{H^2}$. Here C_0 comes from inequalities (1.2).

Step 1. A priori estimates in $H^s(\mathbb{R}^2)$. First, we prove the following a priori estimate:

PROPOSITION 3.1. *Let $s > 2$ and $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$. There exists a constant C such that if (u, θ) is a solution of* (1.3), *then*

$$\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2 + C_0^{-1}\int_0^t \|\nabla\theta(\tau)\|_{H^s}^2 \, d\tau \leq (\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2) \exp\Big\{C\int_0^t G(\tau)\, d\tau\Big\}, \tag{3.1}$$

with $G(\tau) = 1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^2}^2$.

Proof. First, we will obtain an H^1 estimate. The straightforward energy estimate for (1.3) and Gronwall's inequality give

$$\|\theta\|_2^2 + \int_0^t C_0^{-1}\|\nabla\theta(\tau)\|_2^2 \, d\tau \leq \|\theta_0\|_2^2, \quad \|u\|_2 \leq \|u_0\|_2 + \int_0^t \|\theta(\tau)\|_2 \, d\tau,$$

so

$$\|u\|_2 \leq C, \quad \|\theta\|_2 \leq C, \quad \int_0^t \|\nabla\theta(\tau)\|_2^2 d\tau \leq C, \quad \forall t \leq T. \tag{3.2}$$

Let $p > 2$. Multiplying the second equation of (1.3) by $|\theta|^{p-2}\theta$ and integrating by parts leads to

$$\frac{1}{p}\frac{d}{dt}\|\theta\|_p^p + (p-1)\int_{\mathbb{R}^2} \kappa(\theta)|\nabla\theta|^2|\theta|^{p-2}\,dx = 0.$$

Thus we have $\|\theta\|_p \le \|\theta_0\|_p$, which implies

$$\|\theta\|_\infty \le \|\theta_0\|_\infty. \tag{3.3}$$

It is well-known that u can be recovered from the vorticity ω via the Biot–Savart law:

$$u = \text{P.V.}\, K * \omega, \qquad K(x) = \frac{1}{2\pi|x|^2}(-x_2, x_1).$$

Thus $\|\nabla u\|_2 \simeq \|\omega\|_2$ and $\|\Delta u\|_2 \simeq \|\nabla\omega\|_2$. The vorticity equation is given by

$$\partial_t\omega + u\cdot\nabla\omega = -\partial_1\theta. \tag{3.4}$$

Hence, the energy estimate and Gronwall's inequality give

$$\|\omega\|_2 \le \|\omega_0\|_2 + \int_0^t \|\nabla\theta(\tau)\|_2\,d\tau,$$

which implies

$$\|\nabla u\|_2 \le C. \tag{3.5}$$

For the high order energy estimate for θ, it follows from [17] that the quantity $\Theta = K(\theta) = \int_0^\theta \kappa(z)\,dz$ satisfies the following simple equation:

$$\begin{cases} k'(\Theta)(\partial_t\Theta + u\cdot\nabla\Theta) - \Delta\Theta = 0, \\ \Theta(0,x) = K(\theta_0(x)), \end{cases} \tag{3.6}$$

with k an increasing smooth such that $k(\Theta) = k(K(\theta)) = \theta$ and

$$K'(\theta) = \kappa(\theta), \quad k'(\Theta) = (K'(\theta))^{-1} = \frac{1}{\kappa(\theta)}.$$

By the energy estimate (for more details, see [17, Step 2 in Section 4, Proof of Theorem 1.2]), we finally deduce that

$$\frac{1}{2C_0}\int_{\mathbb{R}^2} \Theta_t^2(t)\,dx + \frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\Theta|^2\,dx \le C\big(1 + \|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\big)\|\nabla\Theta\|_{L^2}^2,$$

from which, (3.2), (3.5) and Gronwall's inequality, it follows that

$$\|\nabla\Theta\|_{L_T^\infty(L^2)} + \|\Theta_t\|_{L^2(Q_T)} \le C\|\nabla\Theta_0\|_{L^2} \le C\|\theta_0\|_{H^1}. \tag{3.7}$$

From (3.6) and Gagliardo–Nirenberg's inequality, we get

$$\|\nabla^2\Theta\|_{L^2} \le C\|\Delta\Theta\|_{L^2} \le C\|\Theta_t\|_{L^2} + \tfrac{1}{2}\|\nabla\Theta\|_{H^1} + C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\Theta\|_{L^2}.$$

Moreover, we have

$$\|\nabla\theta\|_{L^2} = \|k'(\Theta)\nabla\Theta\|_{L^2} \le C\|\nabla\Theta\|_{L^2},$$
$$\|\nabla^2\theta\|_{L^2} = \|k'(\Theta)\nabla^2\Theta + k''(\Theta)\nabla\Theta\otimes\nabla\Theta\|_{L^2} \le C(1+\|\nabla\Theta\|_{L^2})\|\nabla^2\Theta\|_{L^2}.$$

Thus we infer that

$$\|\nabla\theta\|_2 \le C, \quad \int_0^t \|\Delta\theta\|_2^2 \le C, \qquad \forall t \le T. \tag{3.8}$$

Next we will get an H^s estimate. Applying Λ^s to the velocity equation and computing the $L^2(\mathbb{R}^2)$ inner product with $\Lambda^s u$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_2^2 = -\int_{\mathbb{R}^2} \Lambda^s u[\Lambda^s, u]\cdot\nabla u\,dx + \int_{\mathbb{R}^2} \Lambda^s u \Lambda^s(\theta e_2)\,dx,$$

where we have used the fact $\operatorname{div} u = 0$. It follows from Hölder's inequality and Lemma 2.1 that

$$\begin{aligned}\frac{d}{dt}\|u\|_{H^s}^2 &\le 2\|\theta\|_{H^s}\|u\|_{H^s} + C\|u\|_{H^s}^2\|\nabla u\|_\infty \\ &\le (\|\theta\|_{H^s}^2 + \|u\|_{H^s}^2)(1 + C\|\nabla u\|_\infty).\end{aligned} \tag{3.9}$$

Similarly, applying Λ^s to the temperature equation and taking the $L^2(\mathbb{R}^2)$ inner product with $\Lambda^s\theta$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_2^2 + \int_{\mathbb{R}^2}\kappa(\theta)|\Lambda^s\nabla\theta|^2\,dx \\ &\quad = -\int_{\mathbb{R}^2}\Lambda^s\theta[\Lambda^s, u]\cdot\nabla\theta\,dx - \int_{\mathbb{R}^2}\Lambda^s\nabla\theta\cdot[\Lambda^s, \kappa(\theta)-\kappa(0)]\nabla\theta\,dx.\end{aligned} \tag{3.10}$$

Obviously,

$$\Big|\int_{\mathbb{R}^2}\kappa(\theta)|\Lambda^s\nabla\theta|^2\,dx\Big| \ge C_0^{-1}\|\nabla\theta\|_{H^s}, \tag{3.11}$$

and by Lemmas 2.1 and 2.2,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\Lambda^s\nabla\theta\cdot[\Lambda^s, \kappa(\theta)-\kappa(0)]\nabla\theta\,dx\Big| \\ &\quad \le \|\nabla\theta\|_{H^s}\big\{(1+\|\theta\|_\infty)^{1+[s]}\|\theta\|_{H^s}\|\nabla\theta\|_\infty + \|\nabla\theta\|_\infty\|\theta\|_{H^s}\big\} \\ &\quad \le C\|\theta\|_{H^s}\|\nabla\theta\|_{H^s}\|\nabla\theta\|_\infty \le \frac{C_0^{-1}}{4}\|\nabla\theta\|_{H^s}^2 + C\|\theta\|_{H^s}^2\|\nabla\theta\|_\infty^2,\end{aligned} \tag{3.12}$$

where in the second inequality the estimates (3.3) and $\|\theta_0\|_\infty \le C\|\theta_0\|_{H^2}$ are used. For the last term of the right hand side of (3.10), we have

$$\Big|\int_{\mathbb{R}^2}\Lambda^s\theta[\Lambda^s, u]\cdot\nabla\theta\,dx\Big| \le \|\Lambda^s\theta\|_4\|[\Lambda^s, u]\cdot\nabla\theta\|_{4/3}.$$

Using the Gagliardo–Nirenberg inequality in 2-D, we obtain

$$\|\Lambda^s\theta\|_4 \le C\|\Lambda^s\theta\|_2^{1/2}\|\Lambda^s\nabla\theta\|_2^{1/2} = C\|\theta\|_{H^s}^{1/2}\|\nabla\theta\|_{H^s}^{1/2}.$$

Lemma 2.1 and the Gagliardo–Nirenberg inequality give

$$\begin{aligned}\|[\Lambda^s,u]\cdot\nabla\theta\|_{4/3} &\le C\big(\|\nabla u\|_2\|\nabla\theta\|_{H^{s-1,4}} + \|u\|_{H^s}\|\nabla\theta\|_4\big)\\ &\le C\big(\|\nabla u\|_2\|\theta\|_{H^s}^{1/2}\|\nabla\theta\|_{H^s}^{1/2} + \|u\|_{H^s}\|\nabla\theta\|_2^{1/2}\|\Delta\theta\|_2^{1/2}\big).\end{aligned}$$

Collecting the above three estimates, we finally get

$$\begin{aligned}(3.13)\qquad &\Big|\int_{\mathbb{R}^2}\Lambda^s\theta[\Lambda^s,u]\cdot\nabla\theta\,dx\Big|\\ &\le \frac{C_0^{-1}}{4}\|\nabla\theta\|_{H^s}^2 + C(\|\theta\|_{H^s}^2+\|u\|_{H^s}^2)(\|\nabla u\|_2^2+\|\nabla\theta\|_2^2+\|\Delta\theta\|_2^2).\end{aligned}$$

Combining (3.10) with (3.11)–(3.13) yields

$$\begin{aligned}&\frac{d}{dt}\|\theta\|_{H^s}^2 + C_0^{-1}\|\nabla\theta\|_{H^s}^2\\ &\qquad\le C(\|\theta\|_{H^s}^2+\|u\|_{H^s}^2)(\|\nabla\theta\|_\infty^2+\|\nabla u\|_2^2+\|\nabla\theta\|_2^2+\|\Delta\theta\|_2^2).\end{aligned}$$

This estimate together with (3.9) leads to

$$\begin{aligned}(3.14)\qquad &\frac{d}{dt}(\|u\|_{H^s}^2+\|\theta\|_{H^s}^2) + C_0^{-1}\|\nabla\theta\|_{H^s}^2\\ &\le C(\|\theta\|_{H^s}^2+\|u\|_{H^s}^2)(\|\nabla u\|_\infty+\|\nabla\theta\|_\infty^2+\|\nabla u\|_2^2+\|\nabla\theta\|_2^2+\|\Delta\theta\|_2^2).\end{aligned}$$

By Gronwall's inequality, we deduce

$$\begin{aligned}E_n &\triangleq \|u\|_{H^s}^2+\|\theta\|_{H^s}^2 + C_0^{-1}\int_0^t\|\nabla\theta(\tau)\|_{H^s}^2\,d\tau\\ &\le (\|u_0\|_{H^s}^2+\|\theta_0\|_{H^s}^2)\\ &\quad\times\exp\Big(C\int_0^t(1+\|\nabla u\|_\infty+\|\nabla\theta\|_\infty^2+\|\nabla u\|_2^2+\|\nabla\theta\|_2^2+\|\Delta\theta\|_2^2)\,d\tau\Big).\end{aligned}$$

This inequality combined with (3.5) and (3.8) implies (3.1). ∎

Step 2. Local well-posedness. Here, we construct local in time solutions.

THEOREM 3.2. *Let $s>2$ and $(u_0,\theta_0)\in H^s(\mathbb{R}^2)$. Then there exist $T>0$ and a unique solution (u,θ) on $[0,T)$ of the Boussinesq system (1.3) such that*

$$u\in C([0,T];H^s(\mathbb{R}^2)),\quad \theta\in C([0,T];H^s(\mathbb{R}^2))\cap L^2(0,T;H^{s+1}(\mathbb{R}^2)).$$

Furthermore,

$$\|u\|^2_{\tilde L^\infty(0,t;H^s)} + \|\theta\|^2_{\tilde L^\infty(0,t;H^s)} + \|\nabla\theta\|^2_{\tilde L^2(0,t;H^s)} \le (\|u_0\|^2_{H^s} + \|\theta_0\|^2_{H^s}) \exp\Big(\int_0^t Z(\tau)\,d\tau\Big), \tag{3.15}$$

with $Z(\tau) = F(\|\theta\|_{L^\infty})(1 + \|\nabla u(t)\|^2_{L^\infty} + \|\nabla\theta(t)\|^2_{L^\infty})$, *where* $F(\cdot)$ *is a non-decreasing function on* $\mathbb{R}^+$.

Proof. We modify the proof in [19, Theorem 3.1] using Friedrichs' method to construct approximate solutions. Define the projector operator P_n by

$$\mathcal{F}(P_n f)(\xi) = \chi_{B_n}\mathcal{F}(f)(\xi), \qquad \mathcal{F}f(\xi) = \int_{\mathbb{R}^2} f(x)e^{-ix\xi}\,dx,$$

where χ_{B_n} is the characteristic function on the ball B_n centered at the origin with radius n. The approximate system of (1.3) is

$$\begin{cases} \partial_t u_n + P_n\mathcal{P}(P_n u_n \cdot \nabla P_n u_n) = \mathcal{P}(P_n\theta_n e_2), \\ \partial_t\theta_n - P_n\nabla\cdot(\kappa(P_n\theta_n)\nabla P_n\theta_n) + P_n\mathcal{P}(P_n u_n \cdot \nabla P_n\theta_n) = 0, \\ u_n(0,x) = P_n u_0(x), \quad \theta_n(0,x) = P_n\theta_0(x). \end{cases} \tag{3.16}$$

Here $\mathcal{P}$ denotes the Helmholtz projection operator onto the divergence-free fields, which is given by

$$\mathcal{P} = (\delta_{ij} + \mathcal{R}_i\mathcal{R}_j)_{1\le i,j\le 2}$$

with Riesz transform $\mathcal{R}_i$ defined by

$$\mathcal{F}(\mathcal{R}_i f)(\xi) = \frac{i\xi_i}{|\xi|}\mathcal{F}f(\xi).$$

It is clear that $P_n\mathcal{P} = \mathcal{P}P_n$. It is known that system (3.16) has a unique solution $(u_n,\theta_n) \in C([0,T_n];L^2(\mathbb{R}^2))$ for some $T_n > 0$. Thanks to $P_n^2 = P_n$, $(P_n u_n, P_n\theta_n)$ is also a solution of (3.16), so $(P_n u_n, P_n\theta_n) = (u_n,\theta_n)$. Thus approximate system (3.16) can be rewritten as

$$\begin{cases} \partial_t u_n + P_n\mathcal{P}(u_n \cdot \nabla u_n) = \mathcal{P}(\theta_n e_2), \\ \partial_t\theta_n - P_n\nabla\cdot(\kappa(\theta_n)\nabla\theta_n) + P_n\mathcal{P}(u_n \cdot \nabla\theta_n) = 0, \\ u_n(0,x) = P_n u_0(x), \quad \theta_n(0,x) = P_n\theta_0(x). \end{cases} \tag{3.17}$$

Next we will show energy estimates. Applying the operator Δ_j to (3.17) yields

$$\begin{cases} \partial_t\Delta_j u_n + P_n\mathcal{P}\Delta_j(u_n \cdot \nabla u_n) = \mathcal{P}\Delta_j(\theta_n e_2), \\ \partial_t\Delta_j\theta_n - P_n\Delta_j\nabla\cdot(\kappa_n\nabla\theta_n) + P_n\mathcal{P}\Delta_j(u_n \cdot \nabla\theta_n) = 0, \\ \Delta_j u_n(0,x) = P_n\Delta_j u_0(x), \quad \Delta_j\theta_n(0,x) = P_n\Delta_j\theta_0(x). \end{cases} \tag{3.18}$$

Multiplying both sides of the first equation in (3.18) by $\Delta_j u_n$ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_j u_n\|_2^2 &= -\langle [\Delta_j, u_n]\cdot\nabla u_n), \Delta_j u_n\rangle + \langle \mathcal{P}\Delta_j(\theta_n e_2), \Delta_j u_n\rangle \\
&\le \|[\Delta_j, u_n]\cdot\nabla u_n\|_2\|\Delta_j u_n\|_2 + C\|\Delta_j\theta_n\|_2\|\Delta_j u_n\|_2 \\
&\le \|[\Delta_j, u_n]\cdot\nabla u_n\|_2^2 + C\|\Delta_j\theta_n\|_2^2 + C\|\Delta_j u_n\|_2^2.
\end{aligned} \tag{3.19}$$

Here we have used the fact that $\operatorname{div} u_n = 0$. Similarly, from the second equation of (3.18) and $\operatorname{div} u_n = 0$ we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta_n\|_2^2 &= -\langle \Delta_j(\kappa_n\nabla\theta_n), \Delta_j\nabla\theta_n\rangle - \langle \Delta_j(u_n\cdot\nabla\theta_n), \Delta_j\theta_n\rangle \\
&= -\langle \kappa_n\Delta_j\nabla\theta_n, \Delta_j\nabla\theta_n\rangle - \langle [\Delta_j, \kappa_n]\nabla\theta_n, \Delta_j\nabla\theta_n\rangle \\
&\quad - \langle [\Delta_j, u_n]\cdot\nabla\theta_n, \Delta_j\theta_n\rangle.
\end{aligned}$$

This equality implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta_n\|_2^2 + \frac{C_0^{-1}}{2}\|\Delta_j\nabla\theta_n\|_2^2 \\
&\le C\|[\Delta_j, \kappa_n - \kappa_n(0)]\nabla\theta_n\|_2^2 + C\|\Delta_j\theta_n\|_2^2 + C\|[\Delta_j, u_n]\cdot\nabla\theta_n\|_2^2.
\end{aligned} \tag{3.20}$$

Summing up (3.19) and (3.20) yields

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta_j u_n\|_2^2 + \|\Delta_j\theta_n\|_2^2) + C_0^{-1}\|\Delta_j\nabla\theta_n\|_2^2 \\
&\qquad \le C\big(\|\Delta_j u_n\|_2^2 + \|\Delta_j\theta_n\|_2^2 + \|[\Delta_j, \kappa_n - \kappa_n(0)]\nabla\theta_n\|_2^2 \\
&\qquad\qquad + \|[\Delta_j, u_n]\cdot\nabla u_n\|_2^2 + \|[\Delta_j, u_n]\cdot\nabla\theta_n\|_2^2\big).
\end{aligned}$$

Applying Gronwall's lemma, it follows that

$$\begin{aligned}
&\|\Delta_j u_n\|_{L_t^\infty(L^2)}^2 + \|\Delta_j\theta_n\|_{L_t^\infty(L^2)}^2 + \int_0^t \|\Delta_j\nabla\theta_n(\tau)\|_2^2\,d\tau \\
&\quad \le e^{Ct}\Big\{\|\Delta_j u_0\|_2^2 + \|\Delta_j\theta_0\|_2^2 + C\int_0^t e^{-C\tau}\big(\|[\Delta_j, \kappa_n - \kappa_n(0)]\nabla\theta_n(\tau)\|_2^2 \\
&\qquad + \|[\Delta_j, u_n]\cdot\nabla u_n(\tau)\|_2^2 + \|[\Delta_j, u_n]\cdot\nabla\theta_n(\tau)\|_2^2\big)\,d\tau\Big\}.
\end{aligned}$$

According to Lemma 2.5, we have, for $s > 2$,

$$\begin{aligned}
&\|u_n\|_{\tilde{L}_t^\infty(H^s)}^2 + \|\theta_n\|_{\tilde{L}_t^\infty(H^s)}^2 + \|\nabla\theta_n\|_{\tilde{L}_t^2(H^s)}^2 \\
&\quad \le e^{Ct}(\|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2) + Ce^{Ct}\int_0^t e^{-C\tau}\big\{(1 + \|\theta_n(\tau)\|_\infty)^{2[s]+2} \\
&\qquad \times (\|\nabla u_n(\tau)\|_\infty^2 + \|\nabla\theta_n(\tau)\|_\infty^2)(\|u_n\|_{H^s}^2 + \|\theta_n\|_{H^s}^2)\big\}\,d\tau,
\end{aligned}$$

whence, owing to Gronwall's inequality, we get

$$\|u_n\|^2_{\tilde{L}^\infty_t(H^s)} + \|\theta_n\|^2_{\tilde{L}^\infty_t(H^s)} + \|\nabla\theta_n\|^2_{\tilde{L}^2_t(H^s)} \le (\|u_0\|^2_{H^s} + \|\theta_0\|^2_{H^s}) \exp\Big(C\int_0^t Z_n(\tau)\,d\tau\Big),$$

with $Z_n(t) = F(\|\theta_n(t)\|_{L^\infty})(1 + \|\nabla u_n(t)\|^2_{L^\infty} + \|\nabla\theta_n(t)\|^2_{L^\infty})$.

These a priori estimates are sufficient to show the convergence of the sequence (u_n, θ_n) towards a unique solution of problem (1.3). We refer the reader to [19] for more details. ∎

Step 3. Global well-posedness. Let us prove the following blow-up criterion first.

THEOREM 3.3. *Let* $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$, $s > 2$. *Suppose that* $u \in C([0,T]; H^s(\mathbb{R}^2))$, $\theta \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+1}(\mathbb{R}^2))$ *is the smooth solution to* (1.3). *If the vorticity* ω *corresponding to the solution* u *satisfies*

$$\int_0^T \|\omega(\tau)\|_\infty\,d\tau < \infty,$$

then the solution (u, θ) *can be extended beyond* $t = T$.

Proof. Using Lemma 2.3 with $f = \nabla\theta$, we deduce

$$\|\nabla\theta\|^2_\infty \le C(1 + \|\theta\|_{H^1})^2 \log(e + \|\theta\|^2_{H^s}).$$

Applying Lemma 2.4 , we obtain

$$\|\nabla u\|_\infty \le C(1 + \|\omega\|_\infty)\log(e + \|u\|^2_{H^s}).$$

So, by Theorem 3.1, we have

$$\|u\|^2_{H^s} + \|\theta\|^2_{H^s} \le (\|u_0\|^2_{H^s} + \|\theta_0\|^2_{H^s}) \times \exp\Big(CT + C\int_0^t (1 + \|\theta\|^2_{H^1} + \|\omega(\tau)\|_\infty)\log(e + \|u(\tau)\|^2_{H^s} + \|\theta(\tau)\|^2_{H^s})\,d\tau\Big).$$

Setting $E(t) \triangleq \log(e + \|u(t)\|^2_{H^s} + \|\theta(t)\|^2_{H^s})$, the above inequality implies

$$E(t) \le E(0) + CT + C\int_0^t (1 + \|\theta\|^2_{H^1} + \|\omega(\tau)\|_\infty)E(\tau)\,d\tau$$

for all $0 < t < T$. Applying Gronwall's inequality and (3.2), we obtain

$$E(t) \le (E(0) + CT)\exp\Big(C\int_0^t (1 + \|\omega(\tau)\|_\infty)\,d\tau\Big).$$

This completes the proof. ∎

Now let us turn to global well-posedness; we just need to show that

$$\int_0^T \|\omega\|_\infty < \infty. \tag{3.21}$$

In fact, recall the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = -\partial_1 \theta.$$

Let $p > 2$. Multiplying the vorticity equation by $|\omega|^{p-2}\omega$ and integrating by parts leads to

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p = \int_{\mathbb{R}^2} \partial_1\theta|\omega|^{p-2}\omega\,dx \le \|\omega\|_p^{p-1}\|\nabla\theta\|_p.$$

where, in the last inequality, we have used Hölder's inequality. Thus we have

$$\frac{d}{dt}\|\omega\|_p \le \|\nabla\theta\|_p.$$

By integrating in time over $[0, T]$, we deduce

$$\|\omega\|_p \le \|\omega_0\|_p + \int_0^T \|\nabla\theta(\tau)\|_p\,d\tau.$$

This implies that

$$\|\omega\|_\infty \le \|\omega_0\|_\infty + \int_0^T \|\nabla\theta(\tau)\|_\infty\,d\tau.$$

It follows from [19, Proposition 5.1] that

$$\int_0^T \|\nabla\theta\|_\infty < \infty.$$

Therefore estimate (3.21) holds true.

This completes the proof of Theorem 1.1.

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Qionglei Chen
Institute of Applied Physics
and Computational Mathematics
P.O. Box 8009
Beijing 100088, P.R. China
E-mail: chen_qionglei@iapcm.ac.cn

Liya Jiang
Zhejiang University of Technology
18, Chaowang Road
Hangzhou 310014, P.R. China
E-mail: mathjly@163.com

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