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ABSENCE OF GLOBAL SOLUTIONS TO A CLASS OF NONLINEAR PARABOLIC INEQUALITIES

BY

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Abstract. We study the absence of nonnegative global solutions to parabolic inequalities of the type $u_t \geq -(-\Delta)^{\beta/2}u - V(x)u + h(x,t)u^p$, where $(-\Delta)^{\beta/2}$, $0 < \beta \leq 2$, is the $\beta/2$ fractional power of the Laplacian. We give a sufficient condition which implies that the only global solution is trivial if $p > 1$ is small. Among other properties, we derive a necessary condition for the existence of local and global nonnegative solutions to the above problem for the function V satisfying $V_+(x) \sim a|x|^{-b}$, where $a \geq 0$, $b > 0$, $p > 1$ and $V_+(x) := \max\{V(x), 0\}$. We show that the existence of solutions depends on the behavior at infinity of both initial data and h.

In addition to our main results, we also discuss the nonexistence of solutions for some degenerate parabolic inequalities like $u_t \geq \Delta u^m + u^p$ and $u_t \geq \Delta_p u + h(x,t)u^p$. The approach is based upon a duality argument combined with an appropriate choice of a test function. First we obtain an a priori estimate and then we use a scaling argument to prove our nonexistence results.

1. Introduction and main results. The broad goal of this paper is to discuss the nonexistence of nonnegative solutions to a class of nonlinear parabolic inequalities of the type

$$u_t \geq -(-\Delta)^{\beta/2}u - V(x)u + h(x,t)u^p \tag{1.1}$$

in $\mathbb{R}^N \times (0,T)$, $0 < T \leq \infty$, subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Here $(-\Delta)^{\beta/2}$, $0 < \beta \leq 2$, is the $\beta/2$ fractional power of the Laplacian. The function $h > 0$ and the potential $V \geq 0$ are locally bounded and satisfy some growth conditions at infinity which we shall specify later. The initial data $u_0 \geq 0$ is locally integrable and is such that a local solution to (1.1)–(1.2) exists.

Our initial intention is to find a sufficient condition which asserts that any possible local solution to (1.1)–(1.2) ceases to exist after a finite time. Our analysis is divided into two parts. The first part is devoted to the nonexistence of global solutions in the case where the exponent p is sufficiently small, say $1 < p \leq p_c$. In the second part we investigate the relationship

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between the nonexistence result and the behavior at infinity of both the initial data and the function h, where $p > p_c$ and the potential V behaves like $a|x|^{-b}$, $a \geq 0$, $b > 0$, at infinity.

The problem of blowing up solutions has a long history, which dates back to the pioneering work by Fujita [4] on the nonlinear heat equation

$$u_t = \Delta u + u^p. \tag{1.3}$$

Fujita [4, 5] proved that (1.3) has no global positive solutions if $1 < p \leq p_c := 1 + 2/N$. On the other hand, we can choose $\delta > 0$ such that (1.3) has a global solution whenever $0 \leq u_0(x) \leq \delta e^{-k|x|^2}$ and $p > p_c$. The number p_c is called the *critical exponent*.

In [6] Galaktionov showed that the critical exponent for the porous medium equation

$$u_t = \operatorname{div}(u^m \nabla u) + u^p, \qquad m > 0,$$

is $m + 1 + 2/N$. For the problem

$$u_t = \Delta u^m + |x|^\sigma t^s u^{1+p}, \qquad t > 0,\ x \in \mathbb{R}^N,$$

the critical exponent is $p_c = (m-1)(s-1) + (2 + 2s + \sigma)/N > 0$ (see [21]). We refer the reader to [22] and [1] for other results in this direction.

The first result concerning the blow up of solutions to evolution equations with the fractional power of the Laplacian is due to Sugitani [23] who generalized the results of Fujita to the problem

$$u_t = -(-\Delta)^{\beta/2} u + u^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{1.4}$$

In this case the critical exponent is $1 + \beta/N$. Later Guedda and Kirane [10] discussed the absence of global solutions to

$$u_t = -(-\Delta)^{\beta/2} u + h(t) u^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{1.5}$$

where $h(t)$ behaves like t^σ, $\sigma > -1$. Using the method of [4] they showed that nontrivial solutions are not global if $1 < p < 1 + \beta(\sigma + 1)/N$. The proof is based on a reduction of equation (1.5) to an ordinary differential inequality satisfied by $\overline{u}(t) := \int_{\mathbb{R}^N} p(x,t) u(x,\cdot)\, dx$, where p is the fundamental solution of $L_\beta := \partial/\partial t + (-\Delta)^{\beta/2}$.

More recently parabolic-hyperbolic equations and systems associated with the fractional power of the Laplacian have been investigated by Guedda and Kirane [11]. They proved, among other results, that the only global solution to

$$u_t = -(-\Delta)^{\beta/2} u + t^s |x|^\sigma u^p$$

is the trivial one if $1 < p \leq p_c := 1 + (s + \beta(1+\sigma))/N$.

The constant p_c will appear also in the study of (1.1) with $h(x,t) = t^s |x|^\sigma$. Concerning (1.1) with equality instead of inequality, Zhang [26, 27] studied the problem

$$u_t = \Delta u - V(x)u + u^p.$$

In the first paper [26] it is shown that if the potential V satisfies

$$0 \le V(x) \le \frac{a}{1+|x|^b}, \quad a > 0,\ b > 2,$$

then all positive solutions blow up at a finite time if $1 < p < 1 + 2/N$. The second paper [27] deals with the problem in which the potential V behaves like

$$V(x) \sim \pm\frac{a}{1+|x|^b}, \quad a > 0,\ b > 0,$$

except for the case

$$V(x) \sim -\frac{a}{1+|x|^2}, \quad a > 0.$$

In this paper we will prove that in this case the problem has no global solution for any $1 < p \le 1 + 2/N$ and we give a partial answer to the open questions [27, Remark 1.1, pp. 190–191]. We also obtain a sufficient condition for the local and global nonexistence of solutions for any $p > 1$.

To understand the influence of both u_0 and h on the nonexistence of solutions, Baras and Kersner [2] proved, among other results, that if

$$\lim_{|x|\to\infty} u_0(x)^{p-1}h(x)|x|^2 = \infty,$$

then no global solution to

$$u_t = \Delta u + h(x)u^p, \quad u(x,0) = u_0(x),$$

exists.

As was mentioned at the beginning of this introduction, we are interested in the nonexistence of global solutions to (1.1)–(1.2). We will make the following assumptions.

There exist $\gamma > 0$ and $l > 0$ such that for any compact $\Omega \subset \mathbb{R}^N$,

$$\lim_{R\to\infty} R^{\gamma(N+\beta)-l(p'-1)} \int_\Omega V(R^\gamma y)^{p'}\, dy = 0, \quad p' = \frac{p}{p-1}, \tag{1.6}$$

and for any $0 < t_1 < t_2$,

$$\frac{h(R^\gamma x, R^{\beta\gamma} t)}{R^l} \ge C \tag{1.7}$$

uniformly in $\Omega \times (t_1, t_2)$.

A classical example is $h(x,t) = t^\tau |x|^\sigma$ and $V(x) = a/(1+|x|^b)$ where $\sigma > \beta(p-1)$, $\tau > 0$ and $b > N(p-1)/p$.

We shall prove the following result concerning the absence of global solutions to (1.1)–(1.2).

THEOREM 1.1. *Let $\gamma > 0$. Assume that conditions* (1.6) *and* (1.7) *are satisfied where*

$$(p-1)N/\beta - 1 \le \gamma l. \tag{1.8}$$

Then Problem (1.1)–(1.2) *has the only global nonnegative solution $u \equiv 0$.*

By a *global nonnegative solution* to (1.1)–(1.2) we mean a locally integrable function $hu^p \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$ such that

$$\int_{\mathbb{R}^N} u_0\zeta(\cdot,0) + \int_{\mathbb{R}^{N+1}_+} hu^p\zeta \le \int_{\mathbb{R}^{N+1}_+} u(-\Delta)^{\beta/2}\zeta - \int_{\mathbb{R}^{N+1}_+} u\zeta_t + \int_{\mathbb{R}^{N+1}_+} V(x)u\zeta \tag{1.9}$$

for any positive $\zeta \in C_0^\infty(\mathbb{R}^{N+1}_+)$. Here $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \mathbb{R}_+$ and the integral $\int_{\mathbb{R}^N} u_0\zeta(\cdot,0)$ is understood in the weak sense, i.e.,

$$\int_{\mathbb{R}^N} u(\cdot,t)\zeta(\cdot,t) \to \int_{\mathbb{R}^N} u_0\zeta(\cdot,0) \quad \text{as } t \to 0^+,\ \forall \zeta \in C_{\mathrm{c}}(\mathbb{R}^{N+1}_+).$$

Theorem 1.1 will be proved in Section 2. We shall see from the proof that the positivity condition on the initial data can be relaxed to

$$\int_{\mathbb{R}^N} u_0 \ge 0$$

if we consider the inequality

$$u_t \ge -(-\Delta)^{\beta/2}u - Vu + h(x,t)|u|^p.$$

In this introduction we have restricted our presentation to $V \ge 0$. The general case where V is not necessarily nonnegative is also considered.

In Section 3, we study the nonexistence of local and global solutions to

$$u_t \ge -(-\Delta)^{\beta/2}u + h(x,t)u^p, \tag{1.10}$$

completing in this way the results of [23], [2], [10], [11], [19], [20].

THEOREM 1.2. *Let $p > 1$. There is no local nonnegative solution u to* (1.10) *on $(0,T)$ such that $u(\cdot,0) = u_0$ if*

$$\lim_{|x|\to\infty} u_0(x)^{p-1}h(x,t) = \infty \quad \textit{for any } t \in (0,T).$$

Assume that $h(x,t) \ge g(x)$ for any $t \ge 0$. Then there is no global nonnegative solution if

$$\lim_{|x|\to\infty} u_0(x)^{p-1}g(x)|x|^\beta = \infty.$$

In fact, the proof of the above theorem leads us to a necessary condition for local solvability. This condition is given by

$$\liminf_{|x|\to\infty} u_0(x)^{p-1} g(x) < \frac{2}{p-1}\cdot\frac{1}{T}.$$

This means that if $L := \liminf_{|x|\to\infty} u_0(x)^{p-1}g(x) > 0$, then the maximal interval of existence is included in $\left(0, \frac{2}{p-1}\frac{1}{L}\right)$. For nonglobal solvability, we will show the existence of a positive constant $L_\star$ such that if

$$\liminf_{|x|\to\infty} u_0(x)^{p-1} g(x)|x|^\beta > L_\star, \tag{1.11}$$

then any possible local solution to (1.10) where $h(x,t) \geq g(x)$ is not global. For the problem

$$u_t \geq \Delta u^m + |x|^l u^p,$$

the constant $L_\star$ is equal to λ_1 if $0 < m \leq 1$. Here λ_1 is the first eigenvalue of the Laplacian in the unit ball and l is a real strictly larger than -2. In this case (1.11) reads

$$\liminf_{|x|\to\infty} u_0(x)^{p-m}|x|^{l+2} > \lambda_1.$$

This will be proved in Section 4. The same result was obtained in [14], [24] for

$$u_t = \Delta u + u^p.$$

It is worth noting that those papers were preceded by the work of Baras and Kersner [2] where the local nonexistence of solutions was also studied.

The technique we use is based on a duality argument and a judicious choice of test functions [2], [3], [15]. The main results of this paper were announced in [9].

2. Blow up of solutions to a semilinear parabolic problem. In the present section we will give the proof of Theorem 1.1, and we discuss an extension to a more general case where the positivity of V is not required. We first analyze the absence of global nonnegative solutions to

$$\begin{cases} u_t \geq -(-\Delta)^{\beta/2}u - V(x)u + h(x,t)u^p, & (x,t) \in \mathbb{R}^{N+1}_+, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

where the potential $V \geq 0$ is a locally Hölder, continuous function. For the function h we assume throughout this section that hypothesis (1.7) is satisfied; that is, there exist $\gamma, l > 0$ such that

$$\frac{h(R^\gamma y, R^{\beta\gamma} t)}{R^l} \geq C \tag{2.2}$$

uniformly in $\Omega \times (t_1, t_2)$.

For the convenience of the reader we recall Theorem 1.1 below.

Theorem 2.1. *Assume that* (2.2) *holds where*

$$1 < p \le 1 + \frac{\gamma\beta + l}{\gamma N}, \tag{2.3}$$

and V satisfies

$$\lim_{R\to\infty} R^{\gamma(N+\beta)-l(p'-1)} \int_{\Omega} V(R^\gamma x)^{p'}\,dx = 0, \tag{2.4}$$

where $p' = p/(p-1)$. Then Problem (2.1) *has the only global nonnegative solution $u \equiv 0$.*

Proof. Without loss of generality we may assume that $\Omega \subset \{x \in \mathbb{R}^N; |x| \le 2\}$. Let u be a global nonnegative solution and ζ be a smooth nonnegative test function such that

$$\int_{\mathbb{R}^{N+1}_+} \frac{|\zeta_t|^{p'}}{\zeta^{p'-1}} + \int_{\mathbb{R}^{N+1}_+} \frac{|(-\Delta)^{\beta/2}\zeta|^{p'}}{\zeta^{p'-1}} < \infty. \tag{2.5}$$

According to (2.1) we have

$$\int_{\mathbb{R}^N} u_0\zeta(\cdot,0) + \int_{\mathbb{R}^{N+1}_+} hu^p\zeta \le \int_{\mathbb{R}^{N+1}_+} u(-\Delta)^{\beta/2}\zeta - \int_{\mathbb{R}^{N+1}_+} u\zeta_t + \int_{\mathbb{R}^{N+1}_+} Vu\zeta. \tag{2.6}$$

By the Young inequality

$$\begin{aligned}\int_{\mathbb{R}^N} u_0\zeta(\cdot,0) + \int_{\mathbb{R}^{N+1}_+} hu^p\zeta &\le \frac{1}{6}\int_{\mathbb{R}^{N+1}_+} hu^p\zeta + C\int_{\mathbb{R}^{N+1}_+} |(-\Delta)^{\beta/2}\zeta|^{p'}(\zeta h)^{1-p'} \\ &\quad + \frac{1}{6}\int_{\mathbb{R}^{N+1}_+} hu^p\zeta + C\int_{\mathbb{R}^{N+1}_+} |\zeta_t|^{p'}(\zeta h)^{1-p'} \\ &\quad + \frac{1}{6}\int_{\mathbb{R}^{N+1}_+} hu^p\zeta + C\int_{\mathbb{R}^{N+1}_+} h^{1-p'}V^{p'}\zeta\end{aligned}$$

with $C = \frac{p-1}{p}\left(\frac{6}{p}\right)^{1/(p-1)}$. Therefore we get

$$\begin{aligned}&\frac{1}{2C}\int_{\mathbb{R}^{N+1}_+} hu^p\zeta \\ &\le \int_{\mathbb{R}^{N+1}_+} |\zeta_t|^{p'}(h\zeta)^{1-p'} + \int_{\mathbb{R}^{N+1}_+} |(-\Delta)^{\beta/2}\zeta|^{p'}(h\zeta)^{1-p'} + \int_{\mathbb{R}^{N+1}_+} h^{1-p'}V^{p'}\zeta.\end{aligned} \tag{2.7}$$

Next we consider $\phi \in C_0^\infty(\mathbb{R}_+)$ satisfying

$$\phi(r) = \begin{cases} 1 & \text{if } r \le 1, \\ 0 & \text{if } r \ge 2, \end{cases} \tag{2.8}$$

and

$$0 \le \phi \le 1. \tag{2.9}$$

Set

$$\zeta(t,x) = \left(\phi\left(\frac{t^{2\alpha} + |x|^{2\beta\alpha}}{R^2}\right)\right)^{\lambda}, \qquad R > 0,$$

where $\alpha = 1/(\beta\gamma)$ and λ is large enough such that condition (2.5) holds. After the change of variables

$$\tau = tR^{-1/\alpha}, \qquad y = xR^{-1/(\beta\alpha)},$$

using estimate (2.5), we easily obtain

$$\int_{\mathbb{R}_+^{N+1}} hu^p\zeta \le C\Big(R^s + R^{\frac{1}{\alpha\beta}(N+\beta)-l(p'-1)} \int_{|y|\le\sqrt{2}} V(R^{\frac{1}{\alpha\beta}}y)^{p'}\,dy\Big), \tag{2.10}$$

where

$$s = \frac{1}{\alpha\beta}[N + \beta - \beta p' - \alpha\beta l(p'-1)].$$

Since $s \le 0$, by (2.3), we conclude from the hypothesis on V that $hu^p \in L^1(\mathbb{R}_+^{N+1})$. So

$$\lim_{R\to\infty} \int_{\Omega_R} \zeta hu^p = 0$$

where

$$\Omega_R = \{(x,t) \in \mathbb{R}_+^{N+1};\ R^2 \le t^{2\alpha} + |x|^{2\alpha\beta} \le 2R^2\}.$$

On the other hand, according to (2.6) and to the Hölder inequality, one finds that the integral $\int_{\mathbb{R}_+^{N+1}} \zeta hu^p$ is bounded by

$$\Big(\int_{\Omega_R} \zeta hu^p\Big)^{1/p}\Big[\Big(\int_{\Omega_R} |\zeta_t|^{p'}(\zeta h)^{1-p'}\Big)^{1/p'} + \Big(\int_{\Omega_R} |(-\Delta)^{\beta/2}\zeta|^{p'}(h\zeta)^{1-p'}\Big)^{1/p'}\Big] + \Big(\int_{\mathbb{R}_+^{N+1}} \zeta hu^p\Big)^{1/p}\Big(\int_{\mathbb{R}_+^{N+1}} h^{1-p'}V^{p'}\zeta\Big)^{1/p'}.$$

Passing to the limit as $R \to \infty$ shows that $\int_{\mathbb{R}_+^{N+1}} hu^p = 0$. Thus $u = 0$, which ends the proof. ∎

REMARK 2.1. In the case where $\Omega \subset \{x \in \mathbb{R}^N;\ |x| \le r_0\}$, $r_0 > 0$, we may choose the function ϕ such that $\operatorname{supp}\phi \subset [0, r_0)$.

The above proof leads to the following theorem.

THEOREM 2.2. *Let $1 < p \le 1 + \beta/N$. Assume that the function h satisfies* (2.2) *for some $l > \gamma\beta/(p'-1)$. Suppose*

$$0 \le V(x) \le \frac{a}{1+|x|^b} =: w(x), \tag{2.11}$$

with $b > N(p-1)/p$, $a > 0$. Then Problem (2.1) *has no global nonnegative solution except the trivial one.*

Proof. According to the preceding theorem we check that V satisfies hypothesis (2.4). So it is sufficient to show that the function

$$I(R) := R^{\gamma(N+\beta)-l(p'-1)} \int_0^{r_0} w(R^\gamma r)^{p'} r^{N-1} dr, \quad r_0 = \text{const} > 0,$$

tends to 0 as R goes to infinity. A routine calculation yields the estimate

$$I(R) \leq KR^{\beta\gamma - l(p'-1)} \int_0^{r_0^b R^{b\gamma}} \frac{s^{N/b-1}}{(1+s)^{p'}}\, ds.$$

Therefore, since $b > N(p-1)/p$, we have $I(R) \leq CR^{\beta\gamma-l(p'-1)}$, and thus assumption (2.4) holds because $\beta\gamma < l(p'-1)$. ∎

To illustrate this analysis by an example we consider $h(x,t) = t^\tau |x|^\sigma$. Then the condition $l > \gamma\beta/(p'-1)$ reads

$$\beta < (\sigma + \beta\tau)(p'-1).$$

REMARK 2.2. If we have $|u|^p$ instead of u^p, the positivity of the initial data may be replaced by $\int_{\mathbb{R}^N} u_0 \geq 0$. In the same spirit we can obtain the absence of a nontrivial global solution to the problem

$$u_t = -(-\Delta)^{\beta/2} u + h(x) u^p + f(x,t).$$

Here the function f is assumed to be nonnegative. No assumption on the integrability of f or its regularity are required. A similar result can be obtained if we assume

$$\int_{\mathbb{R}_+^{N+1}} f \geq 0.$$

In [20] the problem $u_t = \Delta u + h(x)u^p + \lambda g(x)$ was considered. It was shown that, for example, if $h(x)$ and $g(x)$ are greater respectively than $|x|^m$ and $|x|^{-q}$ for large $|x|$, then no global nonnegative solution can exist whenever

$$1 < p < 1 + \frac{2+m}{q-2}, \qquad 2 < q < N.$$

According to our conclusion, the result still holds if $q \geq N+2$ since $1+\frac{2+m}{q-2} \leq 1 + \frac{2+m}{N}$.

REMARK 2.3. Let us point out that by the above proofs the positivity of V is not necessary. For the general case the nonexistence result is an immediate consequence of Theorem 2.2.

THEOREM 2.3. *Assume that* (2.2) *holds where*

$$1 < p \leq 1 + \frac{\gamma\beta + l}{\gamma N}, \tag{2.12}$$

and V satisfies

$$\lim_{R\to\infty} R^{\gamma(N+\beta)-l(p'-1)} \int\limits_{|x|\le r_0} V_+(R^\gamma x)^{p'}\,dx = 0, \tag{2.13}$$

where $V_+ = \max\{V,0\}$. *Then the only global nonnegative solution to Problem* (2.1) *is* $u\equiv 0$.

COROLLARY 2.1. *Assume $V\le 0$. Let $1<p\le 1+\beta/N$. Then there exists no nontrivial solution to*

$$u_t = -(-\Delta)^{\beta/2}u - Vu + u^p, \qquad u\ge 0.$$

REMARK 2.4. This corollary proves in particular that the critical exponent $1+2/N$ of the problem

$$u_t = \Delta u - Vu + u^p,$$

where

$$-\delta\,\frac{1}{1+|x|^b} \le V(x) \le 0, \qquad b>2,$$

belongs to the blow up case [27].

REMARK 2.5. The condition $l>0$ is important if $N(p-1)/p<b<2$, since it is easily shown that there exists a global positive solution to

$$u_t \ge \Delta u - \frac{c}{1+|x|^b}\,u + h(t)|u|^p,$$

of the form

$$U(x,t) = \frac{h(t)^{1/(1-p)}}{(A+|x|^2)^{b/(2(p-1))}},$$

where $0<h(t)\le C$ and $h'(t)\le 0$ for any $t\ge 0$.

Before closing this section we note that the above result can be interpreted as a necessary condition on the exponent p for global solvability. But can we obtain a global solution to (2.1) if $p>p_{\mathrm{c}}$? In the following section we shall see that the answer depends on the behavior at infinity of both h and the initial data.

3. Nonexistence of local and global nonnegative solutions to the heat inequality. In this section we turn our attention to the nonexistence of solution to

$$\begin{cases} u_t \ge -(-\Delta)^{\beta/2}u + h(x,t)u^p, & (x,t)\in\mathbb{R}^N\times(0,T),\\ u(x,0)=u_0(x)\ge 0, & x\in\mathbb{R}^N, \end{cases} \tag{3.1}$$

where $p>1$, $0<T\le\infty$, and $u_0\ge 0$ and $h>0$ are locally bounded functions. We have argued in Section 2 that if N is small enough any possible local nonnegative solution ceases to exists in a finite time. On the other hand, it is well known [4], [23] that if p is greater than the critical exponent both global and nonglobal solutions may exist. For Problem (3.1) it is easy to see that if we have

$$h(x,t) = e^{t^a + |x|^b}, \qquad a, b > 0,$$

then condition (2.3) is satisfied for any $l > 0$. Therefore, there is no global nonnegative and nontrivial solution to (3.1) for any $p > 1$. So our intention here is to study the effect of h on the nonexistence of local and global solutions.

This work is motivated by the paper of Baras and Kersner in [2] in which the local solvability of

$$\begin{cases} u_t = \Delta u + h(x)u^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.2}$$

was considered. The authors proved that the result depends on the behavior at infinity of both u_0 and h. In particular, it is shown that no solution to (3.2) exists if

$$\lim_{|x|\to\infty} u_0(x)^{p-1} h(x) = \infty.$$

In this section we extend this result to (3.1). We shall show that existence of global solutions requires suitable behavior of the initial data and h at infinity. To prove Theorem 1.2 we shall need some additional lemmas.

Lemma 3.1. *Let $p > 1$. Assume that u is a nonnegative solution to* (3.1) *on* $(0,T), T < \infty$. *Then, for any $\Phi \in C_0^\infty(\mathbb{R}^N), \Phi \geq 0$, we have*

$$\int_{\mathbb{R}^N} \Phi u_0 \leq (p-1)2^{1/(p-1)} p^{-p/(p-1)} \bigg\{ \left(\frac{p}{p-1}\right)^{p'} T^{-p'} \int_{Q_T} \Phi h^{1-p'} + \int_{Q_T} ((-\Delta)^{\beta/2}\Phi)_+^{p'} (\Phi h)^{1-p'} \bigg\}, \tag{3.3}$$

where $Q_T := \mathbb{R}^N \times [0,T)$, *and*

$$\liminf_{|x|\to\infty} u_0(x)^{p-1} h(x,t) \leq \frac{2}{p-1} \cdot \frac{1}{T} \tag{3.4}$$

for any $0 \leq t < T$.

Proof. Let $\zeta \in C_0^\infty(Q_T)$, $\zeta \geq 0$. One has

$$\int_{\mathbb{R}^N} u_0 \zeta(\cdot,0) + \int_{Q_T} h u^p \leq \int_{Q_T} u(-\zeta_t)_+ + \int_{Q_T} u((-\Delta^{\beta/2})\zeta)_+. \tag{3.5}$$

Using the estimates

$$\int_{Q_T} u(-\zeta_t)_+ \leq \frac{1}{2} \int_{Q_T} u^p h + (p-1)2^{1/(p-1)} p^{-p/(p-1)} \int_{Q_T} (-\zeta_t)_+^{p'} (h\zeta)^{1-p'}$$

and

$$\int\limits_{Q_T} u((-\Delta)^{\beta/2}\zeta)_+$$
$$\leq \frac{1}{2}\int\limits_{Q_T} u^p h\zeta + (p-1)2^{1/(p-1)}p^{-p/(p-1)}\int\limits_{Q_T}((-\Delta)^{\beta/2}\zeta)_+^{p'}(h\zeta)^{1-p'},$$

we deduce

$$\int\limits_{\mathbb{R}^N} u_0\zeta(\cdot,0)$$
$$\leq (p-1)2^{1/(p-1)}p^{-p/(p-1)}\Big[\int\limits_{Q_T}(-\zeta_t)_+^{p'}(h\zeta)^{1-p'} + \int\limits_{Q_T}((-\Delta)^{\beta/2}\zeta)_+^{p'}(h\zeta)^{1-p'}\Big].$$

Next, estimate (3.3) is obtained immediately by taking

$$\zeta(x,t) = (1-t/T)^{p'}\Phi(x),$$

where $\Phi \in C_0^\infty(\mathbb{R}^N)$, $\Phi \geq 0$.

To verify (3.4) we consider

$$\Phi(x) = \varphi(x/R), \qquad R > 0,$$

where $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \varphi \leq 1$, $\operatorname{supp}\varphi \subset \{1 < |x| < 2\}$, and

$$(-\Delta)^{\beta/2}\varphi \leq k\varphi, \qquad k = \text{const.}$$

Together with (3.3) we find that

$$\inf_{|x|>R,\,0\leq t<T}(u_0(x)h(x,t)^{p'-1})\int\limits_{Q_T}\Phi h^{1-p'}$$
$$\leq (p-1)2^{1/(p-1)}p^{-p/(p-1)}\Big[\Big(\frac{p}{p-1}\Big)^{p'}T^{1-p'}\int\limits_{Q_T}\Phi h^{1-p'} + T\frac{C}{R^{\beta p'}}\int\limits_{Q_T}\Phi h^{1-p'}\Big].$$

We then divide by $\int_{Q_T}\Phi h^{1-p'}$ and let $R \to \infty$ to obtain

$$\liminf_{|x|\to\infty} u_0(x)h(x,t)^{p'-1} \leq 2^{1/(p-1)}(p-1)^{-1/(p-1)}T^{1-p'},$$

which completes the proof. ■

From this result one deduces immediately the following corollaries.

Corollary 3.1. *There is no local* (*and then no global*) *solution to* (3.1) *if*

$$\lim_{|x|\to\infty} u_0(x)^{p-1}h(x,t) = \infty.$$

Corollary 3.2. *Assume that* $h(x,t) = h(x)$ *and*

$$\liminf_{|x|\to\infty} u_0(x)^{p-1}h(x) > 0.$$

Then any possible local solution to (3.1) *ceases to exists before a finite time* T_0 *such that*

$$T_0 \leq \frac{2}{p-1} (\liminf_{|x|\to\infty} u_0(x)^{p-1} h(x))^{-1}.$$

In the case where $h(x,t) \geq g(x)$ for $|x|$ large we improve results in Corollaries 3.1 and 3.2. In fact our strategy also gives a necessary condition for the global existence.

Proposition 3.1. *Let u be a global nonnegative solution to* (3.1) *where* $h(x,t) \geq g(x)$ *for* $|x|$ *large. Then*

$$(3.6) \qquad \int_{\mathbb{R}^N} \Phi u_0$$
$$\leq \frac{p}{p-1}\left(\frac{2}{p}\right)^{1/(p-1)} (p-1)^{1/p}\Big\{\int_{\mathbb{R}^N} g^{1-p'}\Phi\Big\}^{1/p'}\Big\{\int_{\mathbb{R}^N} ((-\Delta^{\beta/2})\Phi)_+^{p'}(\Phi g)^{1-p'}\Big\}^{1/p}$$

for any $\Phi \in C_0^\infty(\mathbb{R}^N)$, $\Phi \geq 0$, *and*

$$\liminf_{|x|\to\infty} u_0(x)^{p-1} g(x)|x|^\beta \leq C,$$

where C *is a positive constant depending on* p.

Proof. Since u is also a solution to (3.1) on $(0,T)$ we deduce from the proof of the previous lemma that

$$(3.7) \qquad \int_{\mathbb{R}^N} u_0\Phi \leq K(p)\Big[p'^{p'}T^{1-p'}\int_{\mathbb{R}^N} g^{1-p'}\Phi + T\int_{\mathbb{R}^N} ((-\Delta)^{\beta/2}\Phi)_+^{p'}(\Phi g)^{1-p'}\Big],$$

where

$$K(p) = (p-1)2^{1/(p-1)}p^{-p/(p-1)}.$$

A simple minimization of the right hand side of (3.7) with respect to $T > 0$ yields (3.6).

Next, we take, as above,

$$\Phi(x) = \varphi(x/R).$$

Therefore

$$\int_{|x|>R} u_0\Phi \leq \overline{K}(p)\,\frac{1}{R^{\beta p'/p}}\int_{|x|>R} g^{1-p'}\Phi,$$

and since

$$\inf_{|x|>R}(u_0(x)g(x)^{p'-1}|x|^{\beta p'/p})\int_{|x|>R} g^{1-p'}\Phi|x|^{-\beta p'/p} \leq \int_{|x|>R} u_0\Phi,$$

we deduce that

$$\frac{1}{(2R)^{\beta p'/p}} \inf_{|x|>R} (u_0(x) g(x)^{p'-1} |x|^{\beta p'/p}) \int\limits_{|x|>R} g^{1-p'} \varPhi \le \overline{K}(p) \frac{1}{R^{\beta p'/p}} \int\limits_{|x|>R} g^{1-p'} \varPhi.$$

Therefore the limit

$$\liminf_{|x|\to\infty} u_0(x) g(x) |x|^{\beta p'/p}$$

is finite. ∎

Now we are in a position to state our main results of this section.

THEOREM 3.1. *Assume that*

$$\lim_{|x|\to\infty} u_0(x)^{p-1} h(x,t) = \infty \tag{3.8}$$

for any $t \ge 0$. *Then Problem* (3.1) *has no nonnegative local solution.*

THEOREM 3.2. *Assume that* $h(x,t) \ge g(x)$ *and*

$$\lim_{|x|\to\infty} u_0(x)^{p-1} g(x) |x|^{\beta} = \infty.$$

Then Problem (3.1) *has no nonnegative global solution.*

EXAMPLE. Assume $h(x) \approx |x|^{\sigma}$ for $|x|$ large and $u_0(x) = C_0 |x|^{\tau}$. Then we require σ and τ to satisfy

$$\tau(p-1) + \sigma + \beta > 0.$$

The following result is proved by Kalashnikov [13], [2] for the heat equation and it is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. *Assume that* $g(x)$ *goes to infinity with* $|x|$. *Then no local solution* u *exists such that*

$$u(x,t) \ge a > 0 \quad \text{for any } (x,t).$$

REMARK 3.1. As was mentioned in [20], in general if a solution u to (3.1) is not global, it is not easy to prove, via comparison theorems, that there is no global solution to (3.1) with h_1 and v_0 instead of h and u_0 respectively where $h_1 \ge h$ and $v_0 \ge u_0$. However, it is transparent from Theorem 3.2 that in this case no global nonnegative solution can exist, since

$$u_0(x)^{p-1} h(x) |x|^{\beta} \le v_0(x)^{p-1} h_1(x) |x|^{\beta}.$$

The following result gives a necessary condition for global solvability of (3.1).

PROPOSITION 3.2. *Assume that* (3.1) *has a global solution. Then there exists a positive constant* $C = C(p,N)$ *such that*

$$\int\limits_{|x|<r} u_0 \le C \liminf_{R\to\infty} R^{-\beta/(p-1)} \int\limits_{|x|<R} g^{1-p'} \quad \text{for any } r > 0. \tag{3.9}$$

Proof. We take $\Phi(x) = \varphi(x/R)$ where $\varphi \in C_0^\infty(\mathbb{R}^N)$ is positive, $\varphi = 1$ on $|x| \le 1, \varphi = 0$ on $|x| \ge 2$ and $(-\Delta)^{\beta/2}\varphi \le K\varphi$. Then Proposition 3.1 shows that

$$\int\limits_{|x|<R} u_0 \le C\frac{1}{R^{\beta p'/p}}\Big(\int\limits_{|x|<2R} g^{1-p'}\Big)^{1/p'}\Big(\int\limits_{R<|x|<2R} g^{1-p'}\Big)^{1/p}.$$

Now take $R > r$ and define

$$F(R) = \int\limits_{|x|<R} g^{1-p'}.$$

Then we have

$$R^{\beta p'/p} I \le F(2R)^{1/p'}[F(2R) - F(R)]^{1/p},$$

where

$$I = \text{const} \cdot \int\limits_{|x|<r} u_0.$$

Next we define a sequence $(w_n)_{n\in\mathbb{N}^\star}$ by $w_n = F(2^n R)$. Then

$$2^{\beta np'/p} R^{\beta p'/p} I \le w_{n+1}^{1/p'}(w_{n+1} - w_n)^{1/p},$$

hence

$$2^{\beta np'} R^{\beta p'} I^p \le w_{n+1}^p - w_n^p.$$

Summing these inequalities from 1 to $n = j - 1$, one sees that

$$R^{\beta p'}\,\frac{2^{j\beta p'} - 2^{\beta p'}}{2^{\beta p'} - 1}\,R^{\beta p'} I^p \le w_n^p$$

for any $j \ge 2$, which yields

$$F(2^n R) \ge \text{const} \cdot (2^n R)^{\beta p'/p} I,$$

and this implies the desired estimate. ∎

Remark 3.2. For the case $h = 1$ condition (3.9) can be formulated as

$$\limsup_{R\to\infty} R^{\beta/(p-1)-N} \int\limits_{|x|<R} u_0 < \infty.$$

Therefore if $1 < p < 1 + \beta/N$, we deduce that $u_0 \equiv 0$ (see Section 2), while u_0 is integrable if $p = 1 + \beta/N$.

For instance, if $u_0(x) = a|x|^{-b}, a > 0, b > 0$, then the last condition is equivalent to $b \le \beta/(p-1)$. Now consider $h(x) = |x|^l$; then if $1 < p < 1 + (\beta + l)/N$, no global nontrivial solution to (3.1) exists (see [10]).

Remark 3.3. The proof of Proposition 3.2 produces also a necessary condition for the global existence of solutions to

$$u_t = -(-\Delta)^{\beta/2} u + h(x,t)u^p + f(x).$$

This condition is given by the inequality

$$C \int_{|x|<r} f \le \liminf_{R\to\infty} R^{-\beta/(p-1)} \int_{|x|<R} g^{1-p'} \tag{3.10}$$

for any $r > 0$, where $g(x) \le h(x,t)$ and C is a positive constant.

REMARK 3.4. By using the test function $\zeta = (1-t/T)^{p'}\Phi$ as above, we can also obtain a nonexistence result for the problem

$$u_t \ge -(-\Delta)^{\beta/2}u - V(x)u + h(x)u^p, \qquad u(x,0) = u_0(x), \tag{3.11}$$

in which the potential $V := V_+ - V_-$ satisfies, for $|x| \to \infty$,

$$V_+(x) \sim a|x|^{-b}, \qquad a,b > 0.$$

The above method yields precisely the following estimate:

$$\int_{|x|>R} u_0\Phi \le C\bigg[T^{1-p'} \int_{R<|x|<2R} \frac{\Phi}{h^{p'-1}} + T\bigg\{R^{-\beta p'} \int_{R<|x|<2R} \frac{\Phi}{h^{p'-1}} + \int_{R<|x|<2R} V_+^{p'} \frac{\Phi}{h^{p'-1}}\bigg\}\bigg]$$

for any $T > 0$. As usual we obtain, for $\gamma > 0$ and any $T > 0$,

$$\inf_{|x|>R} u_0(x)h(x)^{p'-1}|x|^\gamma \le CR^\gamma[T^{1-p'} + T(R^{-\beta p'} + R^{-bp'})].$$

A routine minimization with respect to T yields

$$\inf_{|x|>R} u_0(x)h(x)^{p'-1}|x|^\gamma \le CR^{\gamma - \frac{1}{p-1}\inf\{\beta,b\}}.$$

We formulate this conclusion in the following.

PROPOSITION 3.3. *Let $p > 1$ and suppose V satisfies*

$$V_+(x) \sim a|x|^{-b}, \qquad a,b > 0,$$

for $|x|$ large. Then there is no global nonnegative solution to (3.11) *such that*

$$\lim_{|x|\to\infty} u_0(x)^{p-1}h(x)|x|^{\inf\{\beta,b\}} = \infty.$$

REMARK 3.5. From the above results we can derive nonexistence results for the stationary problems of the preceding inequalities. For example there is no nonnegative solution to

$$-(-\Delta)^{\beta/2}u - V(x)u + h(x)u^p = 0,$$

defined on $\mathbb{R}^N$ such that

$$\lim_{|x|\to\infty} u(x)^{p-1}h(x)|x|^{\inf\{\beta,b\}} = \infty,$$

where $V_+(x) \sim a|x|^{-b}$, $a,b > 0$.

4. Porous medium inequalities. A simple model of the problem considered in this section is the following:

$$u_t \geq \Delta u^m + u^p, \tag{4.1}$$

where $p > \max\{1, m\},\ m > 0$. In the case of equality instead of inequality, the problem

$$u_t = \Delta u^m + u^p, \qquad m > 1, \tag{4.2}$$

describes processes with a finite speed of propagation of perturbation [22]. It is known that if the initial data $u_0 \geq 0$ is a bounded continuous function, then Problem (4.2) has a unique, local-in-time, weak, continuous solution $u(x,t) \geq 0$. This solution is not global if $1 < p \leq m + 2/N$ (see [22]). On the other hand, by the same argument as before, if u_0 tends to infinity with $|x|$, there is no local solution.

In [18] Mukai *et al.* presented some properties of solutions to (4.2) where initial data slowly decay near $x = \infty$. For instance, in the case $u_0(x) \sim \lambda |x|^{-a}$ the authors obtained global existence and nonglobal existence in terms of $\lambda > 0$ and $a \geq 0$.

In [21] it is proved that any nontrivial local solution to (4.2) blows up in finite time if $0 < m < 1$ and $1 < p < m + 2/N$. Concerning Problem (4.1), with the help of the argument used in Section 2, we easily get the following.

THEOREM 4.1. *Let* $p > \max\{m, 1\}$. *Assume that*

$$p \leq m + 2/N.$$

Then the only global nonnegative solution to (4.1) *is the trivial one.*

The technique of the preceding section can also be applied to the case where $p > m + 2/N$.

THEOREM 4.2. *Let* $p > \max\{m, 1\}$. *Assume that*

$$\lim_{|x| \to \infty} u_0(x) |x|^{2/(p-m)} = \infty. \tag{4.3}$$

Then any possible local solution $u(x,t)$ *to* (4.1) *such that* $u(x,0) = u_0(x)$ *is not global.*

In this section we shall see that the result of the above theorem still holds if condition (4.3) is satisfied in a weak sense. More precisely, we assume that u_0 satisfies

$$\lim_{|x| \to \infty} u_0(x) |x|^{2/(p-m)} = A \tag{4.4}$$

for some $A \in [0, \infty]$, in the following weak sense: for any $\varphi \in W_0^{1,\infty}(\mathbb{R}^N)$, $\varphi \geq 0$,

$$\lim_{R \to \infty} \int_{\mathbb{R}^N} \varphi(x) |Rx|^{2/(p-m)} u_0(Rx)\, dx = A \int_{\mathbb{R}^N} \varphi(x)\, dx.$$

Define

$$A_\star = \lambda_1^{1/(p-m)} C(p,m),$$

where

$$C(p,m) = \frac{p}{p-1}\left(\frac{2}{p}\right)^{1/(p-m)} (p-m)^{1/p} m^{m/(p(p-m))},$$

and λ_1 is the first eigenvalue of $-\Delta$ in the unit ball B with zero Dirichlet boundary condition. Then we have

PROPOSITION 4.1. *Let $p > m + 2/N$. Assume that u_0 satisfies* (4.4) *where $A > A_\star$. Then Problem* (4.1) *has no global solution.*

Proof. Assume that (4.1) has a global solution with the initial data u_0. Arguing as in the proof of Proposition 3.1 we first deduce that for any $\Phi \in W_0^{1,\infty}(\mathbb{R}^N)$ such that $\Phi \geq 0$ and $\Delta\Phi \in L_0^\infty(\mathbb{R}^N)$, we have

$$(4.5)\qquad \int_{\mathbb{R}^N} \Phi u_0 \leq C(p,m)\Big\{\int_{\mathbb{R}^N} \Phi\Big\}^{1/p'}\Big\{\int_{\mathbb{R}^N} (-\Delta\Phi)_+^{p/(p-m)} \Phi^{-m/(p-m)}\Big\}^{1/p}.$$

Let $\varphi_1 \in C_0^\infty(B)$, $\varphi_1 \geq 0$, be the first eigenfunction of $-\Delta$ in the unit ball B:

$$-\Delta\varphi_1 = \lambda_1\varphi_1.$$

Setting

$$\overline{\varphi}(x) = \begin{cases} \varphi_1(x) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and using (4.5) with $\Phi(x) = \overline{\varphi}(x/R)$ yields

$$R^{2/(p-m)} \int_{\mathbb{R}^N} \overline{\varphi}(x/R) u_0(x)\,dx \leq \lambda_1^{1/(p-1)} C(p,m) \int_{\mathbb{R}^N} \overline{\varphi}(x/R)\,dx.$$

Since

$$\int_{\mathbb{R}^N} \overline{\varphi}(x/R) u_0(x)\,dx = \int_{\mathbb{R}^N} \overline{\varphi}(x/R) u_0(x) |x|^{2/(p-m)} |x|^{-2/(p-m)}\,dx,$$

we infer that

$$\int_{\mathbb{R}^N} \overline{\varphi}(x/R) |x|^{2/(p-m)} u_0(x)\,dx \leq \lambda_1^{1/(p-1)} C(p,m) \int_{\mathbb{R}^N} \overline{\varphi}(x/R)\,dx.$$

Changing the variables $y = Rx$ yields

$$\int_{\mathbb{R}^N} \overline{\varphi}(y) |Ry|^{2/(p-m)} u_0(yR)\,dy \leq \lambda_1^{1/(p-1)} C(p,m) \int_{\mathbb{R}^N} \overline{\varphi}(y)\,dy.$$

Passing to the limit as $R \to \infty$ implies

$$A \leq \lambda_1^{1/(p-1)} C(p,m).$$

This contradicts (4.4) and the proof is finished. ∎

The above result can be easily extended to the inequalities

$$u_t \geq \Delta u^m + |x|^l u^p. \tag{4.6}$$

Here we obtain the nonexistence result for $0 < m \leq 1$ and $-2 < l < 0$.

PROPOSITION 4.2. *Let $0 < m \leq 1$ and $-2 < l < 0$. There is no global nonnegative solution to* (4.6) *such that the initial data satisfies*, *in the weak sense*,

$$\lim_{|x|\to\infty} |x|^{(2+l)/(p-m)} u_0(x) > \lambda_1^{1/(p-m)}.$$

Proof. Assume that u is a global nonnegative solution to (4.6). Put

$$w(t) = R^{-N} \int_{|x|<R} u(x,t)\varphi_1(x/R)\,dx,$$

where φ_1 is the first eigenfunction satisfying $\int_B \varphi_1 = 1$. Using (4.6) we deduce that

$$w_t \geq R^l w^p - \lambda_1 R^{-2} w^m$$

for all $t \geq 0$, thanks to the Jensen and Hölder inequalities. Next a simple analysis of the above ordinary differential inequality yields the estimate

$$w(0) \leq (\lambda_1 R^{-l-2})^{1/(p-m)};$$

that is,

$$\int_{|x|\leq R} R^{-N+(l+2)/(p-m)} u_0(x)\varphi_1(x/R)\,dx \leq \lambda_1^{1/(p-m)}.$$

Now we conclude as in the proof of Proposition 4.1. ■

REMARK 4.1. Let us consider the particular case

$$u_t \geq -(-\Delta)^{\beta/2} u^m + u^p, \qquad p > m, \tag{4.7}$$

where $u_0(x) \sim |x|^{-a}$ for $|x|$ large. Set

$$a^\star = \frac{\beta}{p-m}.$$

According to our discussion, if $0 \leq a < a^\star$ there is no global solution. On the other hand, it is shown in [18] that if $a^\star < a < N$ where $\beta = 2$ and $p > m + 2/N$, then there exist global solutions to $u_t = \Delta u^m + u^p$ such that the limit $\lim_{|x|\to\infty} u(x,0)|x|^a$ is finite and positive. Based on this observation it is natural to address the following question. Can we identify all real a such that the corresponding solution is not global? Is the constant $a^\star$ the threshold between the blow up case and the global existence if $p > m+\beta/N$? For (3.10) the answer is no. In fact $a^\star$ may be infinite as we now show. For simplicity we take $m = 1$ and consider

$$u_t = -(-\Delta)^{\beta/2} u + u^p + \lambda f(x),$$

where $\lambda > 0$, with $f \geq 0$ satisfying

$$\lim_{|x|\to\infty} f(x)^{p-1}|x|^{\beta} = \infty.$$

By the same strategy as in the proof of Theorem 2.1, one easily obtains the nonexistence of global solutions for any nonnegative initial data even if the data has compact support. Now if we assume that $\liminf_{|x|\to\infty} f(x)^{p-1}|x|^{\beta} > 0$, then there exists $\lambda_c > 0$ such that the problem has no global solution for any $\lambda > \lambda_c$. Note that the last condition on f asserts that if $p > 1 + \beta/2$ then

$$\int_{\mathbb{R}^N} \frac{f(x)}{|x|^{N-2}}\, dx = \infty,$$

which is the same condition as in [20] for the case $\beta = 2$. However if $1 < p < \beta/2$ and if we take, for example,

$$f(x) = (1 + |x|)^{-\beta/(p-1)},$$

then the function $f/|x|^{N-2}$ is integrable, but there is no global solution for λ large.

In the next section we shall show that a similar result is valid for an inequality with gradient-dependent diffusivity (so-called p-Laplacian operator).

5. Parabolic inequalities associated to the p-Laplacian operator. In this section we will obtain reasonable assumptions for the nonexistence of global nonnegative solutions to

$$u_t \geq \Delta_p u + h(x,t)u^q, \tag{5.1}$$

where $q > p - 1 > 0$ and Δ_p is the p-Laplacian operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

To this end we will follow the strategy of the previous sections.

The stationary problem was considered by many authors. In that case it is known [12] that if $p - 1 < q < Np/(N-p) - 1$, then there exists no positive radial solution to

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q = 0,$$

and if $q = Np/(N-p) - 1$ then the only positive radial solutions are of the form

$$u_a(x) = C(N,p)(a + |x|^{p/(p-1)})^{(p-N)/p},$$

where a is any positive number. Later Mitidieri and Pokhozhaev [16] showed the nonexistence of nonnegative solutions to

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q \leq 0$$

if one of the following conditions holds:

(a) $p-1 < q \leq N(p-1)/(N-p)$, $p < N$,
(b) $0 \leq 0 \leq p-1$, $N \geq 1$.

For Problem (5.1) with $h = 1$ Mitidieri and Pokhozhaev [17] proved the absence of global solutions if q is not (strictly) larger than a critical exponent. These results are preceded by the work of Galaktionov [6], [7] on the problem

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q.$$

He showed that if $1 \leq p < p-1+p/N$ then no global nontrivial solution exists, while for $q > p-1+p/N$ there is a class of small global solutions. This means that the critical exponent for the last equation is $q_{\mathrm{c}} = p-1+p/N$.

In this section we will combine the argument of [16] and the technique used above to obtain necessary conditions for the existence of local and global solutions to (5.1). Below we assume the function h satisfies condition (2.2). The first result on nonexistence of global nonnegative solutions to (5.1) is formulated below.

Theorem 5.1. *Let $q > p-1 > 1$. Then there exists $N_{\mathrm{c}} = N_{\mathrm{c}}(p,q,l)$ such that for $N \leq N_{\mathrm{c}}$ there are no global solutions to* (5.1).

Proof. Let u be a nonnegative solution to (5.1) with initial data u_0. Without lost of generality we may suppose that $u > 0$. Let $-1 < \alpha < 0$ be a fixed number. Taking $u^\alpha\zeta$ as a test function where $\zeta \in C_0^\infty(\mathbb{R}_+^{N+1})$, $\zeta \geq 0$, one sees that

$$\int_{Q_T} hu^{q+\alpha}\zeta + \frac{1}{1+\alpha}\int_{\mathbb{R}^N} u_0^{1+\alpha}\zeta(\cdot,0) + |\alpha|\int_{Q_T}|\nabla u|^p u^{\alpha-1}\zeta \leq \int_{Q_T}|\nabla u|^{p-1}u^\alpha|\nabla\zeta| - \frac{1}{1+\alpha}\int_{Q_T} u^{\alpha+1}\zeta_t.$$

Making use of the Young inequality once and twice, one finds, for $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$,

$$\begin{aligned}\int_{Q_T} hu^{q+\alpha}\zeta + \frac{1}{1+\alpha}\int_{\mathbb{R}^N} u_0^{1+\alpha}\zeta(\cdot,0) + |\alpha|\int_{Q_T}|\nabla u|^p u^{\alpha-1}\zeta \\ \leq \varepsilon_1\int_{Q_T} u^{\alpha-1}|\nabla u|^p\zeta + C_{\varepsilon_1}\int_{Q_T} u^{\alpha+p-1}|\nabla\zeta|^p\zeta^{1-p} \\ + \varepsilon_2\int_{Q_T} u^{\alpha+q}h\zeta + C_{\varepsilon_2}\int_{Q_T}|\zeta_t|^{(q+\alpha)/(q-1)}(h\zeta)^{-(1+\alpha)/(q-1)},\end{aligned}$$

$$\int\limits_{Q_T} hu^{q+\alpha}\zeta + \frac{1}{1+\alpha}\int\limits_{\mathbb{R}^N} u_0^{1+\alpha}\zeta(\cdot,0) + |\alpha|\int\limits_{Q_T} |\nabla u|^p u^{\alpha-1}\zeta$$
$$\le \varepsilon_1 \int\limits_{Q_T} u^{\alpha-1}|\nabla u|^p\zeta + C_{\varepsilon_1}\bigg[\varepsilon_3\int\limits_{Q_T} u^{\alpha+q}h\zeta + C_{\varepsilon_3}\int\limits_{Q_T}\frac{|\nabla\zeta|^{\tau p}}{h^{\tau-1}\zeta^{\tau p-1}}\bigg]$$
$$+ \varepsilon_2\int\limits_{Q_T} u^{\alpha+q}h\zeta + C_{\varepsilon_2}\int\limits_{Q_T}|\zeta_t|^{(q+\alpha)/(q-1)}(h\zeta)^{-(1+\alpha)/(q-1)},$$

where

$$s(\alpha+p-1) = \alpha+q, \qquad \frac{1}{s}+\frac{1}{\tau} = 1. \tag{5.2}$$

This implies in particular

$$K_1\int\limits_{Q_T} hu^{q+\alpha}\zeta + K_2\int\limits_{Q_T}|\nabla u|^p u^{\alpha-1}\zeta + \frac{1}{1+\alpha}\int\limits_{\mathbb{R}^N} u_0^{1+\alpha}\zeta(\cdot,0) \le \int\limits_{Q_T}\frac{|\nabla\zeta|^{\tau p}}{h^{\tau-1}\zeta^{\tau p-1}} + \int\limits_{Q_T}|\zeta_t|^{(q+\alpha)/(q-1)}(h\zeta)^{-(1+\alpha)/(q-1)}. \tag{5.3}$$

The remainder of the proof is based on an appropriate choice of the test function ζ. As in the first section, we consider

$$\zeta(x,t) = \varphi\bigg(\frac{t^{\gamma_1}+|x|^{\gamma_2}}{R}\bigg),$$

where γ_1,γ_2 are real parameters which will be specified later and the nonnegative function $\varphi\in C_0^\infty(\mathbb{R}_+)$ is defined by (2.8)–(2.9) in Section 2. After the standard change of variables $t\mapsto tR^{-\gamma_1}$ and $x\mapsto xR^{-\gamma_2}$, inequality (5.3) takes the form

$$\int\limits_{Q_T} u^q h\zeta + \int\limits_{\mathbb{R}^N} u_0\zeta(\cdot,0) \le KR^r,$$

where

$$r = \frac{1}{\gamma_1} + \frac{N}{\gamma_2} - \frac{\tau p}{\gamma_2} - l\,\frac{1+\alpha}{q-1}.$$

Here the real l is determined from the asymptotic behavior at infinity of h, that is,

$$h(xR^{\gamma_2}, tR^{\gamma_1}) \sim R^l$$

for R large. The expression of r is obtained with the help of the following assumption on γ_1 and γ_2:

$$l(\tau-1) + \frac{\tau p}{\gamma_2} = \frac{1}{\gamma_1}\frac{q+\alpha}{q-1} + l\,\frac{1+\alpha}{q-1}.$$

Therefore there exists $N_{\mathrm{c}} > 0$ such that if $N \le N_{\mathrm{c}}$ then $r \le 0$ and so $u\equiv 0$, which ends the proof. ■

REMARK 5.1. To conclude we obtain, as in Section 2, an estimate of the integral

$$\int_{Q_T} u^q h\zeta + \int_{\mathbb{R}^N} u_0\zeta(\cdot,0).$$

Since this estimate is the key to nonexistence of local and global solutions if $N > N_{\mathrm{c}}$, we give here its proof.

LEMMA 5.1. *Assume that* $q > \max\{p-1, 1\}$. *Let* u *be a local positive solution to* (5.1). *Then for any* $\alpha \in (-(q+1-p)/p, 0)$, *there exists* $a > 0$ *such that*

$$\int_{Q_T} u^q h\zeta + \int_{\mathbb{R}^N} u_0\zeta(\cdot,0) \le K\bigg\{ \int_{Q_T} \frac{|\zeta_t|^{q'}}{h\zeta^{q'-1}} + \int_{Q_T} \frac{|\nabla\zeta|^{\tau p}}{h^{\tau-1}\zeta^{\tau p-1}} + \int_{Q_T} |\zeta_t|^{(q+\alpha)/(q-1)}(h\zeta)^{-(1+\alpha)/(q-1)} + \int_{Q_T} \frac{|\nabla\zeta|^{pa'}}{\zeta^{pa'-1}h^{a'-1}}\bigg\}$$

for some constant $K > 0$, *where* $a' = a/(a-1)$.

Proof. In fact, since by (5.1),

$$\int_{Q_T} u^q h\zeta + \int_{\mathbb{R}^N} u_0\zeta(\cdot,0) \le \int_{Q_T} |\nabla u|^{p-1}|\nabla\zeta| + \int_{Q_T} u|\zeta_t|,$$

we deduce that

$$\int_{Q_T} u^q h\zeta \le \varepsilon \int_{Q_T} u^q h\zeta + C_\varepsilon \int_{Q_T} \frac{|\zeta_t|^{q'}}{h\zeta^{q'-1}} + \int_{Q_T} |\nabla u|^{p-1}|\nabla\zeta|.$$

The last integral can be written as

$$\int_{Q_T} |\nabla u|^{p-1}|\nabla\zeta| = \int_{Q_T} |\nabla u|^{p-1}u^{(\alpha-1)(p-1)/p}\zeta^{1/p'}u^{(1-\alpha)(p-1)/p}\zeta^{-1/p'}|\nabla\zeta|.$$

Therefore, by the Hölder inequality, we get

$$\begin{aligned}\int_{Q_T} |\nabla u|^{p-1}|\nabla\zeta| &\le \bigg\{\int_{Q_T} |\nabla u|^p u^{\alpha-1}\zeta\bigg\}^{(p-1)/p}\bigg\{\int_{Q_T} u^{(1-\alpha)(p-1)}\frac{|\nabla\zeta|^p}{\zeta^{p-1}}\bigg\}^{1/p}\\ &\le \bigg\{\int_{Q_T} |\nabla u|^p u^{\alpha-1}\zeta\bigg\}^{(p-1)/p}\bigg\{\int_{Q_T} u^{q+\alpha}h\zeta\bigg\}^{1/(ap)}\bigg\{\int_{Q_T}\frac{|\nabla\zeta|^{pa'}}{\zeta^{pa'-1}h^{a'-1}}\bigg\}^{1/(a'p)},\end{aligned}$$

where $(p-1-q)/p < \alpha < 0$ and

$$(1-\alpha)(p-1)a = q+\alpha, \qquad 1/a + 1/a' = 1.$$

Thus

$$(1-\varepsilon)\int_{Q_T} u^q h\zeta \le C_\varepsilon \int_{Q_T} \frac{|\zeta_t|^{q'}}{h\zeta^{q'-1}} + \Big\{\int_{Q_T} |\nabla u|^p u^{\alpha-1}\zeta\Big\}^{(p-1)/p}$$
$$\times \Big\{\int_{Q_T} u^{q+\alpha} h\zeta\Big\}^{1/(ap)} \Big\{\int_{Q_T} \frac{|\nabla\zeta|^{pa'}}{\zeta^{pa'-1}h^{a'-1}}\Big\}^{1/(a'p)}.$$

Next using estimate (5.3), we deduce for some positive constant K the desired estimate

$$\int_{Q_T} u^q h\zeta + \int_{\mathbb{R}^N} u_0\zeta(\cdot,0)$$
$$\le K\Bigg\{\int_{Q_T} \frac{|\zeta_t|^{q'}}{h\zeta^{q'-1}} + \Bigg[\int_{Q_T} \frac{|\nabla\zeta|^{\tau p}}{h^{\tau-1}\zeta^{\tau p-1}}$$
$$+ \int_{Q_T} |\zeta_t|^{(q+\alpha)/(q-1)}(h\zeta)^{-(1+\alpha)/(q-1)}\Bigg]^{1/p'+1/(ap)} \Big\{\int_{Q_T} \frac{|\nabla\zeta|^{pa'}}{\zeta^{pa'-1}h^{a'-1}}\Big\}^{1/(a'p)}\Bigg\}.$$

Note that, since $1/p' + 1/(ap) + 1/(a'p) = 1$, this estimate leads to

$$\int_{Q_T} u^q h\zeta + \int_{\mathbb{R}^N} u_0\zeta(\cdot,0) \le K\Bigg\{\int_{Q_T} \frac{|\zeta_t|^{q'}}{h\zeta^{q'-1}} + \int_{Q_T} \frac{|\nabla\zeta|^{\tau p}}{h^{\tau-1}\zeta^{\tau p-1}}$$
$$+ \int_{Q_T} |\zeta_t|^{(q+\alpha)/(q-1)}(h\zeta)^{-(1+\alpha)/(q-1)} + \int_{Q_T} \frac{|\nabla\zeta|^{pa'}}{\zeta^{pa'-1}h^{a'-1}}\Bigg\}. \ \blacksquare$$

REMARK 5.2. When $h = 1$ the nonexistence result is very simple to formulate. In this case γ_1 and γ_2 satisfy $\gamma_1/\gamma_2 = (q+\alpha)/(\tau p(q-1))$; that is,

$$\frac{\gamma_1}{\gamma_2} = \frac{q-p+1}{p(q-1)}.$$

Thus

$$\int_{Q_T} u^q h\zeta \le CR^{r_1}.$$

Therefore if

$$q \le p - 1 + p/N$$

then the exponent r_1 is nonpositive, which leads to the conclusion $u \equiv 0$.

In the remainder of this section we argue as in Section 3 to exhibit necessary conditions for local and global existence of solutions to

$$u_t \ge \operatorname{div}(|\nabla u|^{p-2}\nabla u) + h(x)u^q, \quad u(x,0) = u_0(x). \tag{5.4}$$

Since the results are the analogues of those obtained before, they are stated without proofs.

Put
$$g(x) = \inf\{h(x), h(x)^{(\tau-1)(q-1)/(1+\alpha)}\},$$
where τ is given by (5.2).

THEOREM 5.2. *Assume that* $q > \max\{p-1, 1\}$. *Let* u *be a local nonnegative solution to* (5.4) *defined in* $(0, T)$. *Then for any fixed* α *such that* $(p+1-q)/p < \alpha < 0$ *there exists* $C_0 > 0$ *such that*
$$\liminf_{|x|\to\infty} u_0(x)^{q-1} g(x) \le C_0/T.$$

It is not hard to obtain explicitly the expression for C_0.

Let us now consider the function
$$h(x) = (1 + |x|^2)^a, \qquad a > 0,$$
and the initial data defined by
$$u_0(x) = T^{-1/(q-1)} A(1 + |x|^2)^{-1/(b(q-1))}.$$
Then we have

COROLLARY 5.1. *Let* $p \ge 2$ *and* $q > p-1$. *Assume that*
$$A \le C_0^{1/(q-1)}.$$
Then any solution u *to* (5.4) *such that* $u(\cdot, 0) = u_0$ *ceases to exist after a finite time not longer than* T.

The following result gives a sufficient condition for nonglobal solvability.

THEOREM 5.3. *Let* $q > \max\{1, p-1\}$. *Assume that*
$$\lim_{|x|\to\infty} u_0(x)^{q-1} g(x) |x|^{p(q-1)/(q-p+1)} = \infty.$$
Then any local solution to (5.4) *is not global.*

REMARK 5.3. In [16] the authors constructed an explicit stationary solution in the case $h = 1$. This solution is given by
$$u(x) = \frac{a}{(1 + |x|^{p'})^{(p-1)/(q-p+1)}},$$
where the parameter $a > 0$ is small enough, namely
$$a^{q-p+1} < \left(\frac{p}{q-p+1}\right)^{p-1} \frac{q(N-p) - N(p-1)}{q-p+1} =: a_{N,p,q},$$
and satisfies
$$|x|^{p/(q-p+1)} u(x) \to a \qquad \text{as } |x| \to \infty.$$

It is easily seen [12] that the function
$$u(x) = a_{N,p,q} |x|^{-p/(q-p+1)}$$
is a solution to $\Delta_p u + u^q = 0$ in $\mathbb{R}^N \setminus \{0\}$. Note that this solution is locally integrable if $q > p - 1 + p/N$.

REMARK 5.4. If we have $-u^q$ instead of u^q, Gmira [8] showed that the problem $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q$ has a global solution u such that $\lim_{|x|\to\infty} u(x,t)|x|^{p/(q-p+1)} = \infty$ for any t.

We finish this section with an extension of Theorem 3.2.

PROPOSITION 5.1. *Let* $q > \max\{p-1, 1\}$ *and* $(p+1-q)/p < \alpha < 0$. *Assume that the initial data* u_0 *produces a global solution to* (5.4). *Then for any* $\gamma > 0$ *there exists a positive constant* C *such that*

$$\int_{|x|\le\gamma} u_0 \le C \liminf_{R\to\infty} R^{p(1+\alpha)/(q-p+1)} \int_{|x|<R} g^{-(1+\alpha)/(q-1)}.$$

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