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HEREDITARILY INDECOMPOSABLE CONTINUA WITH EXACTLY n AUTOHOMEOMORPHISMS

BY

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Abstract. The main goal of this paper is to construct, for every $n, m \in \mathbb{N}$, a hereditarily indecomposable continuum X_{nm} of dimension m which has exactly n autohomeomorphisms.

1. Introduction. All spaces considered are assumed to be metrizable separable. Our terminology follows [6] and [10]. A continuum X is *hereditarily indecomposable*, abbreviated HI, if for any two intersecting subcontinua K, L of X, either $K \subset L$ or $L \subset K$. For a continuum X, let $\mathcal{G}(X)$ denotes the group of all homeomorphisms of X onto X. A continuum X is *rigid* if the identity 1_X is the only homeomorphism of X onto X, i.e., $\mathcal{G}(X) = \{1_X\}$. In [5] H. Cook gave an example of a rigid, 1-dimensional, HI continuum. Recently M. Reńska [18] constructed, for every $m \in \mathbb{N}$, an HI rigid m-dimensional Cantor manifold. The main goal of this paper is to prove the following theorem.

1.1. Theorem. *For every $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$ there exists an HI continuum X_{nm} such that $\dim X_{nm} = m$ and the group $\mathcal{G}(X_{nm})$ of homeomorphisms of X_{nm} onto X_{nm} is a cyclic group of order n.*

A homeomorphism $h : X \to X$ is *stable* if there exist homeomorphisms $h_0, h_1, \ldots, h_n$ such that $h = h_n h_{n-1} \ldots h_1 h_0$ and for every $i \le n$ there exists a nonempty open set U_i such that $h_i|U_i$ is the identity. The continua X_{nm} constructed in Theorem 1.1 have the property that the set of stable homeomorphisms of X_{nm} onto X_{nm} is degenerate and is not dense in the space $\mathcal{G}(X_{nm})$ for $n > 1$. The next theorem shows that there exist HI continua with $2^{\aleph_0}$ homeomorphisms, each of which is stable (moreover, it is the identity on some open nonempty subset).

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1.2. THEOREM. *For every $m \in \mathbb{N} \cup \{\infty\}$ there exists an HI continuum Y_m with* $\dim Y_m = m$ *such that the group $\mathcal{G}(Y_m)$ of homeomorphisms of Y_m onto Y_m has cardinality $2^{\aleph_0}$ and there exists a nonempty open subset U_m of Y_m such that for every $h \in \mathcal{G}(Y_m)$, $h|U_m = 1_{U_m}$.*

A space X is *strongly infinite-dimensional* (abbreviated SID) if there exists an infinite sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X such that if L_i is a partition between A_i and B_i in X for $i = 1, 2, \ldots$ then $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. An SID space X is *hereditarily SID* if every subset of X of positive dimension is SID. An infinite-dimensional continuum X is a *Cantor manifold* if all closed sets which disconnect X are infinite-dimensional. The first hereditarily SID compactum was constructed by Rubin [19] (cf. [6, Problem 6.1.G]); while the first example of an SID compactum all of whose nontrivial subcontinua are infinite-dimensional was given earlier by Henderson [7]. In [18] M. Reńska constructed a rigid HI hereditarily SID Cantor manifold. We will prove the following theorem.

1.3. THEOREM. *For every $n \in \mathbb{N}$ there exists an HI continuum Z_n, all of whose nontrivial subcontinua are strongly infinite-dimensional, such that the group $\mathcal{G}(Z_n)$ of homeomorphisms of Z_n onto Z_n is a cyclic group of order n.*

In our constructions we apply some ideas of [3], [18] and [16].

It is an interesting question whether the spaces X_{nm} and Y_m satisfying the conditions of Theorems 1.1 and 1.2 can be m-dimensional Cantor manifolds (for $m > 1$) and whether the spaces Z_n from Theorem 1.3 can be infinite-dimensional Cantor manifolds.

2. Preliminaries. The first HI continuum, now called the pseudo-arc, was constructed by B. Knaster [9] in 1922. The pseudo-arc, which will be denoted by P, is an HI one-dimensional chainable continuum (unique, up to homeomorphism); and it is the only (up to homeomorphism) nondegenerate, homogeneous, chainable continuum. The pseudo-arc P is also hereditarily equivalent, i.e., every nontrivial subcontinuum of P is homeomorphic to P (cf. [10, §48, X], or [13]).

The first examples of HI continua of arbitrary dimension n, where $n \in \{2, 3, \ldots, \infty\}$, were constructed by R. H. Bing [2].

The *composant* of a point x in a continuum X is the union of all proper subcontinua of X containing x. If X is a nontrivial HI continuum, then (see [10, §48, VI])

(a) every composant of X is a connected F_σ-subset of X, both dense and boundary in X,

(b) different composants of X are disjoint, and

(c) (Mazurkiewicz's theorem) X has continuum many different composants.

A subcontinuum K of a continuum X is *terminal* if every subcontinuum of X which intersects both K and its complement must contain K. A continuous mapping from a continuum X onto Y is called *atomic* if every fiber of f is a terminal subcontinuum of X.

In our constructions we will apply the method of condensation of singularities, which goes back to Anderson and Choquet [1]. Namely, we will need the following construction, based on the technique of Maćkowiak [14], [15] and described in detail in [17] (cf. also [4]).

2.1. THEOREM. *Let X be a continuum, $\{Z_i : i \in \mathbb{N}\}$ a sequence of compacta, $\{A_i : i \in \mathbb{N}\}$ a sequence of 0-dimensional compact disjoint subsets of X, and suppose each Z_i admits a continuous map onto A_i with connected fibers. Then there exist a continuum $L(X, Z_i, A_i)$ and an atomic mapping $p : L(X, Z_i, A_i) \to X$ such that*

(i) *$p|p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) : p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) \to X \setminus \bigcup_{i=1}^{\infty} A_i$ is a homeomorphism,*

(ii) *$p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i)$ is dense in $L(X, Z_i, A_i)$,*

(iii) *$p^{-1}(A_i)$ is homeomorphic to Z_i for every $i \in \mathbb{N}$ (hence $p^{-1}(a)$ is homeomorphic to a component of Z_i if $a \in A_i$),*

(iv) *if n and m are natural numbers such that $\dim X \le n$ and $\dim Z_i \le m$ for every $i \in \mathbb{N}$ then $\dim L(X, Z_i, A_i) \le \max(n, m)$,*

(v) *if $C(x)$ is the composant of x in $L(X, Z_i, A_i)$ then $C(x) = p^{-1}(C(p(x)))$, where $C(p(x))$ is the composant of $p(x)$ in X.*

The existence of the space $L(X, Z_i, A_i)$ which admits an atomic mapping $p : L(X, Z_i, A_i) \to X$ with properties (i)–(iv) follows from [17, Theorem 3.2] and property (v) follows from the atomicity of p (see [17, Lemma 2.8]).

We will also need the following auxiliary facts.

2.2. LEMMA. *For every $m \in \mathbb{N}$ there exists an infinite family of pairwise nonhomeomorphic HI m-dimensional Cantor manifolds.*

For $m = 1$ such a family of cardinality $2^{\aleph_0}$ was constructed by R. H. Bing [3]. For $m = 2, 3, \ldots$, the existence of such a family follows, for example, from the following lemma proved by M. Reńska in [18]: for every m-dimensional HI continuum K there exists an HI m-dimensional Cantor manifold M such that K is not embeddable into M. Indeed, let K and M be two m-dimensional Cantor manifolds such that K does not embed in M and let $a_1, a_2, \ldots$ be points of M such that a_k and a_l belong to different composants of M if $k \neq l$. Then $K_j = L(M, Z_i, A_i)$, where $Z_i = K$ and $A_i = \{a_i\}$ for $i = 1, \ldots, j$ and $Z_i = \emptyset = A_i$ for $i > j$, is an m-dimensional Cantor manifold exactly j of whose composants do not embed in M (see Theorem 2.1).

Thus K_j is not homeomorphic to K_l for $j \neq l$. Let us add that, as proved in [18], there also exists a family of cardinality $2^{\aleph_0}$ consisting of topologically different HI rigid m-dimensional Cantor manifolds.

2.3. LEMMA (H. Cook [5]). *There exists a one-dimensional HI continuum no two of whose nondegenerate subcontinua are homeomorphic.*

2.4. LEMMA. *There exists an infinite family of topologically different HI hereditarily SID Cantor manifolds.*

The existence of such a family follows from Corollary 4.3 of [16] stating that for every hereditarily SID compactum K there exists an HI hereditarily SID Cantor manifold which does not embed in K. Moreover, as proved in [16], there exists such a family of cardinality $2^{\aleph_0}$.

2.5. LEMMA (W. Lewis [12]). *For every $n \in \mathbb{N}$ there exists a homeomorphism r of the pseudo-arc of period n. Moreover, for each $n \in \mathbb{N}$ there exists an embedding of the pseudo-arc in the plane such that r is the restriction of a period n rotation of the plane.*

2.6. LEMMA. *Let U be an open subset of the pseudo-arc P such that $P \setminus \overline{U} \neq \emptyset$. Then there exists a family $\{h_t : t \in T\}$ of homeomorphisms of P onto P, where $|T| = 2^{\aleph_0}$, such that $h_{t'} \neq h_t$ if $t' \neq t$ and $h_t|U = 1_U$ for every t.*

This lemma follows immediately from Theorem 8 in [11], stating that if p and q are distinct points of $P \setminus U$, where U is open in P, such that the subcontinuum M irreducible between p and q does not intersect $\mathrm{cl}(U)$, then there is a homeomorphism $h : P \to P$ with $h(p) = q$ and $h|U = 1_U$ (cf. also [8, Theorem]).

2.7. LEMMA. *Let $p : X \to Y$ and $\widetilde{p} : \widetilde{X} \to Y$ be mappings between continua such that p is atomic and for every $y \in Y$ with $\widetilde{p}^{-1}(y)$ nondegenerate there exists an open neighborhood U of y in Y, a homeomorphism h of $\widetilde{p}^{-1}(\overline{U})$ onto a subset of X and a homeomorphism g of $ph(\widetilde{p}^{-1}(\overline{U}))$ onto $\overline{U}$ such that $\widetilde{p}(x) = gph(x)$ for every $x \in \widetilde{X}$. Then $\widetilde{p}$ is atomic.*

Proof. Take $y \in Y$ such that $\widetilde{p}^{-1}(y)$ is nondegenerate and let U, h and g be as above. Let L be any continuum in $\widetilde{X}$ such that $L \cap \widetilde{p}^{-1}(y) \neq \emptyset \neq L \setminus \widetilde{p}^{-1}(y)$. We will show that $L \supset \widetilde{p}^{-1}(y)$.

(a) First consider the case when $L \subset \widetilde{p}^{-1}(\overline{U})$. Then $h(L)$ is a continuum in $h(\widetilde{p}^{-1}(\overline{U}))$ such that $h(L) \cap (g \circ p)^{-1}(y) \neq \emptyset \neq h(L) \setminus (g \circ p)^{-1}(y)$. Since $g \circ p$ is atomic as the composition of an atomic map and a homeomorphism, we have $h(L) \supset (g \circ p)^{-1}(y)$. Thus $L \supset h^{-1}p^{-1}g^{-1}(y) = \widetilde{p}^{-1}(y)$.

(b) Suppose now that $L \not\subset \widetilde{p}^{-1}(\overline{U})$. Then $L \cap \widetilde{p}^{-1}(\overline{U})$ intersects the boundary of $\widetilde{p}^{-1}(\overline{U})$ in $\widetilde{Y}$. Let $y_0 \in L \cap \widetilde{p}^{-1}(y)$. Then the component K of

$L \cap \widetilde{p}^{-1}(\overline{U})$ containing y_0 intersects the boundary of $L \cap \widetilde{p}^{-1}(\overline{U})$ in L (by Janiszewski's theorem, see [10, §47, III]). Thus $K \setminus \widetilde{p}^{-1}(y) \neq \emptyset \neq K \cap \widetilde{p}^{-1}(y)$ and $K \subset \widetilde{p}^{-1}(\overline{U})$, hence, by case (a), $K \supset \widetilde{p}^{-1}(y)$. Since $L \supset K$, this finishes the proof.

3. The proofs

Proof of Theorem 1.1. Fix $n, m \in \mathbb{N}$. By Lemma 2.5, there exist a pseudo-arc $P \subset \mathbb{R}^2$ and a homeomorphism $r : P \to P$ of period n, which is the restriction of a period n rotation of the plane around $(0,0)$ (so the point $(0,0) \in P$ is a fixed point of r). Let $V_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \lambda \cos\alpha$ and $x_2 = \lambda \sin\alpha$ for some $0 < \lambda < \infty$ and $0 < \alpha < 2\pi/n\}$, $P_0 = P \cap V_0$ and $P_k = r^k(P_0)$ for $k = 0, 1, \ldots, n-1$. Then every P_k is open in P and $P \setminus \bigcup_{k=0}^{n-1} P_k$ is a 0-dimensional boundary subset of P.

Let $A_0 = \{a_1, a_2, \ldots\}$ be a countable dense subset of P_0 such that a_i and a_j are in the same composant of P if and only if $i = j$. Put $A_k = r^k(A_0)$ for $k = 0, 1, \ldots, n-1$ and let $A = \bigcup_{k=0}^{n-1} A_k$. The set A is dense in P. Since a homeomorphic image of a composant of P is a composant of P, every composant of P contains at most n points of A.

Consider now three cases. If $2 \leq m < \infty$ then let $K_1, K_2, \ldots$ be a sequence of topologically different HI m-dimensional Cantor manifolds (see Lemma 2.2). If $m = 1$ then, by Cook's Lemma 2.3, one can find a sequence $K_1, K_2, \ldots$ of HI one-dimensional continua such that: if L and L' are two different nondegenerate subcontinua of K_i and K_j respectively, where $i, j \in \mathbb{N}$, then L and L' are not homeomorphic; in particular, no subcontinuum of any K_i is homeomorphic to the pseudo-arc. If $m = \infty$ then let K_i be any i-dimensional HI Cantor manifold, for $i = 1, 2, \ldots$

Let $M_{nm} = L(P, K_i, \{a_i\})$ be an HI continuum and $p : M_{nm} \to P$ be an atomic mapping satisfying the conditions of Theorem 2.1. We can assume additionally that $M_{nm} \subset P \times I^\infty$, where $I = [0,1]$, and that p is the restriction of the projection of $P \times I^\infty$ onto P. Moreover, we can assume that $p^{-1}(y) = (y, (0, 0, \ldots))$ for every $y \in P \setminus P_0$.

Indeed, assume that $M_{nm} \subset I^\infty$ and $p : M_{nm} \to P$ is as in Theorem 2.1, and for $x, y \in \mathbb{R}^2$ let $\varrho(x,y) = \min(\varrho_e(x,y), 1)$, where ϱ_e the Euclidean metric in the plane. Put $f(x) = (p(x), \varrho(p(x), \mathbb{R}^2 \setminus V_0) \cdot x)$ for $x \in M_{nm}$. Then f is continuous and one-to-one, hence it is a homeomorphism of M_{nm} onto $f(M_{nm}) \subset P \times I^\infty$. Thus we can replace M_{nm} by $f(M_{nm})$ and p by the restriction of the projection of $P \times I^\infty$ onto P.

Let $\overline{r}(y,t) = (r(y), t)$ for $(y,t) \in P \times I^\infty$. For $k = 0, 1, \ldots, n-1$, let $\widetilde{P}_k = \overline{r}^k(p^{-1}(\overline{P}_0))$ and $X_{nm} = \bigcup_{k=0}^{n-1} \widetilde{P}_k$.

Let $\widetilde{p} : X_{nm} \to P$ be the restriction of the projection of $P \times I^\infty$ onto P. The mapping $\widetilde{r} = \overline{r}|X_{nm}$ is a period n homeomorphism of X_{nm} onto X_{nm}, which is the restriction of the product of r and the identity. Thus,

(1) $$\widetilde{p} \circ \widetilde{r}^k = r^k \circ \widetilde{p} \quad \text{for every } k = 1, \dots, n-1,$$

and

(2) $$p(x) = \widetilde{p}(x) \quad \text{for } x \in \widetilde{P}_0.$$

The map $\widetilde{p}$ is atomic by Lemma 2.7. Indeed, we apply Lemma 2.7 to $\widetilde{X} = X_{nm}$, $X = M_{nm}$, $Y = P$ and let $\widetilde{p} : \widetilde{X}_{nm} \to P$ and $p : M_{nm} \to P$ be as above. For every $y \in P$ such that $\widetilde{p}^{-1}(y)$ is nondegenerate there exists $k \in \{0, 1, \dots, n-1\}$ such that $y \in r^k(P_0)$. If we put $U = \widetilde{p}^{-1}(r^k(P_0))$, $h(x) = (\widetilde{r}^k)^{-1}(x)$ for $x \in \overline{U}$ and $g = r^k$ then $gph(y) = r^k(p(\widetilde{r}^k)^{-1}(y)) = \widetilde{p}(y)$ by (1) and (2). Thus the assumptions of Lemma 2.7 are satisfied. It follows that $\widetilde{p}$ is atomic.

The space X_{nm} is an HI continuum, being the preimage of an HI continuum under the atomic mapping $\widetilde{p}$ with HI fibers (see [15]). By Theorem 2.1(iv), $\dim M_{nm} = m$, so $\dim X_{nm} = m$ by the sum theorem.

Note that $\widetilde{p}|\widetilde{p}^{-1}(P \setminus A) : \widetilde{p}^{-1}(P \setminus A) \to P \setminus A$ is one-to-one, so $\widetilde{p}^{-1}(P \setminus A)$ is a one-dimensional set homeomorphic to $P \setminus A$.

On the other hand, if $t \in A$, then $t = r^k(a_i)$ for some $i \in \mathbb{N}$ and $k \in \{0, 1, \dots, n-1\}$; hence $\widetilde{p}^{-1}(t)$ is homeomorphic to K_i. Note that by Theorem 2.1(v) every composant of X_{nm} is the preimage under $\widetilde{p}$ of a composant of P, hence it is the union of a one-dimensional subset homeomorphic to P with finitely many points removed and of at most n disjoint m-dimensional Cantor manifolds homeomorphic to some K_i. In particular, if $t \in A$ then $\widetilde{p}^{-1}(t)$ is a maximal m-dimensional Cantor manifold which is a proper subset of X_{nm} and homeomorphic to some K_i (that is, there is no m-dimensional Cantor manifold contained in a certain composant, homeomorphic to $\widetilde{p}^{-1}(t)$ and containing $\widetilde{p}^{-1}(t)$). For $m > 1$ this follows from the fact that $\widetilde{p}^{-1}(t)$ is obviously a maximal m-dimensional Cantor manifold which is a proper subset of X_{nm}. For $m = 1$, $\widetilde{p}^{-1}(t)$ is a maximal proper subcontinuum of X_{nm} homeomorphic to $\widetilde{p}^{-1}(t)$, since no subcontinuum of $\widetilde{p}^{-1}(t)$ embeds in P.

We will show that $1_{X_{nm}} = \widetilde{r}^0, \widetilde{r}, \widetilde{r}^2, \dots, \widetilde{r}^{n-1}$ are the only homeomorphisms of X_{nm} onto X_{nm}, so $\mathcal{G}(X_{nm})$ is a cyclic group of order n.

Let h be an arbitrary homeomorphism of X_{nm} onto X_{nm}. The image under h of a composant C of X_{nm} is a composant of X_{nm}. Since h must map maximal m-dimensional Cantor manifolds lying in C and homeomorphic to K_i onto maximal m-dimensional Cantor manifolds in $h(C)$ homeomorphic to K_i, and since no two different K_i's are homeomorphic, we have

(3) $$\text{for every } t \in A,\ h(\widetilde{p}^{-1}(t)) = \widetilde{r}^k(\widetilde{p}^{-1}(t)) \text{ for some } k \in \{0, 1, \dots, n-1\}.$$

Thus the mapping $\overline{h} : P \to P$, where $\overline{h}(t) = \widetilde{p}(h(\widetilde{p}^{-1}(t)))$, is well defined. From the upper semicontinuity of $\widetilde{p}^{-1}$ it follows that $\overline{h}$ is continuous (in fact, it is a homeomorphism). By (3) we have

(4) for every $t \in A$, $\overline{h}(t) = r^k(t)$ for some $k \in \{0, 1, \ldots, n-1\}$.

For every $k \in \{0, 1, \ldots, n-1\}$ let $D_k = \{t \in P : \overline{h}(t) = r^k(t)\}$. It is easy to see that every D_k is closed and $D_k \cap D_l = \{(0,0)\}$ for every $k \neq l$. By (4) the set $\bigcup_{k=0}^{n-1} D_k$ is dense in P, hence we have $P = \bigcup_{k=0}^{n-1} D_k$. Since P is connected, every D_k is connected. From indecomposability of P we have $P = D_{k_0}$ for some k_0, so $\overline{h} = r^{k_0}$. Since $\widetilde{p}|\widetilde{p}^{-1}(P \setminus A)$ is one-to-one, h coincides with $\widetilde{r}^{k_0}$ on $\widetilde{p}^{-1}(P \setminus A)$. Since the latter set is dense in X_{nm}, h coincides with $\widetilde{r}^{k_0}$ on the whole space. This ends the proof.

Proof of Theorem 1.2. From the proof of Theorem 1.1 it follows that if $n > 1$ then the continuum M_{nm} and the mapping $p : M_{nm} \to P$ obtained during the construction of X_{nm} have the following properties:

(a) M_{nm} is HI and $\dim M_{nm} = m$,

(b) if $U_m = p^{-1}(P_0)$, then U_m is an open subset of M_{nm} such that every homeomorphism of M_{nm} onto M_{nm} is the identity on $\overline{U}_m$,

(c) if $V_m = p^{-1}(P \setminus \overline{P}_0)$, then V_m is an open nonempty subset of M_{nm} such that $\overline{V}_n \cup \overline{U}_n = M_{nm}$ and $p : V_n \to P \setminus \overline{P}_0$ is a homeomorphism (recall that $p^{-1}(y) = (y, (0, 0, \ldots))$ for all $y \in P \setminus P_0$).

Set $Y_m = M_{nm}$, where n is any fixed natural number > 1. Then Y_m satisfies the conditions of Theorem 1.2. Indeed, by (a) and (b) it suffices to show that Y_m has continuum many different autohomeomorphisms. Applying Lemma 2.6 for $U = P_0$ we obtain a family $\{h_t : t \in T\}$ of different homeomorphisms of P onto P, where $|T| = 2^{\aleph_0}$, such that $h_t|P_0 = 1_{P_0}$. Define $\widetilde{h}_t : Y_m \to Y_m$ in the following way: if $x \in \overline{V}_m$, then $\widetilde{h}_t(x) = p^{-1}h_t p(x)$ and if $x \in \overline{U}_m$ then $\widetilde{h}_t(x) = x$. It is easy to see that $p^{-1}h_t p(x) = x$ for $x \in \overline{V}_m \cap \overline{U}_m$. It follows that $\widetilde{h}_t$ is a homeomorphism of Y_m onto Y_m. Since $p|p^{-1}(K)$ is one-to-one and $h_{t'} \neq h_t$ for $t' \neq t$, we have $\widetilde{h}_{t'} \neq \widetilde{h}_t$ for $t' \neq t$. This ends the proof.

Proof of Theorem 1.3. We use the idea and notation of the proof of Theorem 1.1. First we divide P_0 into two dense 0-dimensional subsets P_0' and P_1' such that P_0' is the union of countably many disjoint sets $F_1, F_2, \ldots$ closed in P. Then we choose a countable dense subset $A_0 = \{a_1, a_2, \ldots\}$ of P_1'. Now, let $B_1, B_2, \ldots$ be a sequence of closed disjoint 0-dimensional subsets of P defined by $B_{2i-1} = F_i$ and $B_{2i} = \{a_i\}$ for $i = 1, 2, \ldots$ Let $K_1, K_2, \ldots$ be a sequence of topologically different HI hereditarily SID Cantor manifolds (see Lemma 2.4).

Let $M = L(P, K_i \times B_i, B_i)$ be an HI continuum and $p : M \to P$ be an atomic mapping satisfying the conditions of Theorem 2.1. As in the proof of Theorem 1.1, we can assume additionally that $M \subset P \times I^\infty$, p is the restriction of the projection of $P \times I^\infty$ onto P and $p^{-1}(y) = (y, (0, 0, \ldots))$ for every $y \in P \setminus P_0$.

Let $\overline{r}$ be the product of r and the identity. For $k = 0, 1, \ldots, n-1$, put $\widetilde{P}_k = \overline{r}^k(p^{-1}(\overline{P}_0))$ and $Z_n = \bigcup_{k=0}^{n-1} \widetilde{P}_k$.

Let $\widetilde{p} : Z_n \to P$ be the restriction of the projection of $P \times I^\infty$ onto P. The map $\widetilde{p}$ is atomic by Lemma 2.7. The mapping $\widetilde{r} = \overline{r}|Z_m$ is a period n homeomorphism of Z_n onto Z_n, which is the restriction of the product of r and the identity. Since $\widetilde{p}$ is an atomic mapping with HI fibers onto an HI continuum, Z_n is HI.

To prove that all nontrivial subcontinua of Z_n are SID, take any nontrivial continuum L contained in Z_n. Note that $B = \bigcup_{k=1}^{n-1} r^k(\bigcup_{i=1}^\infty B_i)$ is a 0-dimensional subset of P such that $P \setminus B$ is 0-dimensional and $\widetilde{p}|\widetilde{p}^{-1}(P \setminus B)$ is one-to-one. Thus $\widetilde{p}^{-1}(P \setminus B)$ is a 0-dimensional set homeomorphic to $P \setminus B$. It follows that L must intersect one of the sets $\widetilde{p}^{-1}(b)$ for $b \in B$. If $L \subset \widetilde{p}^{-1}(b)$ for some $b \in B$, then L is SID, since $\widetilde{p}^{-1}(b)$ is homeomorphic to an HI hereditarily SID Cantor manifold. If, for some $b \in B$, L intersects both $\widetilde{p}^{-1}(b)$ and its complement, then $L \supset \widetilde{p}^{-1}(b)$, by the atomicity of $\widetilde{p}$. It follows that L is SID.

To prove that $1, \widetilde{r}, \widetilde{r}^2, \ldots, \widetilde{r}^{n-1}$ are the only homeomorphisms of Z_n onto Z_n, we modify the reasoning in the proof of Theorem 1.1. First we note that a subcontinuum Z of Z_n is a maximal infinite-dimensional Cantor manifold in Z_n if and only if it is equal to $\widetilde{p}^{-1}(b)$ for some $b \in B$. Indeed, if $b \in B$, then $\widetilde{p}^{-1}(b)$ is homeomorphic to some K_i. Moreover, if $Z \subset Z_n$ is a continuum such that $\widetilde{p}(Z)$ contains two different points x and y, then one can find a partition L between x and y in P disjoint from B (see [6, Theorem 1.5.13]). Then $\widetilde{p}^{-1}(L) \cap Z$ is a partition of Z homeomorphic to a subset of L, hence it is one-dimensional. This implies that Z is not an infinite-dimensional Cantor manifold.

Let h be a homeomorphism of Z_n onto Z_n. Then, for every $b \in B$, h maps $\widetilde{p}^{-1}(b)$ onto a maximal infinite-dimensional Cantor manifold in Z_n, so there exists $b' \in B$ such that $h(\widetilde{p}^{-1}(b)) = \widetilde{p}^{-1}(b')$. Moreover, since every $\widetilde{p}^{-1}(a_i)$ is homeomorphic to K_{2i} and no two different K_i's are homeomorphic, for every $i \in \mathbb{N}$ and $l \in \{0, 1, \ldots, n-1\}$ we have

$$(5) \qquad h(\widetilde{r}^l(\widetilde{p}^{-1}(a_i))) = \widetilde{r}^s(\widetilde{p}^{-1}(a_i)) \quad \text{for some } s \in \{0, 1, \ldots, n-1\}.$$

It follows that the induced continuous mapping $\overline{h} : P \to P$, where $\overline{h} = \widetilde{p}(h(\widetilde{p}^{-1}(t)))$, has the property:

$$(6) \qquad \text{for every } t \in \bigcup_{k=0}^{n-1} r^k(A_0),\ \overline{h}(t) = r^k(t) \text{ for some } k \in \{0, 1, \ldots, n-1\}.$$

Next, as in the proof of Theorem 1.1, we put $D_k = \{t \in P : h(t) = r^k(t)\}$ for $k \in \{0, 1, \ldots, n-1\}$ and show that $P = D_{k_0}$ for some k_0, which implies that $\overline{h} = r^{k_0}$. Since $\widetilde{p}|\widetilde{p}^{-1}(P \setminus B)$ is one-to-one, h coincides with $\widetilde{r}^{k_0}$ on $\widetilde{p}^{-1}(P \setminus B)$. Since the latter set is dense in Z_n, h coincides with $\widetilde{r}^{k_0}$ on the whole Z_n.

3.1. Remark. Note that there exist continuum many topologically different spaces X_{nm} (respectively, Y_m, Z_n) satisfying the conditions of Theorem 1.1 (resp., 1.2, 1.3). Indeed, if we replace in the proof of Theorem 1.1 (resp., 1.2, 1.3) K_1 by a continuum K_1' nonhomeomorphic to any of $K_1, K_2, \ldots$, then we obtain a topologically different continuum. Since we can choose K_1' from an appropriate family of cardinality 2^{∞} satisfying the conditions of Lemma 2.2, 2.3 or 2.4 (see Sec. 2), we can obtain in this way continuum many nonhomeomorphic continua.

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